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MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

BICOMPLEX PALEY-WIENER THEOREM

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Abstract. In this paper, we study the bicomplex version of the Paley-Wiener theorem and the Cauchy integral formula in the upper half-plane.

Вивчається теорема Пейлі-Вінера та інтегральна формула Коші в верхній півплощині у випадку бікомплексних чисел.

1. Introduction

The study of bicomplex numbers started in 1892 when Segre [36] found that the property of commutativity had been missing from the skew field of quaternions. The quaternions were first introduced by W. R. Hamilton in 1844. The study of bicomplex numbers has always been an active field of research. Segre was inspired by the works of Hamilton, and then he introduced a new number system called the bicomplex numbers.

The work of J. D. Riley in [30] has further developed the theory of functions with bicomplex variables. Also, without forgetting to mention the work of G. B. Price [28], who provided us with a very powerful method to study holomorphic functions with bicomplex variables.

We denote the set of bicomplex numbers by $\mathbb{BC}$ and define it as follows:

$$\mathbb{BC} = \{z_1 + jz_2 \mid z_1, z_2 \in \mathbb{C}\},$$

where $\mathbb{C}$ is the set of complex numbers. Therefore, bicomplex numbers are sometimes called complex numbers with complex coefficients. The set of complex numbers has the imaginary unit i. In $\mathbb{BC}$, there are two imaginary units, i and j, which commute, i.e., $ij = ji$, and satisfy $i^2 = j^2 = -1$. Bicomplex numbers can be added and multiplied, and both operations are commutative and associative.

There are three types of conjugations on the set of bicomplex numbers. These are the "bar conjugation," the "$\star$-conjugation," and the "$\dagger$-conjugation", which are given as follows:

— $Z = \bar{z_1} + j\bar{z_2}$ the bar conjugation.
— $Z = z_1 - jz_2$ the $\dagger$-conjugation.
— $Z^* = \bar{z_1} - j\bar{z_2}$ the $\star$-conjugation.

Here $z_1, z_2 \in \mathbb{C}(i)$ and $\bar{z_1}$ and $\bar{z_2}$ are the classical conjugates of z_1 and z_2. Due to these three types of conjugations, three types of moduli arise in the set of bicomplex numbers. Among the three, we use only the moduli that arise due to the $\star$-conjugation. For more details on these conjugations, one can see [6, p. 8]. We will provide more details on this in later sections.

Another important set of numbers is the set of hyperbolic numbers, which can be defined independently of $\mathbb{BC}$. We denote the set of hyperbolic numbers by

$$\mathbb{D} = \{a + kb \mid a, b \in \mathbb{R}\},$$

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where k is called the hyperbolic unit with $k^2 = 1$. The set of hyperbolic numbers is also called split-real numbers, Lorentz numbers, perplex numbers, etc. These were first introduced by Cockle [9]. While working with $\mathbb{BC}$, we encounter $ij = k$. Thus, we realize that there exists a subset of the bicomplex numbers that is isomorphic to the set of split-real numbers. Thus, we can define the set of split-real numbers as

$$\mathbb{D} = \{a + ijb \mid a, b \in \mathbb{R}\}.$$

The following subset of $\mathbb{D}$, which is given as

$$\mathbb{D}^+ = \{a + kb \mid a^2 - b^2 \geq 0, a \geq 0\},$$

will be especially useful later.

Another important feature of the bicomplex numbers $\mathbb{BC}$ is the presence of idempotent units e and $e^\dagger$, which makes it possible to represent the bicomplex numbers in their idempotent form. The idempotent units are also called special zero divisors as

$$e = \frac{1 + ij}{2} \qquad \text{and} \qquad e^\dagger = \frac{1 - ij}{2}$$

Observe that $e.e^\dagger = 0$ and $1 = e + e^\dagger$. Also,

$$e^2 = e \qquad \text{and} \qquad (e^\dagger)^2 = e^\dagger.$$

Thus any bicomplex number Z can be represented as $Z = e\beta_1 + e^\dagger\beta_2$, where β_1 and β_2 are complex numbers. This is called representing a bicomplex number in terms of its idempotent units. Next, we describe a few representations of $Z \in \mathbb{BC}$. Any $Z \in \mathbb{BC}$ can be written as

$$Z = z_1 + jz_2 \tag{1.1}$$
$$= e\beta_1 + e^\dagger\beta_2 \tag{1.2}$$
$$= a_1 + ia_2 + ja_3 + ka_4. \tag{1.3}$$

The equation (1.1) determines Z as an element of $\mathbb{C}^2(i)$, while the equation (1.3) identifies Z as an element of $\mathbb{R}^4$, equation (1.2) is the idempotent representation of Z see [12, Page 7] for more details. As we have mentioned earlier, among the three moduli that arise due to the three conjugations, we use the modulus that arises due to the $*$-conjugation and call it the k-modulus. Let us discuss this in more detail.

Definition 1.1. [6] The k-modulus is denoted by $\|\cdot\|_k$. It is hyperbolic-valued and is defined as

$$\|Z\|_k^2 = Z \cdot Z^*.$$

Using the idempotent decompositions of $Z \in \mathbb{BC}$, we see that

$$\|Z\|_k^2 = e|\beta_1|_1^2 + e^\dagger|\beta_2|_2^2,$$

where $|\cdot|_1$ and $|\cdot|_2$ are the classical complex components of $\|\cdot\|_k$. This means we have a map $\|\cdot\|_k : \mathbb{BC} \longrightarrow \mathbb{D}^+$, which satisfies all the properties of a norm and hence is called a hyperbolic-valued norm, or a $\mathbb{D}$-norm, or k-norm.

The next definition presents the upper half-plane in $\mathbb{BC}$.

Definition 1.2. [22] We denote the upper half-plane in $\mathbb{BC}$ by $\Pi^+_{\mathbb{BC}}$ and it is defined as

$$\Pi^+_{\mathbb{BC}} = \{Z \in \mathbb{BC} : Z = z_1 + jz_2 \ or\ Z = e\beta_1 + e^\dagger\beta_2 : (\beta_1, \beta_2) \in \Pi^+ \times \Pi^+\},$$

where $\Pi^+ = \{z \in \mathbb{C}(i) : z = x + iy \text{ and } y > 0 \in \mathbb{C}(i)\}$ is the upper half-plane in $\mathbb{C}$.

