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EXISTENCE OF CLASSICAL SOLUTIONS FOR A CLASS OF $(p(x), q(x))$-LAPLACIAN SYSTEMS

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Abstract. In this paper we investigate a class of $(p(x), q(x))$-Laplacian systems for existence of global classical solutions. We give conditions under which the considered equations have at least one, at least two and at least three classical solutions. To prove our main results we propose a new approach based on the use of fixed points for the sum of two operators.

У цій статті ми досліджуємо клас $(p(x), q(x))$ - лапласівських систем на предмет існування глобальних класичних рішень. Ми наводимо умови, при яких розглянуті рівняння мають принаймні одне, принаймні два і принаймні три класичних рішення. Щоб довести наші основні результати, ми пропонуємо новий підхід, заснований на використанні нерухомих точок для суми двох операторів.

1. Introduction

Many problems of the real world and applied sciences such as elastic mechanics, fluid dynamics, electrorheological fluids, image processing, flow in porous media, calculus of variations, nonlinear elasticity theory, heterogeneous porous media models lead directly to partial differential equations involving variable exponent conditions (such as the $p(x)-$Laplacian), see for example [7, 19, 30]. It is the reason for which there is an intensive research on this subject in the last decades, see for example the papers [1, 9, 22, 15, 18, 23, 24, 25, 26, 27] and the references therein.

In this paper, we investigate the following $(p(x), q(x))$-Laplacian system

$$\begin{aligned} -\Delta_{p(x)}u &= f_1(x, u, v, \nabla_x u, \nabla_x v) \\ -\Delta_{q(x)}v &= f_2(x, u, v, \nabla_x u, \nabla_x v), \quad x \in \mathbb{R}^n, \end{aligned} \tag{1.1}$$

where

(H1): $p, q \in \mathcal{C}^1(\mathbb{R}^n)$, $1 < p, q \le B$ on $\mathbb{R}^n$ for some constant $B > 2$, $n \in \mathbb{N}$.
(H2): $f \in \mathcal{C}(\mathbb{R}^{3n+2})$,

$$\begin{aligned} 0 &\le |f_l(x, u, v, \nabla_x u, \nabla_x v)| \\ &\le a_{l1}(x) + a_{l2}(x)|u(x)|^{b_{l1}(x)} + a_{l3}(x)|v(x)|^{b_{l2}(x)} + a_{l4}(x)|u(x)|^{b_{l3}(x)}|v(x)|^{b_{l4}(x)} \\ &\quad + \sum_{i,j=1}^{n} a_{l5ij}(x)|u_{x_i}(x)|^{b_{l5ij}(x)}|v_{x_j}(x)|^{b_{l6ij}(x)} + \sum_{i=1}^{n} a_{l6i}(x)|u_{x_i}(x)|^{b_{l7i}(x)} \\ &\quad + \sum_{i=1}^{n} a_{l7i}(x)|v_{x_i}(x)|^{b_{l8i}(x)}, \end{aligned}$$

$x \in \mathbb{R}^n$, $u, v \in \mathcal{C}^1(\mathbb{R}^n)$,

$$\begin{aligned} 0 &\le a_{lk}, a_{l5ij}, a_{l6i}, a_{l7i} \le B, \\ 0 &< b_{lk}, b_{l5ij}, b_{l6ij}, b_{l7i}, b_{l8i} \le B \end{aligned}$$

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on $\mathbb{R}^n$, $l \in \{1, 2\}$, $k \in \{1, \dots, 4\}$, $i, j \in \{1, \dots, n\}$.

Here

$$\begin{aligned} \Delta_{p(x)} u &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(|\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_j} \right), \\ \Delta_{q(x)} v &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(|\nabla v|^{q(x)-2} \frac{\partial v}{\partial x_j} \right) \end{aligned}$$

for $u, v \in \mathcal{C}^2(\mathbb{R}^n)$. Our aim is to investigate the problem (1.1) for existence and nonuniqueness of global classical solutions. Our work is motivated by the interest of researchers in many mathematical questions related to partial differential equations involving $p(x)$-Laplacian. In fact, existence and multiplicity of solutions for an elliptic system involving the $p(x)$-Laplacian are obtained in [28]. In [3], some sufficient conditions for the existence of non-trivial solutions for a strongly coupled nonlinear elliptic system on the whole space $\mathbb{R}^N$ involving the $p(x)-$Laplacien operator were established. Existence of positive smooth solutions for a class of singular $(p(x), q(x))-$Laplacian systems was investigated in [2] by using sub and supersolution method. In [4], the techniques of Young measure-valued solutions are used to prove the existence of weak solutions for a class of nonlinear $p(x)-$Laplace system. In the paper [5], the authors study the existence and asymptotic behavior of positive solutions for a class of elliptic systems involving $(p(x), q(x))$-Laplacian systems using sub-super solutions method, with respect to symmetry conditions. In the non-stationary case, the sub-super solutions method is used in [6] to study the existence of weak positive solutions for a class of the $(p(x), q(x))$-Laplacian. Some other references on partial differential equations involving $p(x)$-Laplacian and coupled $p(x)$-Laplacian are [20, 21] and [11, 12], respectively.

