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NORM-PEAK MULTILINEAR MAPPINGS ON $\mathbb{R}^n$ WITH A CERTAIN NORM

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Abstract. Let $n \geq 2$. A continuous n-linear mapping T from a Banach space E into a Banach space F is called *norm-peak* if there is unique $(x_1, \ldots, x_n) \in E^n$ such that $\|x_1\| = \cdots = \|x_n\| = 1$ and T attains its norm only at $(\pm x_1, \ldots, \pm x_n)$.

Let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm satisfying that $\{W_1, \ldots, W_n\}$ forms a basis and the set of all extreme points of $B_{\mathbb{R}^n_{\|\cdot\|}}$ is $\{\pm W_1, \ldots, \pm W_n\}$.

In this note, we characterize all norm-peak multilinear mapping from $\mathbb{R}^n_{\|\cdot\|}$ into F.

Нехай $N \geq 2$. Неперервне n-лінійне відображення T з банахового простору E в банахів простор F називається *відображенням з піковим значунням норми*, якщо існує єдиний $(x_1, \ldots, x_n)$ в E^n такий, що $\|x_1\| = \cdots = \|x_n\| = 1$ і T досягає своєї норми тільки при $(\pm x_1, \ldots, \pm x_n)$.

Нехай $\mathbb{R}^n_{|/\cdot|/} = \mathbb{R}^n$ із нормою, що задовольняє тому, що $\{W_1, \ldots, W_n\}$ утворює базис, і $\{\pm W_1, \ldots, \pm w_n\}$ — множина всіх екстремальних точок множини $B_{\mathbb{R}^n_{\|\cdot\|}}$.

В цій статті описуються всі полілінійні відображення з піковими значеннями норми з $\mathbb{R}^n_{\|\cdot\|}$ в F.

1. Introduction

In 1961, Bishop and Phelps [3] showed that the set of norm-attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. Lindenstrauss [14] studied norm-attaining operators. The problem of denseness of norm-attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm-attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [2], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm-attaining multilinear forms. Choi and Kim [4] showed that the Radon-Nikodym property is also sufficient for the denseness of norm-attaining polynomials. Acosta [1] studied norm attaining multilinear mappings. Jiménez-Sevilla and Payá [7] studied the denseness of norm-attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces. Payá and Saleh [15] presented new sufficient conditions for the denseness of norm-attaining multilinear forms. Note that the norm denseness problem of the set of norm-attaining forms in the spaces of all continuous multilinear forms is closely related to sets with the Radon-Nikodym property. It is also linked to a broader topic of optimization on infinite dimensional normed spaces and variational principles (see Stegall [17], Finet and Georgiev [6]).

Let $n \in \mathbb{N}$, $n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE : F)$ the Banach space of all continuous n-linear mappings from E into a Banach space F endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} \|T(x_1, \cdots, x_n)\|$. $\mathcal{L}_s(^nE : F)$ denote the closed subspace of all continuous symmetric n-linear mappings. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $\|T(x_1, \ldots, x_n)\| = \|T\|$.

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For $T \in \mathcal{L}(^nE:F)$, we define

$$\text{Norm}(T) = \Big\{(x_1,\ldots,x_n) \in E^n : (x_1,\ldots,x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. Note that $(x_1,\ldots,x_n) \in \text{Norm}(T)$ if and only if $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$. Indeed, if $(x_1,\ldots,x_n) \in \text{Norm}(T)$, then

$$\|T(\epsilon_1 x_1,\ldots,\epsilon_n x_n)\| = \|\epsilon_1\cdots\epsilon_n T(x_1,\ldots,x_n)\| = \|T(x_1,\ldots,x_n)\| = \|T\|,$$

which shows that $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1,\ldots,\epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1,\ldots,n)$, then

$$(x_1,\ldots,x_n) = \Big(\epsilon_1(\epsilon_1 x_1),\ldots,\epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that the norming sets can be empty, finite or infinite.