By using the consequence of the idempotent decompositions, it is very easy to see that

$$\Pi^+_{\mathbb{BC}} = e\Pi^+ + e^\dagger\Pi^+. \tag{1.4}$$

Definition 1.3. [29] A set $G \subset \mathbb{BC}$ is called a product type set if $G = eG_1 + e^\dagger G_2$, where $G_1 = \Pi_{1,i}(G)$ and $G_2 = \Pi_{2,i}(G)$, and $\Pi_{1,i}(G)$ and $\Pi_{2,i}(G)$ are the idempotent projections of G on G_1 and G_2, respectively. That is, a set $G \subset \mathbb{BC}$ is said to be of product type in $\mathbb{BC}$ if $\Pi_{1,i}(G)$ and $\Pi_{2,i}(G)$ are open and connected sets in the complex plane.

Definition 1.4. [6] Let a and b be two hyperbolic numbers such that $a \leq b$. Then a hyperbolic interval, denoted by $[a,b]_\mathbb{D}$, is defined as

$$[a,b]_\mathbb{D} = \{\mathfrak{z} \in \mathbb{D}\ ;\ a \leq \mathfrak{z} \leq b\}.$$

Definition 1.5. [16, 29] Let $G \subset \mathbb{BC}$ be such that $G = eG_1 + e^\dagger G_2$, where G_1 and G_2 are the idempotent components of the domain G in the complex plane. Then, a bicomplex function $F : G \subset \mathbb{BC} \longrightarrow \mathbb{BC}$ is said to be of product type if $F(Z) = ef_1(\beta_1) + e^\dagger f_2(\beta_2)$, where each $f_k : G_k \longrightarrow \mathbb{C}$ for $k = 1, 2$ are complex-valued functions such that $F(e\beta_1 + e^\dagger\beta_2) = ef_1(\beta_1) + e^\dagger f_2(\beta_2)$, for all $e\beta_1 + e^\dagger\beta_2 \in G$.

Definition 1.6. [24] Let $\mathfrak{M}$ be a σ-algebra on a set G. A hyperbolic real-valued bicomplex function $m = em_1 + e^\dagger m_2$ defined on G is called a hyperbolic measure if m_1 and m_2 are real measures on $\mathfrak{M}$.

Definition 1.7. [6] Let F be a bicomplex product type function defined on a domain $G \subset \mathbb{BC}$. Then

$$F(e\beta_1 + e^\dagger\beta_2) = eF_1(\beta_1) + e^\dagger F_2(\beta_2), \tag{1.5}$$

where $G \subset \mathbb{BC}$, is a bicomplex domain of product type defined in Definition 1.5.

It is worth noting that any Z in the upper half plane $\Pi^+_{\mathbb{BC}}$ can also be written as

$$Z = e\beta_1 + e^\dagger\beta_2 \in\ \Pi^+_{\mathbb{BC}}\ \text{ if and only if }\ \beta_1 = z_1 - iz_2 \in\ \Pi^+ \text{ and } \beta_2 = z_1 + iz_2 \in\ \Pi^+.$$

Then a simple elaboration shows that

$$\beta_1 = z_1 - iz_2 = (x_0 + ix_1) - i(x_2 + ix_3) = (x_0 + x_3) + i(x_1 - x_2).$$

Thus,

$$\beta_1 \in\ \Pi^+\ \text{ if and only if }\ x_1 - x_2 > 0 \tag{1.6}$$

and

$$\beta_2 = z_1 + iz_2 = (x_0 + ix_1) + i(x_2 + ix_3) = (x_0 - x_3) + i(x_1 + x_2).$$

Therefore,

$$\beta_2 \in\ \Pi^+\ \text{ if and only if }\ x_1 + x_2 > 0. \tag{1.7}$$

Hence equations (1.6) and (1.7) imply that $\beta_1, \beta_2 \in \Pi^+$ if and only if $x_1 > |x_2|$.

Now we denote the boundary of the bicomplex upper half-plane by $\partial\Pi^+_{\mathbb{BC}}$; it is defined as

$$\begin{aligned}\partial\Pi^+_{\mathbb{BC}} =& e\partial\Pi^+ + e^\dagger\partial\Pi^+\\ =& e\mathbb{R} + e^\dagger\mathbb{R}\\ =& \mathbb{D},\end{aligned} \tag{1.8}$$

which is the set of hyperbolic numbers. Here $\partial\Pi^+$ is the boundary of the idempotent components of the bicomplex upper half-plane as given in Definition 1.2.

Then we define the $\mathbb{D}-$integral of F on the boundary of the bicomplex upper half-plane $\Pi^+_{\mathbb{BC}}$ by

$$\int_{\partial\Pi^+_{\mathbb{BC}}} F(Z)dZ \odot dZ^\dagger = e\int_{-\infty}^{\infty} F_1(\beta_1)d\beta_1 + e^\dagger\int_{-\infty}^{\infty} F_2(\beta_2)d\beta_2\ \ , \qquad Z \in \Pi^+_{\mathbb{BC}}.$$

Using this definition of $\mathbb{D}-$integral, we say that a bicomplex function F on $\partial\Pi^{+}_{\mathbb{BC}}$ is $\mathbb{D}-$square integrable if

$$\int_{\partial\Pi^{+}_{\mathbb{BC}}} \|F\|_k^2 dm < \infty,$$

where dm is the four-dimensional Lebesgue measure such that $dm = edm_1 + e^{\dagger}dm_2$.

