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SOME REMARKS ON STATISTICAL COMPLETENESS IN METRIC SPACES

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Abstract. In this paper, we study statistical convergence of sequences in metric spaces and derive some results on statistically Cauchy sequence and statistical completeness. We also generalize Cantor's intersection theorem in the statistical setting.

У цій статті ми вивчаємо статистичну збіжність послідовностей в метричних просторах і отримуємо деякі результати про статистичні фундаментальні послідовності і статистичну повноту. Ми також узагальнюємо теорему Кантора про перетин в статистичному сенсі.

1. Introduction

The idea of statistical convergence, which grew out from the usual convergence using the asymptotic density of a set, was first developed by H. Fast [8] and H. Steinhaus [23] in 1951. However, Zygmund's monograph [27] in 1935 contained the concept of statistical convergence under the name "almost convergence" for the first time. Over the years, several authors have employed statistical convergence under a variety of names. In 1959, Schoenberg [22] used statistical convergence under the name of D-convergence and established a connection with a summability approach. Statistical convergence was abundantly used by numerous authors in diverse areas including number theory [7], measure theory [17], probability theory [10], optimization theory [19], and approximation theory [11].

Generalizations of statistical convergence in different spaces has also been studied ([15, 16]). In 1980, Salat [20] obtained some results on statistically convergent sequences in the real number setting. Kostyrko et al. [14] characterized the set of all statistical limit points of a sequence in terms of F_σ-sets and discontinuity points of distribution functions of the sequence. Fridy [9] obtained a statistical analogue of limit point results and completeness theorems by distinguishing between statistical limit point and statistical cluster point. Di Maio et al. [6] studied statistical convergences in topological and uniform spaces and demonstrated how these convergences can be used in selection theory, function spaces, and hyperspaces. Ilkhan and Kara [12] introduced the statistical Bourbaki-Cauchy sequence and also proved some results in quasi-metric spaces [13]. Bilalov and Nazarova [1] introduced p-strong convergence and proved the equivalence of statistical convergence to the p-strong convergence and statistical fundamentality, followed by the derivation of Tauberian theorems involving statistical convergence in metric spaces. Debnath & Rakshit [5] introduced I-statistical limit points and I-statistical cluster points of a real number sequence and studied some of its basic properties. Tripathy & Hazarika [26] studied the notion of I-monotonic sequences and introduced the notion of I-convergence for series of real or complex numbers and also derived some results on them.

Savas & Debnath [21] presented a novel idea of lacunary statistically $\phi-$convergence and its associated sequence space denoted by $S_\theta - \phi$ as a generalization of the statistical convergence and lacunary statistical convergence by using the lacunary sequence 'θ' and

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Orlicz function 'ϕ' and investigated some of its basic properties and relations. Choudhury & Debnath [3] developed the concept of quasi statistical convergence and strong quasi statistical summability in gradual normed linear spaces and investigated some of its basic properties and their interrelationship. They also introduced the concept of gradual quasi statistical Cauchy sequences. Choudhury et al. [4] introduced the notion of $I-$deferred statistical limit point, cluster point, limit superior, limit inferior and analyzed various properties of these concepts based on $I-$deferred statistical convergence.

In this work, we have done a thorough study of statistical convergence and derived, in Theorem 3.1, that statistical convergence is not metrizable. We then investigate a relationship between usual Cauchy and statistically Cauchy sequences in a metric space. It has been proved in Proposition 3.3, that every usual Cauchy sequence is statistically Cauchy in a metric space. Then, a counter-example 3.5 is given to prove that a statistically Cauchy sequence may not be Cauchy in the usual sense. Following this, a necessary and sufficient condition for a sequence to be statistically Cauchy is derived in Proposition 3.6. Then Corollary 3.7 shows that a uniformly continuous function maps every statistically Cauchy sequence onto a statistically Cauchy sequence. The equivalence of usual completeness and statistical completeness is proved in Theorem 3.8. In the last section, in Theorem 3.9, we generalize the Cantor's intersection theorem for metric spaces in the statistical convergence setting.