Examples. (a) Let $(a_k)_{k\in\mathbb{N}}$ be a real sequence such that $a_k > 0$ for all k and $\sum_{k=1}^{\infty} a_k = 1$. Let $n \geq 2$ and

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} a_i\ x_i^{(1)}\cdots x_i^{(n)} \in \mathcal{L}(^nc_0).$$

We claim that Norm$(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that Norm$(T) \neq \emptyset$. Let $\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big)\Big| \leq \sum_{i=1}^{\infty} a_i\ |x_i^{(1)}|\cdots|x_i^{(n)}| \leq \sum_{i=1}^{\infty} a_i = 1,$$

which shows that $|x_i^{(1)}| = \cdots = |x_i^{(n)}| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, Norm$(T) = \emptyset$.

(b) Let

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) = x_1^{(1)}\cdots x_1^{(n)} \in \mathcal{L}(^nc_0).$$

Then, $\|T\| = 1$ and

$$\begin{aligned}\text{Norm}(T) \quad = \quad & \Big\{\Big((\pm 1, x_2^{(1)}, x_3^{(1)},\ldots),\ldots,(\pm 1, x_2^{(n)}, x_3^{(n)},\ldots)\Big) \in (c_0)^n : \\ & |x_j^{(k)}| \leq 1,\ \text{for } k = 1,\ldots,n, j \geq 2\Big\}.\end{aligned}$$

(c) Let

$$T\Big((x_i^{(1)})_{i\in\mathbb{N}},\ldots,(x_i^{(n)})_{i\in\mathbb{N}}\Big) = x_1^{(1)}\cdots x_1^{(n)} \in \mathcal{L}(^n\ell_1).$$

Then, $\|T\| = 1$ and Norm$(T) = \{(\pm e_1,\ldots,\pm e_1)\}$, where $e_1 = (1,0,0,0,\ldots)$.

(d) Let $T \in \mathcal{L}(^nE)$ and $(x_1,\ldots,x_n) \in \text{Norm}(T)$. By the Hahn-Banach theorem there are $f_j \in E^*$ such that

$$f_j(x_j) = \|f_j\| = 1 \text{ for every } j = 1,\ldots,n.$$

Let ϕ be a permutation on $\{1,\ldots,n\}$. We define $T_{\phi,f_1,\ldots,f_n} \in \mathcal{L}(^{2n}E)$ by

$$T_{\phi,f_1,\ldots,f_n}(y_1,\ldots,y_{2n}) := \prod_{j=1}^{n} f_{\phi(j)}(y_{\phi(j)})\ T(y_{n+1},\ldots,y_{2n}).$$

Then

$$(x_{\phi(1)},\ldots,x_{\phi(n)},x_1,\ldots,x_n) \in \text{Norm}(T_{\phi,f_1,\ldots,f_n}).$$

If Norm$(T) \neq \emptyset$, $T \in \mathcal{L}(^nE:F)$, is called *norm attaining* (see [4]). $T \in \mathcal{L}(^nE:F)$ is called *norm-peak* if Norm$(T) = \{(\pm x_1,\ldots,\pm x_n)\}$ for some $(x_1,\ldots,x_n) \in (S_E)^n$.

For more details about the theory of multilinear mappings on a Banach space, we refer to [5].

For $m \in \mathbb{N}$, let $\ell_1^m := \mathbb{R}^m$ with the ℓ_1-norm and $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm. Note that if E is finite dimensional and $T \in \mathcal{L}(^nE)$, $\mathrm{Norm}(T) \neq \emptyset$ since S_E is compact. Kim [8–12] classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}_s(^2\ell_\infty^2), \mathcal{L}(^2\ell_\infty^2), \mathcal{L}(^2\ell_1^2), \mathcal{L}_s(^2\ell_1^3)$, or $\mathcal{L}_s(^3\ell_1^2)$. Kim [13] has classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$, $\|(x, y)\|_{h(w)} = \max\Big\{|y|, |x|+(1-w)|y|\Big\}$.

Saleh [16] showed that the norm-attaining continuous bilinear forms on $L_1(\mu)$ are norm dense in all continuous bilinear forms if and only if μ is completely atomic. This motivates the study of the norming sets of $T \in \mathcal{L}(^2\ell_1)$, since one can expect many continuous bilinear forms with interesting norming sets in the ℓ_1-setting.