Using equation (1.5), we can say that F is $\mathbb{D}$-square integrable if and only if F_1 and F_2 are square integrable. That is,

$$\int_{-\infty}^{\infty} |F_i|_k^2 dm_i < \infty.$$

We denote the space of all $\mathbb{D}-$square integrable functions on $\partial\Pi^{+}_{\mathbb{BC}}$ by $L^2_k(\partial\Pi^{+}_{\mathbb{BC}})$, see [27] and consequently,

$$L^2_k(\partial\Pi^{+}_{\mathbb{BC}}) = eL^2(-\infty,\infty) + e^{\dagger}L^2(-\infty,\infty). \tag{1.9}$$

The hyperbolic norm of $F \in L^2_k(\partial\Pi^{+}_{\mathbb{BC}})$ is defined as

$$\|F\|^2_{k,2} = e|F_1|^2_{1,2} + e^{\dagger}|F_2|^2_{2,2},$$

where $|\cdot|^2_{1,2}$ and $|\cdot|^2_{2,2}$ are the classical components of the hyperbolic norm. That is,

$$\int_{\partial\Pi^{+}_{\mathbb{BC}}} \|F\|_k^2 dm = e\int_{-\infty}^{\infty} |F_1|^2_{1,2} dm_1 + e^{\dagger}\int_{-\infty}^{\infty} |F_2|^2_{1,2} dm_2.$$

Theorem 1.8. *For $1 \leqslant p \leqslant \infty$, a Cauchy sequence $\{F_n\}$ in $L^p_k(dm)$ with limit F has a pointwise convergent subsequence, almost everywhere to $F(x_0, x_3)$.*

Proof. The proof of the above theorem is quite simple. From the definition of $\mathbb{D}$-square integrable in [27], we have

$$L^p_k(dm) = eL^p(dm_1) + e^{\dagger}L^p(dm_2). \tag{1.10}$$

Let $Z = e\beta_1 + e^{\dagger}\beta_2$. Then, knowing the fact that for every Cauchy sequence $\{F_{n,1}\}$ and $\{F_{n,2}\}$ in $L^p(dm_1)$ and $L^p(dm_2)$ with limits F_1 and F_2, there exist convergent subsequences converging to $F_1(\mathrm{Re}(\beta_1))$ and $F_2(\mathrm{Re}(\beta_2))$, respectively. Thus, the theorem holds for every Cauchy sequence $\{F_n\}$ in $L^p_k(dm)$. $\square$

Corollary 1.9. *Let $\hat{F}$ be the bicomplex Fourier transform of the bicomplex function F. If F lies in L^2_k and $\hat{F} \in L^1_k$, then*

$$F(x_0, x_3) = \int_{\partial\Pi^{+}_{\mathbb{BC}}} \hat{F}(t)\exp\{i(x_0 + kx_3)\}\, dm(t) \ \ a.e.$$

For recent work on bicomplex analysis and its applications, one can refer to [8, 10, 26, 6, 12] and the references therein.

2. Bicomplex Fourier Transforms

The bicomplex Fourier transform for functions of bicomplex variables is studied in [7, 8, 18]. The standard bicomplex Fourier transform is defined as

$$\hat{F}_{\mathbb{BC}} = \frac{1}{\sqrt{2\pi}}\int_{\partial\Pi^{+}_{\mathbb{BC}}} \exp\{-itZ\}F(t)\, dt \odot dt^{\dagger},$$

where $Z = e\beta_1 + e^{\dagger}\beta_2$ and $dt \odot dt^{\dagger}$ is notation to separate the e and $e^{\dagger}$ components when we apply the idempotent decompositions.

Now, using the idempotent units e and $e^\dagger$, we have

$$\begin{aligned}\hat{F}_{\mathbb{BC}} =& \frac{1}{2\pi}\int_{\partial\prod^+_{\mathbb{BC}}} \exp\{-it(e\beta_1 + e^\dagger\beta_2)\}\,dt\odot dt^\dagger\\ =&\frac{1}{2\pi}e\int_{-\infty}^{\infty}\exp\{-it\beta_1\}F_1(t)\,dt + \frac{1}{2\pi}e^\dagger\int_{-\infty}^{\infty}\exp\{-it\beta_2\}F_2(t)\,dt^\dagger\\ =& e\mathcal{F}_1(F_1) + e^\dagger\mathcal{F}_2(F_2).\end{aligned}$$

Example 2.1. [7] Consider $F(t) = \exp\{-\|t\|_k\}$. Then,

$$\hat{F}_{\mathbb{BC}} = \frac{2}{1+Z^2}, \quad \text{where} \quad Z = z_1 + jz_2,$$

with

$$\hat{F}_{\mathbb{BC}} = e\hat{F}_1(z_1) + e^\dagger\hat{F}_2(z_2),$$

$$\hat{F}_1(z_1) = \frac{2}{1+z_1^2} \quad \text{and} \quad \hat{F}_2(z_2) = \frac{2}{1+z_2^2},$$

such that $\hat{F}_1(z_1)$ and $\hat{F}_2(z_2)$ are holomorphic in $-1 < \mathrm{Im}(z_1)$ and $\mathrm{Im}(z_2) < 1$.

The above example shows that, often, $\hat{F}$ can be extended to a function holomorphic in some regions of $\mathbb{BC}$. Next, keeping in mind that $\exp\{itZ\}$ is a holomorphic function of Z, we can expect and discuss a few conditions on F that, when imposed, turn its bicomplex Fourier transform $\hat{F}(t)$ into a holomorphic function in certain regions of $\mathbb{BC}$.

For the above claim, let $\mathfrak{F} \in L^2_k(\partial\prod^+_{\mathbb{BC}}, dm)$ such that $\mathfrak{F}(t) = 0$ on $(-\infty, 0)\times(-\infty, 0)$. Then define

$$F(Z) = \int_{(0,\infty)\times(0,\infty)} \mathfrak{F}(t)\exp\{itZ\}dt\odot dt^\dagger, \tag{2.11}$$

where Z lies in the bicomplex upper half-plane $\prod^+_{\mathbb{BC}}$. Then

$$\begin{aligned}\exp\{itZ\} =& \exp\{it(e\beta_1 + e^\dagger\beta_2)\}\\ =& e\exp\{it\beta_1\} + e^\dagger exp\{it\beta_2\}\\ =& e\exp\{it[(x_0+x_3) + i(x_1-x_2)]\} + e^\dagger\exp\{it[(x_0-x_3)+i(x_1+x_2)]\}.\end{aligned}$$

