Methods of Functional Analysis and Topology
Vol. 30 (2024), no. 1-2, pp. 72–79
https://doi.org/10.31392/MFAT-npu26_1-2.2024.07

HYPERCYCLICITY OF AFFINE COMPOSITION OPERATORS ON ALGEBRAS OF SYMMETRIC ANALYTIC FUNCTIONS

ZORIANA NOVOSAD

Abstract. The paper is devoted to studying the dynamics of affine composition operators on the Fréchet algebras of symmetric analytic functions on ℓ_p. We introduced a class of affine composition operators preserving the symmetry of functions and found necessary and sufficient conditions of hypercyclicity of such operators. Some applications for dynamics of composition operators on the space of entire functions of several complex variables, $H(\mathbb{C}^n)$ are proposed. In particular, we found some conditions of hypercyclicity for a class of polynomial composition operators on $H(\mathbb{C}^n)$.

Стаття присвячена вивченню динаміки афінних композиційних операторів на алгебрах Фреше симетричних аналітичних функцій на ℓ_p. Ведено клас афінних композиційних операторів, що зберігають симетрію функцій, і знайдено необхідні та достатні умови гіперциклічності таких операторів. Пропонуються деякі застосувьння динаміки композиційних операторів в просторі $H(\mathbb{C}^n)$ цілих функцій декількох комплексних змінних. Зокрема, знайжено деякі умови гіперциклічності для класу операторів поліноміальної композиції на $H(\mathbb{C}^n)$.

1. Introduction

The dynamics of bounded linear operators on infinite dimensional topological linear spaces can generally be very complicated. In particular, such an operator may have a dense orbit (that is, to be hypercyclic) or other "chaotic" and "mixing" properties. The first example of a hypercyclic operator on the Fréchet space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$ was constructed by Birkhoff [10] in 1929. The Birkhoff theorem states that the translation operator $T(f)(z) = f(z+1)$ is hypercyclic on $H(\mathbb{C})$. Later, in 1952 MacLane [30] proved that the differentiation operator $D\colon H(\mathbb{C}) \to H(\mathbb{C})$, $D(f) = f'$, $f \in H(\mathbb{C})$, is also hypercyclic. Rolewicz in 1969 observed that the weighted backward shift operator $T\colon \ell_2 \to \ell_2$ given by

$$T(x_1, x_2, x_3, \dots) = \lambda(x_2, x_3, \dots)$$

is hypercyclic on ℓ_2 whenever $|\lambda| > 1$, $\lambda \in \mathbb{C}$. Operators of such a type are hypercyclic in more general situations. For details about various generalizations of Rolewicz weighted backward shift operators we refer the reader to [4, 6, 26, 35, 37].

G. Godefroy and J. Shapiro generalized the Birkhoff result involving what are called convolution operators [22].

Theorem 1.1 ([22]). *Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be a continuous linear operator such that*

$$T(f)(z+b) = T(\tau_b(f))(z)$$

(where for $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, $\tau_b(f)(z) := f(z+b)$). If T is not a multiple of the identity, then T is hypercyclic.

2020 *Mathematics Subject Classification.* Primary 47A16; Secondary 46E10, 46E50.

Keywords. Hypercyclic operators, functional spaces, topological automorphisms, symmetric functions.

This research was supported by the National Research Foundation of Ukraine, 2023.03/0198 "Analysis of the spectra of countably generated algebras of symmetric polynomials and possible applications in quantum mechanics and computer science".

Further generalizations of Theorem 1.1 for spaces of entire functions on finite and infinite dimensional Banach spaces were obtained in [2, 8, 12, 17, 31, 32].

Let us recall that an operator C_Φ on $H(\mathbb{C}^n)$ is said to be a *composition operator* if $C_\Phi f(x) = f(\Phi(x))$ for some analytic map $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$. It is known that only the translation operator T_a, for some $a \neq 0$, is a hypercyclic composition operator on $H(\mathbb{C})$ [9]. However, if $n > 1$, $H(\mathbb{C}^n)$ supports more hypercyclic composition operators.

In [7] Bernal-González established some necessary and sufficient conditions for a composition operator by an affine map to be hypercyclic. In particular, in [7] it is proved that a given affine automorphism $S = A + b$ on $\mathbb{C}^n$, the composition operator $C_S\colon f(x) \mapsto f(S(x))$ is hypercyclic if and only if the linear operator A is bijective and the vector b is not in the range of $A - I$.