An outline of the present paper is as follows: In the next section, we give some existence and multiplicity results about fixed points of the sum of two operators. In Section 3, we give an integral representation and a priori estimates related to solutions of problem (1.1). In Section 4, we prove our main results. Finally, in Section 5, we give an example to illustrate our main results.

2. Sum of operators: existence and multiplicity of fixed points

In this section, we will recall two results which concern the existence of fixed points and nonnegative fixed points for the sum of two operators.

The proof of the following theorem can be found in [10] or [13].

Theorem 2.1. *Let E be a Banach space, Y a closed, convex subset of E, U be any open subset of Y with $0 \in U$. Consider two operators T and S, where*

$$Tx = \varepsilon x, \qquad x \in \overline{U},$$

for $\varepsilon > 1$ and $S : \overline{U} \to E$ is continuous such that

(i): *$(I - S)(\overline{U})$ resides in a compact subset of Y and*
(ii): *$\{x \in \partial U : x = \lambda(I - S)x\} = \emptyset$, for any $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.*

Then there exists $x^* \in \overline{U}$ such that

$$Tx^* + Sx^* = x^*.$$

In the sequel, E is a real Banach space.

Definition 2.2. A closed, convex set $\mathcal{P}$ in E is said to be cone if

(1) $\alpha x \in \mathcal{P}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{P}$,
(2) $x, -x \in \mathcal{P}$ implies $x = 0$.

Definition 2.3. A mapping $K : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.4. Let X and Y be real Banach spaces. A mapping $K : X \to Y$ is said to be expansive if there exists a constant $h > 1$ such that

$$\|Kx - Ky\|_Y \geq h\|x - y\|_X$$

for any $x, y \in X$.

The following result will be used to prove the existence of two nonnegative solutions of the problem (1.1). Its proof is based on the theory of fixed point index for the sum of two operators developed by Mebarki et *al.* in [8] and [14].

Theorem 2.5. [16, 29] *Let $\mathcal{P}$ be a cone of a Banach space E and U_1, U_2 and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Let Ω be a subset of $\mathcal{P}$ such that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$ and $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$. Assume that $T : \Omega \to E$ is an expansive mapping and $S : \overline{U}_3 \to E$ is a completely continuous one such that $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that there exist $w_0 \in \mathcal{P}\backslash\{0\}$ and $\varepsilon > 0$ small enough such that the following conditions hold:*

(i): $Sx \neq (I - T)(x - \lambda w_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_1 \cap (\Omega + \lambda w_0)$,
(ii): $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_2$ *and* $\lambda x \in \Omega$,
(iii): $Sx \neq (I - T)(x - \lambda w_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_3 \cap (\Omega + \lambda w_0)$.

Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that

$$x_1 \in \partial U_2 \cap \Omega \text{ and } x_2 \in (U_3 \setminus \overline{U}_2) \cap \Omega$$

or

$$x_1 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_2 \in (U_3 \setminus \overline{U}_2) \cap \Omega.$$

The following result will be used to prove the existence of three nonnegative solutions (at least two non zeros) of the problem (1.1). More precisely, it will be used to prove Theorem 4.3. For the proof, we use the same arguments used in [16, 29].

Theorem 2.6. *Let $\mathcal{P}$ be a cone of a Banach space E and U_1, U_2, and U_3 three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Let Ω be a subset of $\mathcal{P}$ such that $(U_2 \setminus \overline{U}_1) \cap \Omega \neq \emptyset$ and $(U_3 \setminus \overline{U}_2) \cap \Omega \neq \emptyset$. Assume that $T : \Omega \to E$ is an expansive mapping and $S : \overline{U}_3 \to E$ is a completely continuous one such that $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that there exist $w_0 \in \mathcal{P}\backslash\{0\}$ and $\varepsilon > 0$ small enough such that the following conditions hold:*

(i): $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_1$ *and* $\lambda x \in \Omega$,
(ii): $Sx \neq (I - T)(x - \lambda w_0)$, *for all* $\lambda \geq 0$ *and* $x \in \partial U_2 \cap (\Omega + \lambda w_0)$,
(iii): $Sx \neq (I - T)(\lambda x)$, *for all* $\lambda \geq 1 + \varepsilon$, $x \in \partial U_3$ *and* $\lambda x \in \Omega$.

Then $T + S$ has at least three non trivial fixed points $x_1, x_2, x_3 \in \mathcal{P}$ such that

$$x_1 \in \overline{U}_1 \cap \Omega \text{ and } x_2 \in (U_2 \setminus \overline{U}_1) \cap \Omega \text{ and } x_3 \in (\overline{U}_3) \setminus \overline{U}_2) \cap \Omega.$$