2. Definitions and Preliminaries

Definition 2.1 ([20])**.** The *asymptotic density* of $A \subseteq \mathbb{N}$, denoted by $\delta(A)$, is defined to be $\lim\limits_{n\to\infty} \frac{A(n)}{n}$, whenever the limit exists; where $A(n) := |\{k \in A : k \leq n\}|$. That is,

$$\delta(A) := \lim_{n\to\infty} \frac{A(n)}{n} = \lim_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}$$

Obviously, from the definition of asymptotic density, we immediately infer that $\delta(A) = 0$ provided that A is a finite set of positive integers. On the other hand, the set of squares of positive integers, i.e., the set $\{1, 4, 9, 16, \dots, \}$ is an infinite set having asymptotic density zero. This follows from the observation that the number of square numbers up to n is at most $\sqrt{n}$. Note also that $\delta(\mathbb{N} \setminus A) = 1 - \delta(A)$.

Definition 2.2 ([1, 20])**.** Let (X, d) be a metric space. We say that a sequence $(x_n) \subseteq X$ is *statistically convergent* to $x \in X$, denoted by $x_n \xrightarrow{st} x$ as $n \to \infty$, if $\forall\, \varepsilon > 0$, we have

$$\delta(\{n \in \mathbb{N} : d(x_n, x) \geq \varepsilon\}) = 0.$$

Or equivalently,

$$\delta(\{n \in \mathbb{N} : d(x_n, x) < \varepsilon\}) = 1.$$

If $x_n \xrightarrow{st} x$ as $n \to \infty$, we call x a statistical limit of (x_n).

Theorem 2.3 ([24])**.** *If a sequence (x_n) in a metric space (X, d) is statistically convergent, then its statistical limit is unique.*

Theorem 2.4 ([24])**.** *A convergent sequence in a metric space is also statistically convergent and the statistical limit and the usual limit are the same.*

The converse of Theorem 2.4 is not true in general. This can be established by the following counter-example :

Example 2.5 ([24])**.** Let us consider a sequence (x_n) in the metric space $\mathbb{R}$ with the usual metric such that

$$x_n := \begin{cases} 1, & \text{if } n = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise .} \end{cases}$$

Definition 2.6 ([24])**.** A subset K of $\mathbb{N}$ is said to be *statistically dense* in $\mathbb{N}$ if $\delta(K) = 1$. For example, the set $\{n \in \mathbb{N} : \text{n is not a square}\}$ is statistically dense as the asymptotic density of its complement was seen to be 0.

Lemma 2.7 ([24])**.** *The following statements are true and will be useful in our study.*

(i) *If $A \subseteq B \subseteq C$ and A is statistically dense in B and B is statistically dense in C, then A is statistically dense in C.*

(ii) *If A and B are two sets such that $\delta(A) = 1, \delta(B) = 1$, then $\delta(A \cap B) = 1$ and $\delta(A \cup B) = 1$.*

Theorem 2.8 ([20])**.** *Let (x_n) and (y_n) be two statistically convergent sequences. If $x_n \xrightarrow{st} a$ and $y_n \xrightarrow{st} b$ as $n \to \infty$, and c is any real number, then*

(i) $x_n + y_n \xrightarrow{st} a + b$ *as* $n \to \infty$,

(ii) $c\, x_n \xrightarrow{st} c\, a$ *as* $n \to \infty$.

Theorem 2.9 ([20])**.** *Let (X, d) be a metric space. A sequence (x_l) in X is statistically convergent and $x_l \xrightarrow{st} b$ as $l \to \infty$ if and only if there exists a set $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and $x_{l_n} \to b$ as $n \to \infty$.*

Definition 2.10 ([2])**.** Let $G \neq \emptyset$ be any set in a metric space (X, d) and $l \in X$. Then l belongs to the *statistical closure* of G if there exists a sequence (x_k) of points in G such that $x_k \xrightarrow{st} l$ as $k \to \infty$. We denote the *statistical closure* of a set G by $\overline{G}^{st}$.