In this paper we consider composition operators with affine mappings on the space $H_{bs}(\ell_p)$ of symmetric analytic functions of bounded type on ℓ_p, $1 \leq 1 < \infty$ and find necessary and sufficient conditions for hypercyclicity of such operators. For this purpose we used a method developed in [34] (see also [17, 33]). Also, we find some applications for dynamics of composition operators on the space $H(\mathbb{C}^n)$ of entire functions of several complex variables. In particular, we find some conditions of hypercyclicity for a class of polynomial composition operators on $H(\mathbb{C}^n)$.

For details of the theory of analytic functions on Banach spaces we refer the reader to Dineen's book [20]. All basic information about hypercyclic operators can be found in [24, 25].

2. Preliminary results

Let X be a topological vector space. A continuous linear operator $T\colon X \to X$ is said to be *hypercyclic* if there is a vector $x \in X$ such that the set $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$ of iterates of x is dense in X. The vector x is called a *hypercyclic vector* associated to the hypercyclic operator T. An operator T is called *hereditarily hypercyclic* if there is an increasing sequence of positive integers (n_k) such that for each subsequence (m_k) of (n_k) there is some $x \in X$ such that the set

$$\{T^{m_k}(x)\colon k \in \mathbb{N}\}$$

is dense in X.

An operator T is called *topologically transitive* if for any pair U, V of nonempty open subsets of X there exists some integer $k \geq 0$ such that $T^k(U) \cap V \neq \varnothing$. T is called *weakly mixing* if $T \oplus T\colon X \oplus X \to X \oplus X$ is topologically transitive. It is well known (see e.g. [25, p. 39]) that for separable X the hypercyclicity of T is equivalent to its topological transitivity.

A sufficient condition for hypercyclicity, the well known Hypercyclicity Criterion, independently discovered by Kitai [27] and Gethner and Shapiro [21], has been the fundamental tool for proving hypercyclicity. The following version of the hypercyclicity criterion was given by Bès (see [9]).

Theorem 2.1 (Hypercyclicity Criterion)**.** *Let X be a separable Fréchet space and $T\colon X \to X$ be a linear and continuous operator. Suppose there exist in X dense subsets X_0, Y_0, a sequence (n_k) of positive integers, and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n\colon Y_0 \to X$ so that*

(1) *$T^{n_k}(x) \to 0$ for every $x \in X_0$ as $k \to \infty$.*
(2) *$S_k(y) \to 0$ for every $y \in Y_0$ as $k \to \infty$.*
(3) *$T^{n_k} \circ S_k(y) = y$ for every $y \in Y_0$.*

Then T is hypercyclic.

Actually, T satisfies the Hypercyclicity Criterion if and only if T is hereditarily hypercyclic and if and only if T is weakly mixing [25, p. 76].

From the Universal Comparison Principle (see e.g. [24, Proposition 4]) we have the following simple proposition to be used in the sequel.

Proposition 2.2. *Let T be a (hereditarily) hypercyclic operator on X and A be an isomorphism of X. Then $A^{-1}TA$ is (hereditarily) hypercyclic.*

If A is an isomorphism of X, then $A^{-1}TA$ is said to be an perator *similar* to T. If $T = C_\alpha$ is a composition operator on $H(\mathbb{C}^n)$ for some mapping $\alpha\colon \mathbb{C}^n \to \mathbb{C}^n$ and $A = C_\Phi$ is a composition with an analytic automorphism Φ of $\mathbb{C}^n$, then $A^{-1}TA = C_{\Phi\circ\alpha\circ\Phi^{-1}}$.

Let X be a Banach space with a symmetric basis $(e_i)_{i=1}^\infty$. A function g on X is said to be *symmetric* if for every $x = \sum_{i=1}^\infty x_i e_i \in X$,

$$g(x) = g\Big(\sum_{i=1}^\infty x_i e_i\Big) = g\Big(\sum_{i=1}^\infty x_i e_{\sigma(i)}\Big)$$

for an arbitrary permutation σ on the set $\{1,\ldots,m\}$ for any positive integer m. A sequence of homogeneous polynomials $(P_j)_{j=1}^\infty$, $\deg P_k = k$, is called a *homogeneous algebraic basis* in the algebra of symmetric polynomials if for every symmetric polynomial P of degree n on X there exists a unique polynomial q on $\mathbb{C}^n$ such that

$$P(x) = q(P_1(x),\ldots,P_n(x)).$$

More information about various algebras of symmetric analytic functions and their algebraic bases can be found in [3, 5, 11, 14, 28, 36] and references therein.