Therefore, if $Z\in \prod^+_{\mathbb{BC}}$, then

$$\begin{aligned}\|\exp\{itZ\}\|_k =& \|\ e\exp\{it[(x_0+x_3) + i(x_1-x_2)]\}\\ &+e^\dagger\exp\{it[(x_0-x_3)+i(x_1+x_2)]\|_k\\ &\leqslant e\|\exp\{it(x_0+x_3)\}\exp\{-t(x_1-x_2)\}\|_1\\ &\quad e^\dagger\|\exp\{it(x_0-x_3)\}\exp\{-t(x_1+x_2)\}\|_2\\ &\leqslant e\|\exp\{-t(x_1-x_2)\}\|_1 + e^\dagger\|\exp\{-t(x_1+x_2)\}\|_2\\ &=\ e\|\exp\{-tImg(\beta_1)\}\|_1 + e^\dagger\|\exp\{-tImg(\beta_2)\}\|_2\\ &=\exp\{(e(-tImg\beta_1)) + e^\dagger(-tImg\beta-2)\}\\ &=\exp\{-t(e(x_1-x_2)+e^\dagger(x_1+x_2))\}\\ &=\exp\{-t(x_1-kx_2)\}.\end{aligned}$$

Hence, (2.11) exists and is well-defined.

From equation (2.11),

$$\begin{aligned} F(Z) =& \int_{(0,\infty)\times(0,\infty)} \mathfrak{F}(t) \exp\{itZ\} dt \odot dt^{\dagger} \\ =& e \int_0^{\infty} \mathfrak{F}_1(t) \exp\{it\beta_1\} dt + e^{\dagger} \int_0^{\infty} \mathfrak{F}_2(t) \exp\{it\beta_2\} dt^{\dagger} \\ =& eF_1(\beta_1) + e^{\dagger} F_2(\beta_2). \end{aligned} \tag{2.12}$$

Thus F is holomorphic on $\prod_{\mathbb{BC}}^{+}$, as each F_i is holomorphic in $\prod^{+}$. Here each F_i is defined as

$$F_i(\beta_i) = \int_0^{\infty} \mathfrak{F}_i(t) \exp\{it\beta_1\}.$$

For more details, see [35].

Next, we show that the restrictions of these functions to the horizontal lines in $\prod_{\mathbb{BC}}^{+}$ is bounded in $L_k^2\big(\partial \prod_{\mathbb{BC}}^{+})\big)$. Let $Z \in \prod_{\mathbb{BC}}^{+}$. Then $Z = e\beta_1 + e^{\dagger}\beta_2 = x_0 + ix_1 + jx_2 + kx_3$, and from equation (2.12), we have

$$F(Z) = eF_1(\beta_1) + e^{\dagger} F_2(\beta_2), \tag{2.13}$$

where each F_1 and F_2 are of the form

$$F_1(\beta_1) = \int_0^{\infty} \mathfrak{F}_1(t) \exp\{-t(x_1 - x_2)\} \exp\{it(x_0 + x_3)\} dt, \tag{2.14}$$

$$F_2(\beta_2) = \int_0^{\infty} \mathfrak{F}_2(t) \exp\{-t(x_1 + x_2)\} \exp\{it(x_0 - x_3)\} dt. \tag{2.15}$$

Then F_1 and F_2 are the restrictions to the horizontal lines in $\prod^{+}$ and, from [35], we see that these restrictions form a bounded set in $L^2(-\infty, \infty)$. Hence from equation (2.13), we see that the restrictions of F to the horizontal lines in $\prod_{\mathbb{BC}}^{+}$ form a bounded set in $L_k^2(\partial \prod_{\mathbb{BC}}^{+})$. Thus, the following remark concludes that:

Remark 2.2. The restrictions F_1 and F_2 of F to the horizontal lines also form a bounded set in $L_k^2\big(\partial \prod_{\mathbb{BC}}^{+}\big)$. For more details, we refer to [35].

3. Bicomplex Paley-Wiener Theorem

In this section, we generalize the Paley-Wiener theorem in a bicomplex setting. The basis of the Paley-Wiener theorem lies in the outstanding fact that the converse of the Remark 2.2 is also true.

Theorem 3.1. *Let $F : \prod_{\mathbb{BC}}^{+} \longrightarrow \mathbb{BC}$ be a holomorphic function on $\prod_{\mathbb{BC}^{+}}$ and*

$$\sup_{\substack{\mathbb{D} \\ x_1 > |x_2|}} \frac{1}{2\pi} \int_{\partial \prod_{\mathbb{BC}}^{+}} \|F(Z)\|_k^2 dx_0 = M < \infty.$$

Then there exists $\mathfrak{F} \in L_k^2\big(\partial \prod_{\mathbb{BC}}^{+}\big)$ such that

$$F(Z) = \int_{(0,\infty)\times(0,\infty)} \mathfrak{F}(t) \exp\{itZ\} dt \odot dt^{\dagger},$$

where Z lies in $\prod_{\mathbb{BC}}^{+}$ with $Z = e\beta_1 + e^{\dagger}\beta_2 = x_0 + ix_1 + jx_2 + kx_3$ and

$$\int_{(0,\infty)\times(0,\infty)} \|F(t)\|_k^2 dt = M$$

for some constant M.

Proof. We begin the proof with an assumption that a holomorphic $L_k^2\big(\partial\Pi_{\mathbb{BC}}^+\big)$ function exists, say $\mathfrak{F}$ and let F be a bicomplex holomorphic function defined on the upper half-plane $\Pi_{\mathbb{BC}}^+$. Then

$$F(Z) = eF_1(\beta_1) + e^{\dagger}F_2(\beta_2),$$

where F_l for $l = 1, 2$ is holomorphic on the complex upper half-plane Π^+. Then, by the classical Paley-Wiener Theorem, for each F_1 and $F_2 \in H(\Pi^+)$, there exist $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in $L^2((0,\infty))$ such that each $F_1((x_0+x_3)+i(x_1-x_2))$ and $F_2((x_0-x_3)+i(x_1+x_2))$ are the inverse Fourier transform of $\mathfrak{F}_1\exp\{-(x_1-x_2)t\}$ and $\mathfrak{F}_2\exp\{-(x_1+x_2)t\}$, respectively, that is,

$$F_1(\beta_1) = \mathcal{F}^{-1}\big(\mathfrak{F}_1(t)\exp\{-(x_1-x_2)\}\big) \tag{3.16}$$

and

$$F_2(\beta_2) = \mathcal{F}^{-1}\big(\mathfrak{F}_2(t)\exp\{-(x_1+x_2\}\big). \tag{3.17}$$