3. Some properties of solutions of problem (1.1)

Let $X^1 = \mathcal{C}^2(\mathbb{R}^n)$ be endowed with the norm

$$\|u\|_{X^1} = \max\left\{\sup_{x\in\mathbb{R}^n} |u(x)|, \quad \sup_{x\in\mathbb{R}^n} |u_{x_j}(x)|, \quad \sup_{x\in\mathbb{R}^n} |u_{x_j x_j}(x)|, \quad j \in \{1, \dots, n\}\right\},$$

provided it exists. Let $X = X^1 \times X^1$ be endowed with the norm

$$\|(u, v)\| = \max\{\|u\|_{X^1}, \|v\|_{X^1}\}, \quad (u, v) \in X,$$

provided it exists. For $(u,v) \in X$, we will write $(u,v) \geq 0$ if $u(x) \geq 0$, $v(x) \geq 0$ for any $x \in \mathbb{R}^n$. Set

$$\begin{aligned}
\int_0^{\overline{x_1}} d\overline{s_1} &= \int_0^{x_2} \dots \int_0^{x_n} ds_n \dots ds_2, \quad d\overline{s_1} = ds_n \dots ds_2,\\
\int_0^{\overline{x_j}} d\overline{s_j} &= \int_0^{x_1} \dots \int_0^{x_{j-1}} \int_0^{x_{j+1}} \dots \int_0^{x_n} ds_n \dots ds_{j+1} ds_{j-1} \dots ds_1,\\
&\quad d\overline{s_j} = ds_n \dots ds_{j+1} ds_{j-1} \dots ds_1, \quad j \in \{2, \dots, n-1\},\\
\int_0^{\overline{x_n}} d\overline{s_n} &= \int_0^{x_1} \dots \int_0^{x_{n-1}} ds_{n-1} \dots ds_1, \quad d\overline{s_n} = ds_{n-1} \dots ds_1,\\
\int_0^{x} ds &= \int_0^{x_1} \dots \int_0^{x_n} ds_n \dots ds_1,\\
s_{x_1} &= (x_1, s_2, \dots, s_n),\\
s_{x_j} &= (s_1, \dots, s_{j-1}, x_j, s_{j+1}, \dots, s_n), \quad j \in \{2, \dots, n-1\},\\
s_{x_n} &= (s_1, \dots, s_{n-1}, x_n), \quad s = (s_1, \dots, s_n), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.
\end{aligned}$$

For $(u,v) \in X$, define operators S_1^1, S_1^2 and S_1 as follows:

$$\begin{aligned}
S_1^1(u,v)(x) &= \sum_{j=1}^{n} \int_0^{\overline{x_j}} |\nabla u(s_{x_j})|^{p(s_{x_j})-2} \frac{\partial u}{\partial x_j} d\overline{s_{x_j}}\\
&\quad + \int_0^x f_1(s, u(s), v(s), \nabla_x u(s), \nabla_x v(s)) ds,\\
S_1^2(u,v)(x) &= \sum_{j=1}^{n} \int_0^{\overline{x_j}} |\nabla v(s_{x_j})|^{p(s_{x_j})-2} \frac{\partial v}{\partial x_j} d\overline{s_{x_j}}\\
&\quad + \int_0^x f_2(s, u(s), v(s), \nabla_x u(s), \nabla_x v(s)) ds,\\
S_1(u,v)(x) &= (S_1^1(u,v)(x), S_1^2(u,v)(x)), \quad x \in \mathbb{R}^n.
\end{aligned}$$

Note that if $(u,v) \in X$ satisfies the equation

$$S_1(u,v)(x) = 0, \quad x \in \mathbb{R}^n, \tag{3.2}$$

and we differentiate it with respect to x_1, x_2, ..., x_n, we get that u and v satisfy the first two equations of (1.1). Thus, any $(u,v) \in X$, $(u,v) \geq 0$, that satisfies the equation (3.2), is a solution to the problem (1.1).

Lemma 3.1. *Suppose that $(H1)$ and $(H2)$ hold. If $(u,v) \in X$, $\|(u,v)\| \leq B$, then*

$$|f_l(x, u(x), v(x), \nabla_x u(x), \nabla_x v(x))| \leq (n^4 + 2n + 4) B^{2B+1}, \quad x \in \mathbb{R}^n, \ l \in \{1,2\}.$$

Proof. We have

$$\begin{aligned}
&|f_l(x, u, v, \nabla_x u, \nabla_x v)|\\
\leq\ & a_{l1}(x) + a_{l2}(x)|u(x)|^{b_{l1}(x)} + a_{l3}(x)|v(x)|^{b_{l2}(x)} + a_{l4}(x)|u(x)|^{b_{l3}(x)}|v(x)|^{b_{l4}(x)}\\
&+ \sum_{i,j=1}^{n} a_{l5ij}(x)|u_{x_i}(x)|^{b_{l5ij}(x)}|v_{x_j}(x)|^{b_{l6ij}(x)} + \sum_{i=1}^{n} a_{l6i}(x)|u_{x_i}(x)|^{b_{l7i}(x)}\\
&+ \sum_{i=1}^{n} a_{l7i}(x)|v_{x_i}(x)|^{b_{l8i}(x)}\\
\leq\ & B + B^{B+1} + B^{B+1} + B^{2B+1} + n^4 B^{2B+1} + nB^{B+1} + nB^{B+1}\\
\leq\ & (n^4 + 2n + 4) B^{2B+1}, \quad x \in \mathbb{R}^n, \quad l \in \{1,2\}.
\end{aligned}$$