Definition 2.11 ([2])**.** A non-empty set G in a metric space (X, d) is said to be *statistically closed* if $\overline{G}^{st} \subseteq G$.

Theorem 2.12 ([24])**.** *Union of two* statistically closed *sets is* statistically closed.

Definition 2.13 ([24])**.** Let (X, d) be any metric space. Let $\mathcal{T}_f$ denote the family of all statistically closed sets in X. Then it is easy to check that $\mathcal{T}_f$ is a topology on X, known as the *statistical topology* on X. On the other hand, the topology obtained from the closed subsets of X, which are induced by usual convergence in (X, d), is called the *classical topology* on X.

Theorem 2.14 ([24])**.** *Let F be a non-empty subset in a metric space (X, d). Then F is statistically closed if and only if F is closed, i.e., the statistical topology $\mathcal{T}_f$ on X coincides with the classical topology on X.*

Definition 2.15 ([1, 24])**.** A sequence (x_n) in a metric space (X, d) is said to be a *statistically Cauchy sequence* if for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\delta(\{n \in \mathbb{N} : d(x_n, x_N) \geq \varepsilon\}) = 0 \text{ or } \delta(\{n \in \mathbb{N} : d(x_n, x_N) < \varepsilon\}) = 1.$$

Definition 2.16 ([25])**.** A sequence (F_l) is said to be *statistically increasing* if there exists a subset $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and $F_{l_n} \subseteq F_{l_{n+1}}$, for all $n \in \mathbb{N}$.

Definition 2.17 ([25])**.** A sequence (F_l) is said to be *statistically decreasing* if there exists a subset $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and $F_{l_n} \supseteq F_{l_{n+1}}$, for all $n \in \mathbb{N}$.

Definition 2.18 ([16])**.** A metric space (X, d) is said to be *statistically complete* if every statistically Cauchy sequence in X is statistically convergent in X.

Theorem 2.19 ([18])**.** *Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ be a uniformly continuous function. Then f maps every Cauchy sequence in X onto a Cauchy sequence in Y.*

3. Main Results

3.1. Statistical completeness.

Theorem 3.1. *Let (X,d) be a metric space such that $|X| \geq 2$. Then there is no metric ρ on X such that given any $(x_n) \subseteq X$ and given any $x \in X$ such that $x_n \xrightarrow{st} x$ in (X,d) only if $x_n \to x$ in (X,ρ) as $n \to \infty$.*

Proof. If possible, let ρ be a metric on X such that statistical convergence in (X,d) is equivalent to the topological convergence in (X,ρ). Let $G \neq \emptyset$ be an open set in (X,ρ) and let $x_0 \in G$. If possible, let $G \neq X$. Take any $x \in X \setminus G$. Define a sequence (x_n) in X by

$$x_n := \begin{cases} x, & \text{if } \exists\, k \in \mathbb{N} \text{ such that } n = k^2, \\ x_0, & \text{otherwise} \end{cases}$$

Clearly, $x_n \xrightarrow{st} x_0$ in (X,d) and so $x_n \to x_0$ in (X,ρ) as $n \to \infty$. Since G is an open set in (X,ρ) containing x_0, we have $x_n \in G$ eventually entailing $x \in G$, which is a contradiction. Hence $G = X$ and so the topology generated by ρ is the indiscrete topology on X. This is not possible as the indiscrete topology on X is not metrizable. Hence proved. ☐

Remark 3.2. Theorem 3.1 shows that the statistical convergence is not metrizable.