Then, by the inversion formula, we have

$$\mathfrak{F}_1(t) = \mathcal{F}\{F_1(\beta_1)\exp\{-(x_1-x_2)t\}\} \tag{3.18}$$

$$\mathfrak{F}_2(t) = \mathcal{F}\{F_2(\beta_2)\exp\{-(x_1+x_2)t\}\}. \tag{3.19}$$

Now, from equations (3.18) and (3.19), we get

$$\begin{aligned}
e\mathfrak{F}_1(t) + e^{\dagger}\mathfrak{F}_2(t) =& e\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\beta_1)\exp\{-it(x_0+x_3) - tx_1 + tx_2\}dx_0\right) \\
&+ e^{\dagger}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\beta_2)\exp\{-it(x_0-x_3) - tx_1 + tx_2\}dx_0^{\dagger}\right) \\
=& e\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\beta_1)\exp\{-it\beta_1\}dx_0 + e^{\dagger}\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\beta_2)\exp\{-it\beta_2\}dx_0^{\dagger} \\
=& \frac{1}{2\pi}\int_{\partial\Pi_{\mathbb{BC}}^+} F(Z)\exp\{-itZ\}dZ \odot dZ^{\dagger} \\
=& \mathfrak{F}(Z).
\end{aligned}$$

Thus, for a bicomplex holomorphic function in $\Pi_{\mathbb{BC}}^+$, we assumed the existence of an $L_k^2\big(\partial\Pi_{\mathbb{BC}}^+\big)$ function $\mathfrak{F}$ such that

$$\mathfrak{F}(Z) = \frac{1}{2\pi}\int F(Z)\exp\{-itZ\}dZ \odot dZ^{\dagger}. \tag{3.20}$$

Note that $dZ \odot dZ^{\dagger}$ is a notation to separate the terms when we apply the idempotent decompositions. The integral in (3.20) is the result of choosing a horizontal line in $\Pi_{\mathbb{BC}}^+$, as the equations (3.16) and (3.17) are representations along the horizontal lines in Π^+. Now, we need to show that $\mathfrak{F} \in L_k^2\big((0,\infty)\times(0,\infty)\big)$ is uniquely defined. So, we use the Cauchy theorem here.

For this, let $\curlywedge_\alpha$ be a rectangular path in $\Pi_{\mathbb{BC}}^+$. Then $\curlywedge_\alpha$ being a closed path, can be written as

$$\curlywedge_\alpha = e\curlywedge_{\alpha_1} + e^{\dagger}\curlywedge_{\alpha_2}, \tag{3.21}$$

where $\curlywedge_{\alpha_1}$ and $\curlywedge_{\alpha_2}$ are rectangular paths in $e\Pi^+$ and $e^{\dagger}\Pi^+$, respectively. Using the equation (3.21), we can assume the vertices of $\curlywedge_{\alpha_1}$ as $e(\pm\alpha+i), e^{\dagger}(\pm\alpha+i)$ and $e(\pm\alpha+iy), e^{\dagger}(\pm\alpha+iy)$, let

$$I = \int_{\curlywedge_\alpha} F(Z)\exp\{-itZ\}dZ \odot dZ^{\dagger}.$$

Then,

$$I = e\int_{\curlywedge_{\alpha_1}} F_1(\beta_1)\exp\{-it\beta_1\}d\beta_1 + e^{\dagger}\int_{\curlywedge_{\alpha_2}} F_2(\beta_2)\exp\{-it\beta_2\}d\beta_2, \tag{3.22}$$

where $F = eF_1 + e^{\dagger}F_2$ such that $F_1, F_2 \in H(\prod^+)$ and $Z = e\beta_1 + e^{\dagger}\beta_2$ such that $(\beta_1, \beta_2) \in \prod^+ \times \prod^+$. So, by using Cauchy's theorem, we get

$$I = 0. \tag{3.23}$$

Using the equation (3.22), we have $I = eI_1 + e^{\dagger}I_2$. Solving I_1 for the straight lines $e(\gamma+i)$ to $e(\gamma+iy)$, we get a sequence $(\alpha_{1,j})_{j=1}^{\infty}$ such that $I_1(\alpha_{1,j}) \longrightarrow 0$ and $I_1(-\alpha_{1,j}) \longrightarrow 0$ as $j \to \infty$ in $e\prod^+$.

Similarly, for I_2, we find a sequence $(\alpha_{2,j})_{j=1}^{\infty}$ such that $I_2(\alpha_{2,j}) \longrightarrow 0$ as $j \to \infty$ and $I_2(-\alpha_{2,j}) \longrightarrow 0$ as $j \to \infty$ in $e^{\dagger}\prod^+$. Thus there must be a sequence $\{\alpha_{k,j}\}_{j=1}^{\infty}$ such that $\{\alpha_{k,j}\}_{j=1}^{\infty} = e\{\alpha_{1,j}\}_{j=1}^{\infty} + e^{\dagger}\{\alpha_{2,j}\}_{j=1}^{\infty}$ and

$$I(\alpha_{k,j}) \longrightarrow 0 \ \text{ and } \ I(-\alpha_{k,j}) \longrightarrow 0. \tag{3.24}$$

Proceeding further, define

$$G_j(x_1, x_2, t) = \frac{1}{2\pi}\int_{-\alpha_{k,j}}^{\alpha_{k,j}} F(z_1 + jz_2)\exp\{-it(x_0 + kx_3)\}dx_0.$$

Then, by equations (3.23) and (3.24), we get

$$\lim_{j\to\infty}\{\exp\{-ktx_2 + tx_1\}G_j(x_1, x_2, t) - \exp\{-kt + t\}G_j(1, 1, t)\} = 0. \tag{3.25}$$

Now, let $\hat{F}_{\mathbb{BC}}$ be the bicomplex Fourier transform and writing $F_{x_1,x_2}(x_0, x_3)$ for $F(x_0 + ix_1 + jx_2 + kx_3)$. Then F_{x_1,x_2} lies in $L_k^2(\partial\prod_{\mathbb{BC}}^+)$.