This completes the proof. □

Let

$$B_1 = \left(n^{\frac{B}{2}} + n^4 + 2n + 4\right) B^{2B+1}.$$

Lemma 3.2. *Suppose that $(H1)$ and $(H2)$ hold. If $(u,v) \in X$, $\|(u,v)\| \le B$, then*

$$|S_1^l(u,v)(x)| \le B_1 \prod_{j=1}^{n} (1+|x_j|), \quad x \in \mathbb{R}^n, \quad l \in \{1,2\}.$$

Proof. We have

$$\begin{aligned}
|S_1^1(u,v)(x)| &= \Bigg| \sum_{j=1}^{n} \int_0^{\overline{x_j}} |\nabla u(s_{x_j})|^{p(s_{x_j})-2} \frac{\partial u}{\partial x_j} d\overline{s_{x_j}} \\
&\quad + \int_0^x f_1(s, u(s), v(s), \nabla_x u(s), \nabla_x v(s)) ds \Bigg| \\
&\le \sum_{j=1}^{n} \left| \int_0^{\overline{x_j}} |\nabla u(s_{x_j})|^{p(s_{x_j})-2} \left| \frac{\partial u}{\partial x_j} \right| d\overline{s_{x_j}} \right| \\
&\quad + \left| \int_0^x f_1(s, u(s), v(s), \nabla_x u(s), \nabla_x v(s)) ds \right| \\
&\le \sum_{j=1}^{n} \prod_{k=1, k\neq j}^{n} (1+|x_k|) n^{\frac{B-2}{2}} B^{B-1} + \prod_{j=1}^{n} |x_j| (n^4 + 4 + 2n) B^{2B+1} \\
&\le \left(n^{\frac{B}{2}} + n^4 + 4 + 2n\right) \prod_{j=1}^{n} (1+|x_j|) B^{2B+1} \\
&= B_1 \prod_{j=1}^{n} (1+|x_j|), \quad x \in \mathbb{R}^n.
\end{aligned}$$

As above,

$$|S_1^2(u,v)(x)| \le B_1 \prod_{j=1}^{n} (1+|x_j|), \quad x \in \mathbb{R}^n.$$

This completes the proof. □

Let A be a positive constant and $g \in \mathcal{C}(\mathbb{R}^n)$ be a function such that

$$g > 0 \text{ on } \mathbb{R}^n \backslash \{\{x_j = 0\}_{j=1}^n\},$$

$$g(0, x_2, \dots, x_n) = g(x_1, 0, x_3, \dots, x_n) = g(x_1, \dots, x_{n-1}, 0) = 0, \quad (x_1, \dots, x_n) \in \mathbb{R}^n,$$

and

$$2^{2n+3} \prod_{j=1}^{n} (1 + |x_j| + x_j^2 + |x_j|^3) \left| \int_0^x g(s) ds \right| \le A. \tag{3.3}$$

In the last section, we will give an example of such a function g. For $u, v \in X$, define operators S_2^1, S_2^2 and S_2 as follows:

$$\begin{aligned}
S_2^1(u,v)(x) &= \int_0^x \prod_{j=1}^{n} (x_j - s_j)^2 g(s) S_1^1(u,v)(s) ds, \\
S_2^2(u,v)(x) &= \int_0^x \prod_{j=1}^{n} (x_j - s_j)^2 g(s) S_1^2(u,v)(s) ds, \\
S_2(u,v)(x) &= (S_2^1(u,v)(x), S_2^2(u,v)(x)), \quad x \in \mathbb{R}^n.
\end{aligned}$$

Lemma 3.3. *Suppose that the conditions* $(H1)$ *and* $(H2)$ *are satisfied. If* $(u,v)\in X$, $(u,v)\geq 0$ *on* $\mathbb{R}^n$, *and*

$$S_2(u,v)(x) = (C,C), \quad x\in\mathbb{R}^n,$$

for some constant C, *then* (u,v) *is a solution to the problem* (1.1).

Proof. We have

$$S_2^1(u,v)(x) = S_2^2(u,v)(x) = C, \quad x\in\mathbb{R}^n.$$

We differentiate the last two equations three times in x_1, x_2, …, x_n and we get

$$g(x)S_1^1(u,v)(x) = g(x)S_1^2(u,v)(x) = 0, \quad x\in\mathbb{R}^n.$$

Hence,

$$S_1^1(u,v)(x) = S_1^2(u,v)(x) = 0, \quad x\in\mathbb{R}^n\backslash\{\{x_j=0\}_{j=1}^n\}.$$

Since $S_1^1(u,v)(\cdot)$ and $S_1^2(u,v)(\cdot)$ are continuous on $\mathbb{R}^n$, we find

$$\begin{aligned} 0 &= \lim_{x_1\to 0} S_1^1(u,v)(x_1,x_2,\dots,x_n) = S_1^1(u,v)(0,x_2,\dots,x_n)\\ &= \lim_{x_1\to 0} S_1^2(u,v)(x_1,x_2,\dots,x_n) = S_1^2(u,v)(0,x_2,\dots,x_n)\\ &\cdots\\ &= \lim_{x_n\to 0} S_1^1(u,v)(x_1,x_2,\dots,x_{n-1},x_n) = S_1^1(u,v)(x_1,x_2,\dots,x_{n-1},0)\\ &= \lim_{x_n\to 0} S_1^2(u,v)(x_1,x_2,\dots,x_{n-1},x_n) = S_1^2(u,v)(x_1,x_2,\dots,x_{n-1},0), \end{aligned}$$

$(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$. Therefore

$$S_1^1(u,v)(x) = S_1^2(u,v)(x) = 0, \quad x\in\mathbb{R}^n.$$

This completes the proof. □

Lemma 3.4. *Under hypotheses* $(H1)$ *and* $(H2)$ *and for* $(u,v)\in X$ *with* $\|(u,v)\|\leq B$, *the following estimate holds:*

$$\|S_2(u,v)\| \leq AB_1.$$

Proof. Suppose that $(H1)$ and $(H2)$ are satisfied. Let $(u,v)\in X$, with $\|(u,v)\|\leq B$.