Proposition 3.3. *If a sequence in a metric space (X,d) is Cauchy, then it is statistically Cauchy in (X,d).*

Proof. Let (x_n) be a Cauchy sequence in a metric space (X,d). Then, $\forall\, \varepsilon > 0, \exists\, n_0 \in \mathbb{N}$, such that $d(x_n, x_m) < \varepsilon$ for all $m,n \geq n_0$. In particular, $d(x_n, x_{n_0}) < \varepsilon$ for all $n \geq n_0$. This implies $\{n \in \mathbb{N} : d(x_n, x_{n_0}) \geq \varepsilon\} \subseteq \{1, 2, \cdots, n_0 - 1\}$, so therefore $\delta(\{n \in \mathbb{N} : d(x_n, x_{n_0}) \geq \varepsilon\}) \leq \delta(\{1, 2, \cdots, n_0 - 1\}) = 0$. Hence, (x_n) is statistically Cauchy. ☐

Remark 3.4. The converse of the above theorem need not be true as seen below.

Example 3.5. Consider the sequence (x_k) of real numbers with usual metric d, whose terms are as follows :

$$x_k := \begin{cases} k, & \text{if } k = n^2 \text{ for some } n \in \mathbb{N}, \\ \frac{1}{k}, & \text{otherwise }. \end{cases}$$

Clearly, (x_k) is statistically Cauchy as there exists a subsequence $(\frac{1}{k})$ which is Cauchy. But (x_k) is not Cauchy because for any $k \in \mathbb{N}$, we have

$$d(x_{(k+1)^2}, x_{k^2}) = d((k+1)^2, k^2) = |(k+1)^2 - k^2| = 2k + 1 \to \infty \text{ as } k \to \infty.$$

Proposition 3.6. *Let (X,d) be a metric space. A sequence (x_l) in X is statistically Cauchy if and only if there exists a set $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ with $\delta(L) = 1$ such that (x_{l_n}) is a Cauchy sequence in X.*

Proof. Let $L = \{l_1 < l_2 < \cdots\} \subset \mathbb{N}$ be such that $\delta(L) = 1$ and (x_{l_n}) is a Cauchy sequence in X. Let $\varepsilon > 0$. Since (x_{l_n}) is a Cauchy sequence in X, therefore there exists $n_0 \in \mathbb{N}$ such that $d(x_{l_m}, x_{l_n}) < \varepsilon$ for all $m,n \geq n_0$. In particular,

$$\begin{aligned} & d(x_{l_m}, x_{l_{n_0}}) < \varepsilon,\ \forall\, m \geq n_0 \\ \Rightarrow\ & d(x_{l_m}, x_{l_{n_0}}) < \varepsilon,\ \forall\, m \in \{n_0, n_0+1, n_0+2, \cdots\} \\ \Rightarrow\ & d(x_n, x_{l_{n_0}}) < \varepsilon,\ \forall\, n \in \{l_{n_0}, l_{n_0+1}, l_{n_0+2}, \cdots\} \end{aligned} \tag{3.1}$$

Let $A_\varepsilon := \{n \in \mathbb{N} : d(x_n, x_{l_{n_0}}) \geq \varepsilon\}$. Then, from (3.1), we have

$$A_\varepsilon \subseteq \mathbb{N} \setminus \{l_{n_0}, l_{n_0+1}, l_{n_0+2}, \cdots\}. \tag{3.2}$$

But $\mathbb{N}\setminus\{l_{n_0}, l_{n_0+1}, l_{n_0+2}, \cdots\} = (\mathbb{N}\setminus\{l_1, l_2, \cdots\}) \cup \{l_1, \cdots, l_{n_0-1}\}$. Since $\delta(\{l_1, l_2, \cdots\}) = 1$ and $\delta(\{l_1, \cdots, l_{n_0-1}\}) = 0$, we find that

$$\delta(\mathbb{N}\setminus\{l_{n_0}, l_{n_0+1}, l_{n_0+2}, \cdots\}) = 0.$$

Hence, from (3.2), we get

$$\delta(A_\varepsilon) \leq \delta(\mathbb{N}\setminus\{l_{n_0}, l_{n_0+1}, l_{n_0+2}, \cdots\}) = 0. \text{ So, } \delta(A_\varepsilon) = 0.$$

Thus, $\forall\, \varepsilon > 0\ \exists\, l_{n_0} \in \mathbb{N}$ such that $\delta(\{n \in \mathbb{N} : d(x_n, x_{l_{n_0}}) \geq \varepsilon\}) = 0$. Hence, it is proved that (x_n) is a statistically Cauchy sequence.