By the bicomplex Plancheral theorem [18], we have

$$\lim_{j\to\infty}\int_{\partial\prod_{\mathbb{BC}}^+} \|\hat{F}_{x_1,x_2}(t) - G_j(x_1, x_2, t)\|_k^2 dt \odot dt^{\dagger} = 0.$$

Thus, by Theorem 1.8, the sequence $\{G_j(x_1, x_2, t)\}$ has a pointwise convergent subsequence that converges to $\hat{F}_{x_1,x_2}(t)$ for almost every t. Now, defining

$$\mathfrak{F}(t) = \exp\{-kt + t\}\hat{F}_{1,1}(t). \tag{3.26}$$

From equation (3.25), we have

$$\mathfrak{F}(t) = \exp\{-ktx_2 + tx_1\}\hat{F}_{x_1,x_2}(t). \tag{3.27}$$

Thus again, from the Plancheral Theorem for $\mathbb{BC}$, we have for every $x_1, x_2 \in (0,\infty) \times (0,\infty)$,

$$\begin{aligned}\int_{\partial\prod_{\mathbb{BC}}^+} \exp\{-2(-ktx_2 + tx_1)\}\|\mathfrak{F}(t)\|_k^2 dt \odot dt^{\dagger} &= \int_{\partial\prod_{\mathbb{BC}}^+} \|\hat{F}_{x_1,x_2}(t)\|_k^2 dt \odot dt^{\dagger}\\ &= \frac{1}{2\pi}\int_{\partial\prod_{\mathbb{BC}}^+} \|F_{x_1,x_2}(x_0, x_3)\|_k^2 dx_0\\ &\leqslant M.\end{aligned} \tag{3.28}$$

If we let $x_2, x_1 \to \infty$, then equation (3.28) shows that $\mathfrak{F}(t) = 0$ a.e in $(-\infty, 0) \times (-\infty, 0)$, and if $x_2, x_1 \to 0$, then

$$\int_{(0,\infty)\times(0,\infty)} \|\mathfrak{F}(t)\|_k^2 \; dt \odot dt^{\dagger} \leqslant M. \tag{3.29}$$

Thus

$$F_{x_1,x_2}(x_0, x_3) = \int_{\partial\prod_{\mathbb{BC}}^+} \hat{F}_{x_1,x_2}(t)\exp\{it(x_0, x_3)\}dt \odot dt^{\dagger} \tag{3.30}$$

or

$$\begin{aligned} F(Z) &= \int_{(0,\infty)\times(0,\infty)} \mathfrak{F}(t)\exp\{-(-ktx_2+tx_1)\}\exp\{it(x_0+kx_3)\}dt\odot dt^{\dagger} \\ &= \int_{(0,\infty)\times(0,\infty)} \mathfrak{F}(t)\exp\{itZ\}dt\odot dt^{\dagger} \quad ; \; Z\in\Pi^{+}_{\mathbb{BC}}. \end{aligned}$$

Keeping x_2, x_1 fixed and again applying the bicomplex Plancheral theorem, we obtain

$$\begin{aligned} \frac{1}{2\pi}\int_{\partial\Pi^{+}_{\mathbb{BC}}} \|F(z_1+jz_2)\|_k^2 dx_0 &= \int_{(0,\infty)\times(0,\infty)} \|\mathfrak{F}\|_k^2 \exp\{-2(t(-kx_2+x_1))\}dt\odot dt^{\dagger} \\ &\leqslant \int_{(0,\infty)\times(0,\infty)} \|\mathfrak{F}(t)\|_k^2 dt\odot dt^{\dagger}. \end{aligned}$$

Thus

$$\sup_{\substack{\mathbb{D} \\ 0<x_1,x_2<\infty}} \frac{1}{2\pi}\int_{\partial\Pi^{+}_{\mathbb{BC}}} \|F(x_0+ix_1+jx_2+kx_3)\|_k^2 dx_0 = M \leqslant \int_{(0,\infty)\times(0,\infty)} \|\mathfrak{F}(t)\|_k^2 dt\odot dt^{\dagger}. \tag{3.31}$$

Thus, from equations (3.29) and (3.31), we get

$$\int_{(0,\infty)\times(0,\infty)} \|\mathfrak{F}\|_k^2 dt\odot dt^{\dagger} = M.$$

□

Next, we discuss another class of all bicomplex F of the form

$$F(Z) = \int_{(-A,A)_{\mathbb{D}}} \mathfrak{F}(t)\exp\{itZ\}\, dt\odot dt^{\dagger}, \tag{3.32}$$

where $\mathfrak{F}\in L^2_k(-A,A)_{\mathbb{D}}$, and A is finite and positive. The interval $(-A,A)_{\mathbb{D}}$ is a hyperbolic interval as defined in Definition 1.4.

We proceed by bounding the norm $\|F(Z)\|_k$:

$$\begin{aligned} \|F(Z)\|_k &\leqslant \sqrt{2}\int_{(-A,A)_{\mathbb{D}}} \|\mathfrak{F}(t)\|_k \exp\{-(-ktx_2+tx_1)\}\, dt\odot dt^{\dagger} \\ &\leqslant \sqrt{2}\exp\{A\|(tx_1-ktx_2)\|_k\}\int_{(-A,A)_{\mathbb{D}}} \|\mathfrak{F}(t)\|_k\, dt\odot dt^{\dagger}. \end{aligned} \tag{3.33}$$

Now, define

$$C = \sqrt{2}\int_{(-A,A)_{\mathbb{D}}} \|\mathfrak{F}(t)\|_k\, dt\odot dt^{\dagger}.$$

Since $C<\infty$ (due to the finiteness of the integral and the properties of $\mathfrak{F}$), equation (3.33) becomes

$$\|F(Z)\|_k \leqslant C\exp\{A\|Z\|_k\}. \tag{3.34}$$

We can also prove that F being entire functions that satisfy (3.34) are called bicomplex exponential types. The context of our next theorem is as:
The type of functions in equation (3.32) are exponential functions whose restrictions to the real and kth-axis lie in $L^2_k\big(\partial\Pi^{+}_{\mathbb{BC}}\big)$. We prove that the converse is also true.