Let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm being such that $\{W_1, \ldots, W_n\}$ forms a basis and the set of all extreme points of $B_{\mathbb{R}^n_{\|\cdot\|}}$ is $\{\pm W_1, \ldots, \pm W_n\}$.

In this note, we characterize all norm-peak multilinear mapping from $\mathbb{R}^n_{\|\cdot\|}$ into F.

2. Main results

Throughout this paper we let $\mathbb{R}^n_{\|\cdot\|} = \mathbb{R}^n$ with a norm being such that $\{W_1, \ldots, W_n\}$ forms a basis and the set of all extreme points of $B_{\mathbb{R}^n_{\|\cdot\|}}$ is $\{\pm W_1, \ldots, \pm W_n\}$. Recall that the Krein-Milman theorem states that a compact convex subset of a Hausdorff locally convex topological vector space is equal to the closed convex hull of its extreme points. Using the Krein-Milman theorem, we present an explicit formula for $\|T\|$ for every $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|} : F)$.

Theorem 2.1. *Let F be a Banach space and $n, m \geq 2$. For $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|} : F)$, we have*

$$\|T\| = \max\Big\{\big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\big\| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\}.$$

Proof. Let $M := \max\Big\{\big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\big\| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\}$. Suppose that $(X_1, \ldots, X_m) \in (S_{\mathbb{R}^n_{\|\cdot\|}})^m$. By the Krein-Milman theorem, the closed unit ball of $\mathbb{R}^n_{\|\cdot\|}$ is the closed convex hull of $\{\pm W_1, \ldots, \pm W_n\}$. For $k = 1, \ldots, m$, there are $\delta_i^{(k)} \in \{-1, 1\}$ and $t_i^{(k)} \geq 0$ with $\sum_{i=1}^n t_i^{(k)} \leq 1$ such that

$$X_k = \sum_{i=1}^n \delta_i^{(k)} t_i^{(k)} W_i.$$

It follows that

$$\begin{aligned}
|T(X_1, \ldots, X_m)| &\leq \Big\|T\Big(\sum_{i_1=1}^n \delta_{i_1}^{(1)} t_{i_1}^{(1)} W_{i_1}, \ldots, \sum_{i_m=1}^n \delta_{i_m}^{(m)} t_{i_m}^{(m)} W_{i_m}\Big)\Big\| \\
&\leq \sum_{1 \leq i_k n,\ 1 \leq k \leq m} \big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\big\|\ |\delta_{i_1}^{(1)} t_{i_1}^{(1)}| \cdots |\delta_{i_m}^{(m)} t_{i_m}^{(m)}| \\
&\leq M\Big(\sum_{j=1}^n t_j^{(1)}\Big) \cdots \Big(\sum_{j=1}^n t_j^{(m)}\Big) \\
&\leq M = \max\Big\{\big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\big\| : 1 \leq i_k \leq n,\ 1 \leq k \leq m\Big\} \\
&\leq \|T\|,
\end{aligned}$$

which shows that

$$\|T\| = \sup_{(X_1, \ldots, X_m) \in S_{\mathbb{R}^n_{\|\cdot\|}}} |T(X_1, \ldots, X_m)| \leq M \leq \|T\|.$$

Therefore, $\|T\| = M$. □

Theorem 2.2. *Let F be a Banach space and $n, m \geq 2$. Suppose that $T \in \mathcal{L}(^m\mathbb{R}^n_{\|\cdot\|} : F)$ with $\|T\| = 1$. Then T is norm-peak if and only if there is a unique $(i_1^{'}, \ldots, i_m^{'}) \in \{1, \ldots, n\}^m$ such that for all $(i_1, \ldots, i_m) \in \{1, \ldots, n\}^m \backslash \{(i_1^{'}, \ldots, i_m^{'})\}$,*

$$\Big\|T\big(W_{i_1'}, \ldots, W_{i_m'}\big)\Big\| = 1 > \Big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\Big\|.$$