Conversely, let (x_n) be statistically Cauchy in X. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\delta(\{n \in \mathbb{N} : d(x_n, x_N) \geq \varepsilon\}) = 0$ or $\delta(\{n \in \mathbb{N} : d(x_n, x_N) < \varepsilon\}) = 1$.

Put

$$L_j := \{n \in \mathbb{N} : d(x_n, x_{N_j}) < \frac{1}{j}\}, \tag{3.3}$$

where $j \in \mathbb{N}$.

Since (x_n) is statistically Cauchy, we have, $\delta(L_j) = 1$ for all $j \in \mathbb{N}$. Now, let us define the following sets :

$$L'_1 := L_1 \;;$$
$$L'_j := L'_{j-1} \cap L_j, \qquad \forall\, j = 2, 3, 4, \cdots.$$

Then, clearly,

(1) $L'_1 \supset L'_2 \supset \cdots \supset L'_j \supset L'_{j+1} \supset \cdots$
(2) $\delta(L'_j) = 1,\ \forall\, j = 1, 2, 3, \cdots$, by Lemma 2.7 (ii), since each L'_j is an intersection of two sets, each one having density equals to 1.

Let us choose an arbitrary number $v'_1 \in L'_1$. Then, by (2), there exists $v'_2 > v'_1, v'_2 \in L'_2$ such that for each $n \geq v'_2$, we have $\frac{L'_2(n)}{n} > \frac{1}{2}$, where $L'_j(n) = \{l \in L'_j : l \leq n\}$ and $\lim_{n\to\infty} \frac{L'_j(n)}{n} = \delta(L'_j) = 1$. Further, by (2), there exists $v'_3 > v'_2, v'_3 \in L'_3$ such that for each $n \geq v'_3$, we have $\frac{L'_3(n)}{n} > \frac{2}{3}$. Proceeding similarly by induction, we can construct a sequence $v'_1 < v'_2 < \cdots < v'_j < \cdots$ of positive integers such that $v'_j \in L'_j$ and

$$\frac{L'_j(n)}{n} > \frac{j-1}{j}, \tag{3.4}$$

for each $n \geq v'_j, j \in \mathbb{N}$.

Let us consider the set L as follows. Each natural number of the interval $[1, v'_1)$ belongs to L, further, any natural number of the interval $[v'_j, v'_{j+1})$ belongs to L if and only if it belongs to $L'_j, \forall\, j = 1, 2, 3, \cdots$. i.e.

$$L := [1, v'_1) \cup \left(\bigcup_{j\geq 1}([v'_j, v'_{j+1}) \cap L'_j)\right).$$

According to (1) and (3.4), for each $n, v'_j \leq n < v'_{j+1}$, we get

$$\frac{L(n)}{n} \geq \frac{L'_j(n)}{n} > \frac{j-1}{j}.$$

From this it is obvious that $\delta(L) = 1$. Let $\varepsilon > 0$. Choose a $j \in \mathbb{N}$ such that $\frac{1}{j} < \frac{\varepsilon}{2}$. Let $m \geq n \geq v'_j$ and $m, n \in L$. Then, there exist such natural numbers $r \geq s \geq j$ such that $v'_r \leq m < v'_{r+1}$ and $v'_s \leq n < v'_{s+1}$. But by definition of L, we have $m \in L'_r$ and $n \in L'_s$.