(i): An estimate of $|S_2^1(u,v)(x)|$, $x\in\mathbb{R}^n$, is as follows:

$$\begin{aligned} |S_2^1(u,v)(x)| &= \left|\int_0^x \prod_{j=1}^n (x_j-s_j)^2 g(s) S_1^1(u,v)(s)ds\right|\\ &\leq \left|\int_0^x \prod_{j=1}^n (x_j-s_j)^2 g(s) |S_1^1(u,v)(s)|ds\right|\\ &\leq 2^{2n}B_1\left|\prod_{j=1}^n\int_0^x (x_j^2+s_j^2)\prod_{j=1}^n(1+|s_j|)g(s)da\right|\\ &\leq 2^{2n+1}B_1\prod_{j=1}^n x_j^2(1+|x_j|)\left|\int_0^x g(s)ds\right|\\ &\leq 2^{2n+3}B_1\prod_{j=1}^n(1+|x_j|+x_j^2+|x_j|^3)\left|\int_0^x g(s)ds\right|\\ &\leq AB_1. \end{aligned}$$

(ii): For an estimate of $\left|\frac{\partial}{\partial x_k} S_2^1(u,v)(x)\right|$, $x \in \mathbb{R}^n$, $\quad k \in \{1,\dots,n\}$, we get

$$\begin{aligned}
\left|\frac{\partial}{\partial x_k} S_2^1(u,v)(x)\right| &= 2\left|\int_0^x \prod_{j=1, j\neq k}^n (x_j - s_j)^2 (x_k - s_k) g(s) S_1^1(u,v)(s)ds\right| \\
&\leq 2\left|\int_0^x \prod_{j=1, j\neq k}^n (x_j - s_j)^2 (x_k - s_k) g(s) |S_1^1(u,v)(s)|ds\right| \\
&\leq 2\cdot 2^{2n-2} B_1 \left|\prod_{j=1, j\neq k}^n \int_0^x (x_j^2 + s_j^2) \prod_{j=1}^n (1+|s_j|)(|x_k| + |s_k|) g(s) da\right| \\
&\leq 2^{2n+1} B_1 \prod_{j=1, j\neq k}^n x_j^2 (1+|x_j|)|x_k|(1+|x_k|) \left|\int_0^x g(s)ds\right| \\
&\leq 2^{2n+2} B_1 \prod_{j=1}^n (1+|x_j| + x_j^2 + |x_j|^3) \left|\int_0^x g(s)ds\right| \\
&\leq AB_1.
\end{aligned}$$

(iii): An estimate of $\left|\frac{\partial^2}{\partial x_k^2} S_2^1(u,v)(x)\right|$, $x \in \mathbb{R}^n$, $\quad k \in \{1,\dots,n\}$ is

$$\begin{aligned}
\left|\frac{\partial^2}{\partial x_k^2} S_2^1(u,v)(x)\right| &= 2\left|\int_0^x \prod_{j=1, j\neq k}^n (x_j - s_j)^2 g(s) S_1^1(u,v)(s)ds\right| \\
&\leq 2\left|\int_0^x \prod_{j=1, j\neq k}^n (x_j - s_j)^2 g(s) |S_1^1(u,v)(s)|ds\right| \\
&\leq 2\cdot 2^{2n-2} B_1 \left|\prod_{j=1, j\neq k}^n \int_0^x (x_j^2 + s_j^2) \prod_{j=1}^n (1+|s_j|) g(s) da\right| \\
&\leq 2^{2n} B_1 \prod_{j=1, j\neq k}^n x_j^2 (1+|x_j|)(1+|x_k|) \left|\int_0^x g(s)ds\right| \\
&\leq 2^{2n+3} B_1 \prod_{j=1}^n (1+|x_j| + x_j^2 + |x_j|^3) \left|\int_0^x g(s)ds\right| \\
&\leq AB_1.
\end{aligned}$$

Thus,

$$\|S_2^1(u,v)\|_{X^1} \leq AB_1.$$

As above,

$$\|S_2^2(u,v)\|_{X^1} \leq AB_1.$$

This completes the proof. □

4. Existence and multiplicity of solutions

Our first main result for existence of classical solutions of the problem (1.1) is as follows.

Theorem 4.1. *Under hypothesis* $(H1)$ *and* $(H2)$, *the problem* (1.1) *has at least one bounded solution* $(u,v) \in \mathcal{C}^2(\mathbb{R}^n) \times \mathcal{C}^2(\mathbb{R}^n)$.