Throughout this paper we consider the case where $X = \ell_p$, $1 \le p < \infty$. Let us denote by $\mathcal{P}_s(\ell_p)$ the algebra of all symmetric polynomials on ℓ_p. From [23] we know that the so-called *power polynomials* $(F_k)_{k=1}^\infty$,

$$F_k(x) = \sum_{i=1}^\infty x_i^k, \quad k \ge \lceil p \rceil$$

form an algebraic basis in $\mathcal{P}_s(\ell_p)$, where $\lceil p \rceil$ is the ceil of p.

We denote by $H_s^n(\ell_p)$ the algebra of entire symmetric functions on ℓ_p which is topologically generated by polynomials $F_{\lceil p\rceil},\ldots,F_{\lceil p\rceil+n-1}$. It means that $H_s^n(\ell_p)$ is the completion of the algebraic span of $F_{\lceil p\rceil},\ldots,F_{\lceil p\rceil+n-1}$ in the uniform topology on bounded subsets.

Note that if $(P_j)_{j=1}^\infty$ is a homogeneous algebraic basis in $\mathcal{P}_s(\ell_p)$, then $(P_1,\ldots,P_n)$ is a homogeneous algebraic basis in $H_s^n(\ell_p)$. We will use notations $\mathbf{P} := (P_k)_{k=1}^n$.

For a given algebraic basis $\mathbf{P}$ in $H_s^n(\ell_p)$ the mapping

$$\mathcal{F}_n^{\mathbf{P}}\colon f(t_1,\ldots,t_n) \mapsto f(P_{\lceil p\rceil},\ldots,P_{\lceil p\rceil+n-1})$$

is a topological isomorphism between $H(\mathbb{C}^n)$ and $H_s^n(\ell_p)$ (see e.g. [1]).

To investigate dynamics of composition operators with affine mappings on spaces of symmetric functions, we need to have affine mappings on ℓ_p such that the corresponding composition operators map symmetric functions to symmetric functions. One can check that for a symmetric function f on ℓ_p the function $f(\cdot + y)$ is not symmetric for any fixed $y \in \ell_p$, such that $y \ne 0$. So, the space of symmetric functions is not invariant with respect to the usual translation operator $f(\cdot) \mapsto f(\cdot + y)$. However, there is another, "symmetric" translation on ℓ_p. For given $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, we set

$$x \bullet y := (x_1, y_1, x_2, y_2, \ldots).$$

It is easy to check (see e.g. [14, 15]) that

(1) $\|x \bullet y\| = \|x\| + \|y\|$.

(2) $F_n(x \bullet y) = F_n(x) + F_n(y)$ for any integer $n \ge \lceil p \rceil$.

Also, there is a so-called *multiplicative intertwining* [16] of x and y, $x \diamond y$, as the resulting sequence of ordering the set $\{x_i y_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order. From [16] it is known that for arbitrary $x, y \in \ell_p$ we have

(1) $x \diamond y \in \ell_1$ and $\|x \diamond y\| = \|x\| \cdot \|y\|$.
(2) $F_n(x \diamond y) = F_n(x) \cdot F_n(y)$ for any natural $n \ge \lceil p \rceil$.

Semirings, generated by operations '$\bullet$' and '$\diamond$' were studied in [13, 18, 19].

We will say that the map of the form $x \mapsto a \diamond x \bullet y = (a \diamond x) \bullet y$ is a *symmetric affine operator* and define the operator

$$\mathcal{Q}_y \colon H_s^n(\ell_p) \to H_s^n(\ell_p),$$
$$\mathcal{Q}_y(f)(x) := f(a \diamond x \bullet y),$$

where $a = (a_1, a_2, \ldots) \in \ell_1$, $a \ne 0$. We say that $\mathcal{Q}_y$ is a *symmetric affine composition operator*. Note that

$$F_k(a \diamond x \bullet y) = F_k(a)F_k(x) + F_k(y) \tag{2.1}$$

for every $k \ge \lceil p \rceil$.