Since $r \geq s$, therefore $L'_r \subset L'_s$. It follows that $m, n \in L'_s \subset L_s$. Therefore, using (3.3), we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{N_s}) + d(x_{N_s}, x_n) \\ &< \frac{1}{s} + \frac{1}{s} \leq \frac{1}{j} + \frac{1}{j} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, $d(x_m, x_n) < \varepsilon, \forall\, m, n \in L$ with $m \geq n \geq v'_j$, where $\delta(L) = 1$. Therefore, $(x_n)_{n\in L}$ is a Cauchy sequence. This completes the proof.. □

Corollary 3.7. *Let (X, d) and (Y, ρ) be two metric spaces, and $f : X \to Y$ be a uniformly continuous function. Then f maps every statistically Cauchy sequence of X onto a statistically Cauchy sequence of Y.*

Proof. Let (x_l) be a statistically Cauchy sequence in (X, d). Then, by Proposition 3.6, there exists $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and (x_{l_n}) is a Cauchy sequence in X. Since f is uniformly continuous, by Theorem 2.19, $(f(x_{l_n}))$ is a Cauchy sequence in Y. Thus there exists $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and $(f(x_{l_n}))$ is a Cauchy sequence in Y. Hence, $(f(x_l))$ is a statistically Cauchy sequence in Y. □

Theorem 3.8. *A metric space (X, d) is statistically complete if and only if it is complete.*

Proof. Let (X, d) be a complete metric space and let (x_l) be a statistically Cauchy sequence in (X, d). Therefore, by Proposition 3.6, there exists $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and (x_{l_n}) is a Cauchy sequence in X. Since (X, d) is complete, therefore there exists $x_0 \in X$ such that $x_{l_n} \to x_0$ as $n \to \infty$. Therefore, by Theorem 2.9 , $x_l \xrightarrow{st} x_0 \in X$ as $l \to \infty$. Hence, (X, d) is statistically complete.

Conversely, let (X, d) be statistically complete and let (x_l) be a Cauchy sequence in (X, d). Therefore, by Proposition 3.3, (x_l) is a statistically Cauchy sequence in (X, d). Since (X, d) is statistically complete, so there exists $x_0 \in X$ such that $x_l \xrightarrow{st} x_0$ as $l \to \infty$. Therefore, by Theorem 2.9, there exists a set $L = \{l_1 < l_2 < \cdots < l_n < \cdots\} \subset \mathbb{N}$ such that $\delta(L) = 1$ and $x_{l_n} \to x_0$ as $n \to \infty$. Since (x_l) is a Cauchy sequence and it has a convergent subsequence (x_{l_n}) converging to x_0 in (X, d), therefore $x_l \to x_0 \in X$ as $l \to \infty$. Hence, it follows that (X, d) is a complete metric space. □

3.2. **Generalization of Cantor's Intersection Theorem.**

Theorem 3.9. *A metric space (X, d) is statistically complete if and only if given any statistically decreasing sequence (F_n) of non-empty statistically closed subsets of X with* $\operatorname{diam}(F_n) \xrightarrow{st} 0$ *as $n \to \infty$, there exists $N \subset \mathbb{N}$ with $\delta(N) = 1$ satisfying the property that $\bigcap_{n\in N} F_n$ is a singleton.*

Proof. Let (X, d) be a statistically complete metric space and let (F_n) be a statistically decreasing sequence of non-empty statistically closed subsets of X with $\operatorname{diam}(F_n) \xrightarrow{st} 0$ as $n \to \infty$. Since (F_n) is statistically decreasing, therefore there exists an ordered set $K \subset \mathbb{N}$ with $\delta(K) = 1$ such that $F_m \subseteq F_n$ for all $n \leq m$ where $m, n \in K$. Since F_n is non-empty for all $n \in \mathbb{N}$, we can choose $x_n \in F_n$, for each $n \in K$. This gives rise to a sequence (x_n) in X with the property that $x_m \in F_n$ for all $m \geq n$ and $m, n \in K$. Then $d(x_m, x_n) \leq \operatorname{diam}(F_n) \xrightarrow{st} 0$ as $n \to \infty$, for all $m \geq n$ and $m, n \in K$. It follows that there is an ordered set $N \subset K$ with $\delta(N) = 1$ such that $d(x_m, x_n) \leq \operatorname{diam}(F_n) \to 0$ as $n \to \infty$, for all $m \geq n$ and $m, n \in N$. Thus $(x_n)_{n\in N}$ is a Cauchy sequence in X. Since