Theorem 3.2. *Let F be a bicomplex function of exponential type and*

$$\int_{\partial\Pi^{+}_{\mathbb{BC}}} \|F(x_0+kx_3)\|_k^2 dx_0\odot dx_0^{\dagger} \; < \; \infty. \tag{3.35}$$

Then there exists $\mathfrak{F}\in L^2_k(-A,A)_{\mathbb{D}}$ such that,

$$F(Z) = \int_{(-A,A)_{\mathbb{D}}} \mathfrak{F}(t)\exp\{itZ\}dt\odot dt^{\dagger} \tag{3.36}$$

for all $Z \in \mathbb{BC}$.

Proof. Let $\epsilon_{\mathbb{D}}$ be a number greater than 0, and let $F_{\epsilon_{\mathbb{D}}}(x_0+kx_3) = F(x_0+kx_3)\exp\{-\epsilon_{\mathbb{D}}\|x_0+kx_3\|_k\}$. Then, we show that

$$\lim_{\mathbb{D}} \epsilon_{\mathbb{D}} \to 0 \int_{\partial\Pi^+_{\mathbb{BC}}} F_{\epsilon_{\mathbb{D}}}(x_0+kx_3)\exp\{-it(x_0+kx_3)\}dx_0 \odot dx_0^\dagger = 0, \tag{3.37}$$

where $t \in \partial\Pi^+_{\mathbb{BC}}$ and $\|t\|_k > A$. As we see that $\|F_{\epsilon_{\mathbb{D}}} - F\|_{k,2} \to 0$ as $\epsilon_{\mathbb{D}} \to 0$. The bicomplex Plancheral theorem implies that $\|\hat{F}_{\epsilon_{\mathbb{D}}} - \mathfrak{F}\|_{k,2} \to 0$ as $\epsilon_{\mathbb{D}} \to 0$, where $\mathfrak{F}$ is the bicomplex Fourier transform of F. Thus, equation (3.37) implies that $\mathfrak{F}(t) = 0$ outside $[-A, A]_{\mathbb{D}}$ and hence from Corollary 1.9 , we see that (3.36) holds for almost every $Z = x_0 + kx_3$. Also, the left and right-hand sides of the equation (3.36) represent the entire bicomplex function. Thus, (3.36) holds for every $Z \in \mathbb{BC}$.

Thus, in order to prove the theorem, we shall show that (3.37) holds.

For this, let $\curlywedge_\alpha$ be a bicomplex path, defined as

$$\curlywedge_\alpha(u) = u \ \exp\{i\alpha\},$$

where $u \in [0,\infty) \times [0,\infty)$. Then,

$$\curlywedge_\alpha(u) = e \curlywedge_{\alpha_1} + e^\dagger \curlywedge_{\alpha_2}, \tag{3.38}$$

where $\curlywedge_{\alpha_1}$ and $\curlywedge_{\alpha_2}$ are complex paths. Putting, the half-plane in $\mathbb{BC}$ as

$$\Pi_{\mathbb{BC}(\alpha)} = \{W : Re(W\exp\{i\alpha\}) > A\}$$

and again

$$\Pi_{\mathbb{BC}(\alpha)} = e\Pi_\alpha + e^\dagger\Pi_\alpha, \tag{3.39}$$

where Π_α are decomposition of $\Pi_{\mathbb{BC}(\alpha)}$ in the complex plane. Define,

$$\Omega_\alpha(W) = \int_{\curlywedge_\alpha} F(Z)\exp\{-WZ\}dZ \odot dZ^\dagger. \tag{3.40}$$

Then $\Omega_\alpha(W) = e\Omega_\alpha(W_1) + e^\dagger\Omega_\alpha(W_2)$, where

$$\Omega_\alpha(W_i) = \int_{\curlywedge_{\alpha_1}} F_i(\beta_i)\exp\{-W_i\beta_i\}d\beta_i \quad \text{for} \quad i = 1, 2.$$

Using the complex version of this theorem on [35, Page 375], we see that each $\Omega_\alpha(W_i)$ is holomorphic in the half-plane Π_α, and so $\Omega_\alpha(W)$ is holomorphic in $\Pi_{\mathbb{BC}(\alpha)}$. Also, if $\alpha = 0$, then

$$\Omega_0(W) = \int_{(0,\infty)\times(0,\infty)} F(x_0+kx_3)exp\{-W(x_0+kx_3)\}dx_0 \odot dx_0^\dagger \quad ; \quad ReW > 0$$

and if $\alpha = \pi$,

$$\Omega_\pi(W) = -\int_{(-\infty,0)\times(-\infty,0)} F(x_0+kx_3)exp\{-W(x_0+kx_3)\}dx_0 \odot dx_0^\dagger \quad ; \quad ReW < 0.$$

Thus, Ω_0 and Ω_π are holomorphic in the indicated half-planes in (3.35).

Now, if we see that

$$\begin{aligned}
&\Omega_0(\epsilon_{\mathbb{D}}+it) - \Omega_\pi(-\epsilon_{\mathbb{D}} - it) \\
&\quad = \int_{(0,\infty)\times(0,\infty)} F(x_0 + kx_3)\exp\{-(\epsilon_{\mathbb{D}} + it)(x_0 + kx_3)\}dx_0 + \\
&\quad \int_{(-\infty,0)\times(-\infty,0)} F(x_0 + kx_3)\exp\{-(-\epsilon_{\mathbb{D}} + it)(x_0 + kx_3)dx_0^\dagger \\
&\quad = \int_{\partial\Pi^+_{\mathbb{BC}}} F(x_0 + kx_3)\exp\{-(\epsilon_{\mathbb{D}} + it)(x_0 + kx_3) - (-\epsilon_{\mathbb{D}} + it)(x_0 + kx_3)\}dx_0^\dagger \\
&\quad = \int_{\partial\Pi^+_{\mathbb{BC}}} F(x_0 + kx_3)\exp\{(x_0 + kx_3)[-\epsilon_{\mathbb{D}} - it + \epsilon_{\mathbb{D}} - it]\}dx_0 \odot dx_0^\dagger \\
&\quad = \int_{\partial\Pi^+_{\mathbb{BC}}} F(x_0 + kx_3)\exp\{(x_0 + kx_3)(-it)\}dx_0 \odot dx_0^\dagger,
\end{aligned}$$

then it is sufficient to show that $\Omega_0(\epsilon_{\mathbb{D}}) - \Omega_\pi(-\epsilon_{\mathbb{D}} + it) \to 0$ as $\epsilon_{\mathbb{D}} \to 0$ if $t > A$ and $t < -A$.