Proof. Let $\widetilde{Y}$ denote the set of all equicontinuous families in X with respect to the norm $\|\cdot\|$. Let also, $Y = \overline{\widetilde{Y}}$ be the closure of $\widetilde{Y}$ and

$$U = \{(u,v) \in Y: \quad \|(u,v)\| < B \quad \text{and if} \quad \|(u,v)\| \geq B/2, \quad \text{then} \quad (u,v) > (B/2, B/2)\}.$$

Note that $\overline{U}$ is a compact set in Y. For $(u,v) \in \overline{U}$ and $\epsilon > 1$, define the operators

$$\begin{aligned} T(u,v)(x) &= \epsilon(u,v)(x), \\ S(u,v)(x) &= (u,v)(x) - \epsilon(u,v)(x) - \epsilon S_2(u,v)(x) + \epsilon(AB_1, AB_1), \quad x \in \mathbb{R}^n. \end{aligned}$$

For $(u,v) \in \overline{U}$, we have

$$\begin{aligned} \|(I-S)(u,v)\| &= \|\epsilon(u,v) + \epsilon S_2(u,v) - \epsilon(AB_1, AB_1)\| \\ &\leq \epsilon\|(u,v)\| + \epsilon\|S_2(u,v)\| + \epsilon\|(AB_1, AB_1)\| \\ &\leq \epsilon B_1 + 2\epsilon AB_1. \end{aligned}$$

Thus, $S : \overline{U} \to X$ is continuous and $(I-S)(\overline{U})$ resides in a compact subset of Y. Now, suppose that there is a $(u,v) \in \partial U$ so that

$$(u,v) = \lambda(I-S)(u,v)$$

or

$$(u,v) = \lambda\epsilon\left[(u,v) + S_2(u,v) - (AB_1, AB_1)\right]$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $\|S_2(u,v)\| \leq AB_1$, we get

$$(u,v)(x) \leq \lambda\epsilon(u,v)(x), \quad x \in \mathbb{R}^n.$$

Since the last inequality holds for any $x \in \mathbb{R}^n$ and $\|(u,v)\| = B$, or $\|(u,v)\| \geq \frac{B}{2}$ and $(u,v) > 0$ on $\mathbb{R}^n$, we get $\lambda\epsilon \geq 1$, which is a contradiction. Consequently

$$\{(u,v) \in \partial U : (u,v) = \lambda_1(I-S)(u,v)\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.1, it follows that the operator $T+S$ has a fixed point $(u^*, v^*) \in Y$. Therefore

$$\begin{aligned} (u^*, v^*)(x) &= T(u^*, v^*)(x) + S(u^*, v^*)(x) \\ &= \epsilon(u^*, v^*)(x) + (u^*, v^*)(x) - \epsilon(u^*, v^*)(x) - \epsilon S_2(u^*, v^*)(x) \\ &\quad + \epsilon(AB_1, AB_1), \end{aligned}$$

$x \in \mathbb{R}^n$, whereupon

$$(AB_1, AB_1) = S_2(u^*, v^*)(x), \quad x \in \mathbb{R}^n.$$

From here and from Lemma 3.3, it follows that (u^*, v^*) is a solution to the problem (1.1). This completes the proof. □

Let r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq B, \ AB_1 < \frac{L}{5}.$$

Here, B and A are the constants which appear in the conditions $(H1)$ and formula (3.3), respectively and $B_1 = \left(n^{\frac{B}{2}} + n^4 + 2n + 4\right) B^{2B+1}$. Our second main result for existence and multiplicity of classical solutions of the problem (1.1) is as follows.

Theorem 4.2. *Suppose $(H1)$ and $(H2)$ hold. Then the problem (1.1) has at least two nonnegative bounded solutions $(u_1, v_1), (u_2, v_2) \in \mathcal{C}^2(\mathbb{R}^n) \times \mathcal{C}^2(\mathbb{R}^n)$.*

Proof. Set $X = \mathcal{C}^2(\mathbb{R}^n) \times \mathcal{C}^2(\mathbb{R}^n)$ and let

$$\widetilde{P} = \{(u,v) \in X : (u,v) \geq 0 \quad \text{on} \quad \mathbb{R}^n\}.$$

With $\mathcal{P}$ we will denote the set of all equicontinuous families in $\widetilde{P}$. For $(u,v) \in X$, define the operators

$$\begin{aligned} T_1(u,v)(x) &= (1+m\epsilon)(u,v)(x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \\ S_3(u,v)(x) &= -\epsilon S_2(u,v)(x) - m\epsilon(u,v)(x) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right), \; x \in \mathbb{R}^n, \end{aligned}$$

where ϵ is a positive constant and $m > 0$ is large enough. Note that any fixed point $(u,v) \in X$ of the operator $T_1 + S_3$ is a solution to the problem (1.1). Define

$$\begin{aligned} \Omega &= \mathcal{P}, \\ U_1 &= \mathcal{P}_r = \{(u,v) \in \mathcal{P} : \|(u,v)\| < r\}, \\ U_2 &= \mathcal{P}_L = \{(u,v) \in \mathcal{P} : \|(u,v)\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{(u,v) \in \mathcal{P} : \|(u,v)\| < R_1\}. \end{aligned}$$

(1) For $(u_1,v_1),(u_2,v_2) \in \Omega$, we have

$$\|T_1(u_1,v_1) - T_1(u_2,v_2)\| = (1+m\epsilon)\|(u_1,v_1) - (u_2,v_2)\|,$$

whereupon $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.