X is statistically complete, so by Theorem 3.8, X is complete and therefore there exists $x \in X$ such that $x_n \to x, n \in N$ as $n \to \infty$. This implies $x \in \overline{F_n} = F_n$ for all $n \in N$. Hence $x \in \bigcap_{n \in N} F_n$. To show the uniqueness of x, assume that $y \in \bigcap_{n \in N} F_n$ be any element. Since $x, y \in F_n$ for all $n \in N$, so $d(x,y) \leq \text{diam}\,(F_n)$ for each $n \in N$. Since for each $n \in N$, $\text{diam}\,(F_n) \to 0$ as $n \to \infty$, we obtain $d(x,y) = 0$ entailing $x = y$. Hence $\bigcap_{n \in N} F_n$ is a singleton.

Conversely, let (X,d) be a metric space satisfying the given hypothesis. In view of Theorem 3.8, it suffices to show that X is complete. So let (x_n) be a Cauchy sequence in X. For each $n \in \mathbb{N}$, define $F_n := \{x_k : k \geq n\}$. Then, $(\overline{F_n})$ is a decreasing (and so statistically decreasing) sequence of closed (and so statistically closed) sets in X. Since (x_n) is a Cauchy sequence, so $d(x_m, x_n) \to 0$ as $m, n \to \infty$, and therefore $diam(F_n) \to 0$ as $n \to \infty$. As a result, $\text{diam}\,(\overline{F_n}) = \text{diam}\,(F_n) \xrightarrow{st} 0$ as $n \to \infty$, therefore by hypothesis, there exist an ordered set $N \subset \mathbb{N}$ with $\delta(N) = 1$ and $x \in X$ such that $\bigcap_{n \in N} \overline{F_n} = \{x\}$. Now, we will show that $x_n \to x$. Since $\bigcap_{n \in N} \overline{F_n} = \{x\}$, so $x \in \overline{F_n}$, $\forall\, n \in N$. On the other hand, $x_n \in F_n$, $\forall\, n \in \mathbb{N}$ by our construction and therefore $x_n \in \overline{F_n}$, $\forall\, n \in \mathbb{N}$. As a result, for each $n \in N$, we have $d(x_n, x) \leq diam(\overline{F_n}) \to 0$ as $n \to \infty$. It follows that $(x_n)_{n \in N} \to x$ as $n \to \infty$. Thus (x_n) is a Cauchy sequence and it has a convergent subsequence $(x_n)_{n \in N} \to x$ as $n \to \infty$, so $x_n \to x, \forall\, n \in \mathbb{N}$ as $n \to \infty$ and $x \in X$. Hence (X,d) is complete and so statistically complete metric space. □

4. Conclusion

In this paper, we have thoroughly studied the concept of statistical convergence and established that statistical convergence is not metrizable. We then investigated the relationship between usual Cauchy and statistically Cauchy sequences in a metric space and obtained that every usual Cauchy sequence is statistically Cauchy but the converse need not be true. A counter-example is given to prove that a statistically Cauchy sequence may not be Cauchy in the usual sense. Following this, we have established a necessary and sufficient condition for a sequence to be statistically Cauchy in a metric space. We also have shown that a uniformly continuous function maps a statistically Cauchy sequence onto a statistically Cauchy sequence. Further, we have proved that usual completeness and statistical completeness are equivalent. Finally, we generalize Cantor's intersection theorem in the statistical convergence setting.

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Conflict of Interest

The authors, Dr. Naba Kanta Sarma and Mr. Sourabh Nath, declare that they have no conflict of interest.

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