Proof. ($\Rightarrow$). Suppose that T is norm-peak. Let $\text{Norm}(T) = \{(\pm X_1, \ldots, \pm X_m)\}$ for some $(X_1, \ldots, X_m) \in (S_{\mathbb{R}^n_{\|\cdot\|}})^m$. By the Krein-Milman theorem, for $k = 1, \ldots, m$, there are $\delta_i^{(k)} \in \{-1, 1\}$ and $t_i^{(k)} \geq 0$ with $\sum_{i=1}^n t_i^{(k)} \leq 1$ such that

$$X_k = \sum_{i=1}^n \delta_i^{(k)} t_i^{(k)} W_i.$$

Claim 1. There is $(i_1^{'}, \ldots, i_m^{'}) \in \{1, \ldots, n\}^m$ such that $\Big\|T\big(W_{i_1'}, \ldots, W_{i_m'}\big)\Big\| = 1$.

Suppose not. Choose $(j_1, \ldots, j_m) \in \{1, \ldots, n\}^m$ such that $t_{j_1}^{(1)} \cdots t_{j_m}^{(m)} \neq 0$. It follows that

$$\begin{aligned}
1 &= \|T(X_1, \ldots, X_n)\| \\
&= \Big\|T\Big(\sum_{i_1=1}^n \delta_{i_1}^{(1)} t_{i_1}^{(1)} W_{i_1}, \ldots, \sum_{i_m=1}^n \delta_{i_m}^{(m)} t_{i_m}^{(m)} W_{i_m}\Big)\Big\| \\
&\leq \sum_{1 \leq i_k \leq n,\ 1 \leq k \leq m} \Big\|T\big(W_{i_1}, \ldots, W_{i_n}\big)\Big\|\ t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} \\
&= \Big\|T\big(W_{j_1}, \ldots, W_{j_n}\big)\Big\|\ t_{j_1}^{(1)} \cdots t_{j_m}^{(m)} \\
&\qquad + \sum_{(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)} \Big\|T\big(W_{i_1}, \ldots, W_{i_n}\big)\Big\|\ t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} \\
&< t_{j_1}^{(1)} \cdots t_{j_m}^{(m)} + \sum_{(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)} \Big\|T\big(W_{i_1}, \ldots, W_{i_n}\big)\Big\|\ t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} \\
&\quad \text{(because } \Big\|T\big(W_{j_1}, \ldots, W_{j_n}\big)\Big\| < 1, t_{j_1}^{(1)} \cdots t_{j_m}^{(m)} \neq 0\text{)} \\
&< t_{j_1}^{(1)} \cdots t_{j_m}^{(m)} + \sum_{(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)} t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} \\
&= (t_1^{(1)} + \cdots + t_n^{(1)}) \cdots (t_1^{(m)} + \cdots + t_n^{(m)}) \\
&\leq 1,
\end{aligned}$$

which is a contradiction. Therefore, the claim 1 holds.

Claim 2. $(i_1^{'}, \ldots, i_m^{'}) \in \{1, \ldots, n\}^m$ is unique.

Let $(i_1, \ldots, i_m) \in \{1, \ldots, n\}^m$ such that $\Big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\Big\| = 1$. Since T is norm-peak and $\big(W_{i_1}, \ldots, W_{i_m}\big), \big(W_{i_1'}, \ldots, W_{i_m'}\big) \in \text{Norm}(T)$, we have

$$\Big\{\big(\pm W_{i_1}, \ldots, \pm W_{i_m}\big)\Big\} = \Big\{\big(\pm W_{i_1'}, \ldots, \pm W_{i_m'}\big)\Big\}.$$

Thus, $i_k = i_k^{'}$ for all $k = 1, \ldots, m$. Therefore, the claim 2 holds.