(2) For $(u,v) \in \overline{\mathcal{P}_{R_1}}$, we get

$$\begin{aligned} \|S_3(u,v)\| &\leq \epsilon\|S_2(u,v)\| + m\epsilon\|(u,v)\| + \epsilon\frac{L}{10} \\ &\leq \epsilon\left(AB_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

Therefore $S_3(\overline{\mathcal{P}_{R_1}})$ is uniformly bounded. Since $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is continuous, we have that $S_3(\overline{\mathcal{P}_{R_1}})$ is equicontinuous. Consequently $S_3 : \overline{\mathcal{P}_{R_1}} \to X$ is completely continuous.

(3) Let $(u_1,v_1) \in \overline{\mathcal{P}_{R_1}}$. Set

$$(u_2,v_2) = (u_1,v_1) + \frac{1}{m}S_2(u_1,v_1) + \left(\frac{L}{5m}, \frac{L}{5m}\right).$$

Note that $S_2^1(u_1,v_1) + \frac{L}{5} \geq 0$, $S_2^2(u_1,v_1) + \frac{L}{5} \geq 0$ on $\mathbb{R}^n$. We have $u_2, v_2 \geq 0$ on $\mathbb{R}^n$. Therefore $(u_2,v_2) \in \Omega$ and

$$-\epsilon m(u_2,v_2) = -\epsilon m(u_1,v_1) - \epsilon S_2(u_1,v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$\begin{aligned} (I - T_1)(u_2,v_2) &= -\epsilon m(u_2,v_2) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) \\ &= S_3(u_1,v_1). \end{aligned}$$

Consequently $S_3(\overline{\mathcal{P}_{R_1}}) \subset (I - T_1)(\Omega)$.

(4) Assume that for any $(u_0,v_0) \in \mathcal{P}\backslash\{0\}$ there exist $\lambda \geq 0$ and $(u,v) \in \partial\mathcal{P}_r \cap (\Omega + \lambda(u_0,v_0))$ or $v \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda(u_0,v_0))$ such that

$$S_3(u,v) = (I - T_1)((u,v) - \lambda(u_0,v_0)).$$

Then

$$-\epsilon S_2(u,v) - m\epsilon(u,v) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -m\epsilon((u,v) - \lambda(u_0,v_0)) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right)$$

or

$$-S_2(u, v) = \lambda m(u_0, v_0) + \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\|S_2 v\| = \left\|\lambda m(u_0, v_0) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| \geq \frac{L}{5}.$$

This is a contradiction.

(5) Let $\epsilon_1 = \frac{2}{5m}$. Suppose that there exist $(u_1, v_1) \in \partial \mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that

$$S_3(u_1, v_1) = (I - T_1)(\lambda_1(u_1, v_1)).$$

Then,

$$-\epsilon S_2(u_1, v_1) - m\epsilon(u_1, v_1) - \epsilon\left(\frac{L}{10}, \frac{L}{10}\right) = -\lambda_1 m\epsilon(u_1, v_1) + \epsilon\left(\frac{L}{10}, \frac{L}{10}\right),$$

or

$$S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right) = (\lambda_1 - 1)m(u_1, v_1).$$

From here,

$$2\frac{L}{5} \geq \left\|S_2(u_1, v_1) + \left(\frac{L}{5}, \frac{L}{5}\right)\right\| = (\lambda_1 - 1)m\|(u_1, v_1)\| = (\lambda_1 - 1)mL$$

and

$$\frac{2}{5m} + 1 > \lambda_1,$$

which is a contradiction.

Therefore all conditions of Theorem 4.2 hold. Hence, the problem (1.1) has at least two solutions (u_1, v_1) and (u_2, v_2) so that

$$\|(u_1, v_1)\| = L < \|(u_2, v_2)\| < R_1$$

or

$$r < \|(u_1, v_1)\| < L < \|(u_2, v_2)\| < R_1.$$ □

In this sequel, we will use the notations of the proof of Theorem 4.2 and we choose the positive constant m such that $\epsilon m r > \frac{2L}{5}$. Then, our third main result for existence and multiplicity of classical solutions of the problem (1.1) is as follows.

Theorem 4.3. *Suppose $(H1)$ and $(H2)$ hold. Then the problem (1.1) has at least three nonnegative bounded solutions $(u_1, v_1), (u_2, v_2), (u_3, v_3) \in \mathcal{C}^2(\mathbb{R}^n) \times \mathcal{C}^2(\mathbb{R}^n)$.*

Proof. (1) Assume that there are $\lambda \geq 1 + \epsilon$, $(u, v) \in \partial U_1$ and $\lambda(u, v) \in \Omega$ so that

$$S_3(u, v) = (I - T_1)(\lambda u, \lambda v).$$

Then

$$-\epsilon S_2(u, v) - m\epsilon(u, v) - \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right) = -m\epsilon\lambda(u, v) + \left(\epsilon\frac{L}{10}, \epsilon\frac{L}{10}\right)$$

or

$$S_2(u, v) = (\lambda - 1)m(u, v) - \left(\frac{L}{5}, \frac{L}{5}\right).$$

Hence,

$$\begin{aligned}\|S_2(u,v)\| &= \left\|m(\lambda-1)(u,v)-\left(\frac{L}{5},\frac{L}{5}\right)\right\| \\ &\geq (\lambda-1)m\|(u,v)\|-\left\|\left(\frac{L}{5},\frac{L}{5}\right)\right\| \\ &\geq \epsilon m\|(u,v)\|-\frac{L}{5} \\ &= \epsilon m r-\frac{L}{5} \\ &> \frac{L}{5},\end{aligned}$$

which is a contradiction. Thus, the condition (**i**) of Theorem 2.6 holds.