This can be shown by using the idempotent decomposition of Ω_0 and Ω_π with the help of idempotents e and $e^\dagger$ and also using the fact that this theorem holds for its complex version.

Therefore,

$$\lim_{\mathbb{D}} \epsilon_{\mathbb{D}} \to 0 \int_{\partial\Pi^+_{\mathbb{BC}}} F_{\epsilon_{\mathbb{D}}}(x_0 + kx_3)\exp\{-it(x_0 + kx_3)\}dx_0 = 0.$$

□

Now, we prove the bicomplex Cauchy integral formula for the upper half-plane. We start with the following statement:

Theorem 3.3. *If $F \in H^p(\Pi^+_{\mathbb{BC}})$, $1 \leqslant p < \infty$, then*

$$F(Z) = \frac{1}{2\pi i}\int_{\partial\Pi^+_{\mathbb{BC}}} \frac{F(W)}{W - Z} dW \odot dW^\dagger \quad ; Z \in \Pi^+_{\mathbb{BC}}$$

and the integral vanishes for all $Z \in \Pi^-_{\mathbb{BC}}$, where $\Pi^-_{\mathbb{BC}}$ represents the bicomplex lower half-plane.

Conversely, if $H \in L^q_k(\partial\Pi^+_{\mathbb{BC}})$ $(1 \leqslant p < \infty)$ and

$$\frac{1}{2\pi i}\int_{\partial\Pi^+_{\mathbb{BC}}} \frac{H(W)}{W - Z} dW \odot dW^\dagger = 0$$

for all $Z \in \Pi^-_{\mathbb{BC}}$. Then for $Z \in \Pi^+_{\mathbb{BC}}$, this integral represents a bicomplex function $F \in H^p(\Pi^+_{\mathbb{BC}})$, where the boundary function

$$F(x_0, x_3) = H(x_0, x_3) \quad a.e.$$

Proof. Since $H^p(\Pi^+_{\mathbb{BC}}) = eH^P(\Pi^+) + e^\dagger H^p(\Pi^+)$ and $F \in H^p(\Pi^+_{\mathbb{BC}})$. Then the bicomplex Cauchy integral Formula, see [29], is given by

$$C(F(Z)) = e\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F_1(W_1)}{W_1 - \beta_1} dW_1 + e^\dagger \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F_2(W_2)}{W_2 - \beta_2} dW_2.$$

That is,

$$C(F(Z)) = eI_1 + e^\dagger I_2,$$

where I_1 and I_2 are the complex Cauchy integrals for $F_1, F_2 \in H^p(\Pi^+)$ which is analytic in both Π^+ and Π^-. Then the bicomplex Cauchy integral is holomorphic in both $\Pi^+_{\mathbb{BC}}$ and $\Pi^-_{\mathbb{BC}}$.

Now, using the idempotent decompositions, we have

$$\begin{aligned}C(F(Z)) - C(F(Z^*)) =& \{e\mathfrak{F}_1(\beta_1) + e^\dagger\mathfrak{F}_2(\beta_2)\} - \{e\mathfrak{F}_1(\bar{\beta_1}) + e^\dagger\mathfrak{F}_2(\bar{\beta_2})\} \\ &= e\{\mathfrak{F}_1(\beta_1) - \mathfrak{F}_1(\bar{\beta_1})\} + e^\dagger\{\mathfrak{F}_2(\beta_2) - \mathfrak{F}_2(\bar{\beta_2})\}.\end{aligned}$$

So, from the complex analogy of this theorem, we have,

$$\begin{aligned}C(F(Z)) - C(F(Z^*)) &= e\{F_1(\beta_1)\} + e^\dagger\{F_2(\beta_2)\} \quad \beta_1, \beta_2 \in \Pi^+ \\ &= F(Z)\ ;\ Z \in \Pi^+_{\mathbb{BC}}.\end{aligned}$$

Thus, $C(F(Z^*))$ is holomorphic for $Z \in \Pi^+_{\mathbb{BC}}$. So, $C(F(Z))$ must be identically constant in $\Pi^-_{\mathbb{BC}}$. Since $C(F(Z)) \longrightarrow 0$ as $Z \longrightarrow \infty$, we have

$$C(F(Z)) = 0.$$

Thus,

$$C(F(Z^*)) = F(Z) \in \Pi^+_{\mathbb{BC}}$$

and

$$C(F(Z)) = 0 \ \in \Pi^-_{\mathbb{BC}}.$$

Conversely, suppose $H \in L^q_k\big(\partial\Pi^+_{\mathbb{BC}}\big)$. Then

$$L^q_k\big(\partial\Pi^+_{\mathbb{BC}}\big) = eL^q(\partial\Pi^+) + e^\dagger L^q(\partial\Pi^+) \tag{3.41}$$

and

$$\frac{1}{2\pi i}\int_{\partial\Pi^+_{\mathbb{BC}}} \frac{H(W)}{W-Z} dW \odot dW^\dagger = 0 \ \ ; Z \in \Pi^+_{\mathbb{BC}}.$$

Since $H \in L^q_k\big(\partial\Pi^+_{\mathbb{BC}}\big)$ and using the decomposition in equation (3.41) and using the fact that the result holds in each $L^q(\partial\Pi^+)$, the theorem follows. ☐

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