($\Leftarrow$). Suppose that there is a unique $(i_1^{'}, \ldots, i_m^{'}) \in \{1, \ldots, n\}^m$ such that for all $(i_1, \ldots, i_m) \in \{1, \ldots, n\}^m \backslash \{(i_1^{'}, \ldots, i_m^{'})\}$,

$$\Big\|T\big(W_{i_1'}, \ldots, W_{i_m'}\big)\Big\| = 1 > \Big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\Big\|.$$

Let $(Y_1, \ldots, Y_m) \in \mathrm{Norm}(T)$. By the Krein-Milman theorem, for $k = 1, \ldots, m$, there are $\epsilon_i^{(k)} \in \{-1, 1\}$ and $s_i^{(k)} \geq 0$ with $\sum_{i=1}^n s_i^{(k)} \leq 1$ such that

$$Y_k = \sum_{i=1}^n \epsilon_i^{(k)} s_i^{(k)} W_i.$$

Claim 3. If $(i_1, \ldots, i_m) \neq (i_1^{'}, \ldots, i_m^{'})$, then $s_{i_1}^{(1)} \cdots s_{i_m}^{(m)} = 0$.

Suppose not. Then there is $(j_1, \ldots, j_m) \neq (i_1^{'}, \ldots, i_m^{'})$ such that $s_{j_1}^{(1)} \cdots s_{j_m}^{(m)} \neq 0$. It follows that

$$\begin{aligned}
1 &= \|T(Y_1, \ldots, Y_m)\| \\
&= \Big\|T\Big(\sum_{i_1=1}^n \epsilon_{i_1}^{(1)} s_{i_1}^{(1)} W_{i_1}, \ldots, \sum_{i_n=1}^n \epsilon_{i_m}^{(m)} s_{i_m}^{(m)} W_{i_m}\Big)\Big\| \\
&\leq \sum_{1 \leq i_k \leq n,\ 1 \leq k \leq m} \Big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\Big\| \ s_{i_1}^{(1)} \cdots s_{i_m}^{(m)} \\
&= \Big\|T\big(W_{j_1}, \ldots, W_{j_m}\big)\Big\| \ s_{j_1}^{(1)} \cdots s_{j_m}^{(m)} \\
&\qquad + \sum_{(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)} \Big\|T\Big(W_{i_1}, \ldots, W_{i_m}\Big)\Big\| \ s_{i_1}^{(1)} \cdots s_{i_m}^{(m)} \\
&< s_{j_1}^{(1)} \cdots s_{j_m}^{(m)} + \sum_{(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)} \Big\|T\big(W_{i_1}, \ldots, W_{i_m}\big)\Big\| \ s_{i_1}^{(1)} \cdots s_{i_m}^{(m)} \\
&\quad \text{(because } \Big\|T\big(W_{j_1}, \ldots, W_{j_m}\big)\Big\| < 1, s_{j_1}^{(1)} \cdots s_{j_m}^{(m)} \neq 0) \\
&< s_{j_1}^{(1)} \cdots s_{j_m}^{(m)} + \sum_{(i_1, \ldots, i_m) \neq (j_1, \ldots, j_m)} s_{i_1}^{(1)} \cdots s_{i_m}^{(m)} \\
&= (s_1^{(1)} + \cdots + s_n^{(1)}) \cdots + (s_1^{(n)} + \cdots + s_n^{(m)}) \\
&\leq 1,
\end{aligned}$$

which is a contradiction. Therefore, the claim 3 holds.

Claim 4. $(Y_1, \ldots, Y_m) \in \Big\{\big(\pm W_{i_1'}, \ldots, \pm W_{i_m'}\big)\Big\}$.

Let $i_1 \in \{1, \ldots, n\} \backslash \{i_1^{'}\}$. Choose $j_2, \ldots, j_m \in \{1, \ldots, n\}$ such that $s_{j_2}^{(2)} \cdots s_{j_m}^{(m)} \neq 0$. Since $(i_1, j_2, \ldots, j_m) \neq (i_1^{'}, \ldots, i_m^{'})$, by Claim 3, then $s_{i_1}^{(1)}(s_{j_2}^{(2)} \cdots s_{j_m}^{(m)}) = 0$. Thus, $s_{i_1}^{(1)} = 0$ and $s_{i_1'}^{(1)} = 1$, so $Y_1 = \pm W_{i_1'}$. Similarly, $Y_m = \pm W_{i_m'}$. Thus, the claim 4 holds. Therefore, $\mathrm{Norm}(T) = \Big\{\big(\pm W_{i_1'}, \ldots, \pm W_{i_m'}\big)\Big\}$, which shows that T is norm-peak. This completes the proof. □

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