(2) Now, assume that there are $\lambda \geq 1+\epsilon$, $(u,v) \in \partial U_3$ and $\lambda(u,v) \in \Omega$ so that

$$S_3(u,v) = (I-T_1)(\lambda u, \lambda v).$$

As above,

$$\begin{aligned}\|S_2(u,v)\| &\geq (\lambda-1)m\|(u,v)\|-\left\|\left(\frac{L}{5},\frac{L}{5}\right)\right\| \\ &\geq \epsilon m\|(u,v)\|-\frac{L}{5} \\ &= \epsilon m R_1-\frac{L}{5} \\ &> \epsilon m r-\frac{L}{5} \\ &> \frac{L}{5},\end{aligned}$$

which is a contradiction. Hence, the condition (**iii**) of Theorem 2.6 holds.

(3) Assume that for any $(u_0,v_0) \in \mathcal{P}\backslash\{0\}$ there exist $\lambda \geq 0$ and $(u,v) \in \partial\mathcal{P}_L \cap (\Omega + \lambda(u_0,v_0))$ such that

$$S_3(u,v) = (I-T_1)((u,v)-\lambda(u_0,v_0)).$$

Then

$$-\epsilon S_2(u,v)-m\epsilon(u,v)-\epsilon\left(\frac{L}{10},\frac{L}{10}\right) = -m\epsilon((u,v)-\lambda(u_0,v_0))+\epsilon\left(\frac{L}{10},\frac{L}{10}\right)$$

or

$$-S_2(u,v) = \lambda m(u_0,v_0)+\left(\frac{L}{5},\frac{L}{5}\right).$$

Hence,

$$\|S_2 v\| = \left\|\lambda m(u_0,v_0)+\left(\frac{L}{5},\frac{L}{5}\right)\right\| \geq \frac{L}{5}.$$

This is a contradiction. Form here it follows that the condition (**ii**) of Theorem 2.6 holds.

Now, by Theorem 2.6, it follows that the problem (1.1) has at least three classical solutions. □

5. An Example

Below, we will illustrate our main results. Let $n = 2$,

$$R_1 = B = 10, \quad L = 5, \quad \alpha = 2, \quad r = 4, \quad m = 10^{250}, \quad A = \frac{1}{10^{100}}, \quad \epsilon = 2.$$

Then

$$B_1 = (2^5 + 2^4 + 8) \cdot 10^{21}$$

and

$$AB_1 = \frac{1}{10^{100}}(2^5 + 2^4 + 8) \cdot 10^{21} < B.$$

Next,

$$r < L < R_1 \le B, \quad AB_1 < \frac{L}{5}.$$

Moreover,

$$\epsilon rm = 2 \cdot 10^{250} \cdot 4 > 2 = \frac{2L}{5}.$$

Take

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{22})}{1 + s^{44}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2 + s^3)h(s) < \infty, \\ -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2 + s^3)l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$(1 + s + s^2 + s^3)\left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}\right) \le C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s \to \pm 1} l(s) = \frac{\pi}{2}$ and by [17] (pp. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(1 + s + s^2)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(x) = Q(x_1) \dots Q(x_n), \quad x \in \mathbb{R}^n.$$

Then there exists a constant $C_2 > 0$ such that

$$2^{2n+3} \prod_{j=1}^{n} \left(1 + |x_j| + x_j^2 + |x_j|^3\right) \left| \int_0^x g_1(s) ds \right| dt_1 \le C_2, \quad x \in \mathbb{R}^n.$$

Let

$$g(x) = \frac{A}{C_2} g_1(x), \quad x \in \mathbb{R}^n.$$

Then

$$2^{2n+3}\prod_{j=1}^{n}\left(1+|x_j|+x_j^2+|x_j|^3\right)\left|\int_0^x g(s)ds\right|\leq A,\quad x\in\mathbb{R}^n,$$

Take

$$p(x)=3+\frac{1}{1+x_1^2},\quad q(x)=3+\frac{1}{(2+x_1^2)(4+x_2^2)},\quad x=(x_1,x_2)\in\mathbb{R}^2.$$

Then for the problem

$$\begin{aligned}-\Delta_{p(x)}u &= u^2+v_{x_1}^3\\ -\Delta_{q(x)}v &= u_{x_1}^2+v_{x_2}^3,\quad x=(x_1,x_2)\in\mathbb{R}^2,\end{aligned}$$

are fulfilled all conditions of Theorem 4.1, Theorem 4.2 and Theorem 4.3.

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