3. Main results

Theorem 3.1. *The symmetric affine composition operator*

$$\mathcal{Q}_y \colon H_s^n(\ell_p) \to H_s^n(\ell_p),$$
$$\mathcal{Q}_y \colon g(x) \to g(a \diamond x \bullet y)$$

$y \in \ell_p$, $(F_{\lceil p \rceil}(y), \ldots, F_{n+\lceil p \rceil-1}(y)) \in \mathbb{C}^n$, is hereditarily hypercyclic on $H_s^n(\ell_p)$ if and only if the following conditions hold:

(i) *all $F_{\lceil p \rceil}(a) \ne 0, \ldots, F_{n+\lceil p \rceil-1}(a) \ne 0$;*
(ii) *for some $\lceil p \rceil \le k \le n + \lceil p \rceil - 1$, $F_k(a) = 1$ and $F_k(y) \ne 0$.*

Proof. For the convince we determine a pair (A, b) as matrix A with complex entries and fixed column vector b in $\mathbb{C}^n$ as follows:

$$A = \begin{pmatrix} F_{\lceil p \rceil}(a) & \cdots & 0 & 0 \\ \vdots & \ddots & \cdots & \vdots \\ 0 & \cdots & F_{n+\lceil p \rceil-2}(a) & 0 \\ 0 & \cdots & 0 & F_{n+\lceil p \rceil-1}(a) \end{pmatrix}$$

and

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-2} \\ b_n \end{pmatrix} = \begin{pmatrix} F_{\lceil p \rceil}(y) \\ \vdots \\ F_{n+\lceil p \rceil-2}(y) \\ F_{n+\lceil p \rceil-1}(y) \end{pmatrix}.$$

We claim that conditions (i) and (ii) of the theorem are equivalent to the property that A is a bijective operator on $\mathbb{C}^n$ and b is not in the range of $A - I$. Indeed, $\det(A) \ne 0$ if and only if $F_{i+\lceil p \rceil-1}(a) \ne 0$, $i = 1, \ldots, n$. Also, if b is not in the range of $A - I$, then $A - I$ must be non-injective, that is, $F_{i+\lceil p \rceil-1}(a) = 1$ for some $1 \le i \le n$ and $b_i = F_{i+\lceil p \rceil-1}(y) \ne 0$. According to [7], the composition operator

$$C_{At+b}(f) = f(At + b), \quad t = (t_1, \ldots, t_n) \in \mathbb{C}^n$$

is hereditarily hypercyclic on $H(\mathbb{C}^n)$.

Using the isomorphism $\mathcal{F}_n^{\mathbf{F}} \colon f(t_1, \ldots, t_n) \mapsto f(F_{\lceil p \rceil}, \ldots, F_{\lceil p \rceil+n-1})$ from $H(\mathbb{C}^n)$ to $H_s^n(\ell_p)$ we can represent any $g \in H_s^n(\ell_p)$ as

$$g(x) = f(F_{\lceil p \rceil}(x), \ldots, F_{\lceil p \rceil+n-1}(x)),$$

where $f = \left(\mathcal{F}_n^{\mathbf{F}}\right)^{-1}(g)$. By Proposition 2.2, the affine composition operator

$$\mathcal{Q}_y(g) = \mathcal{F}_n^{\mathbf{F}}\Big(C_{At+b}\Big(\left(\mathcal{F}_n^{\mathbf{F}}\right)^{-1}(g)\big(F_{\lceil p\rceil}, \ldots, F_{\lceil p\rceil+n-1}\big)\Big)\Big)$$

is hereditarily hypercyclic.

Conversely, if $\mathcal{Q}_y$ is hypercyclic on $H_s^n(\ell_p)$, then C_{At+b} is hypercyclic on $H(\mathbb{C}^n)$ and by [7], A is bijective and b is not in the range of $A - I$. But as we proved, it is equivalent to conditions (i) and (ii) of the theorem. □

Like in [34], this result can be extended to the infinite dimensional case.

Theorem 3.2. *Let a and y be vectors in ℓ_p such that*

(i) $F_n(a) \neq 0$ *for every* $n \geq \lceil p\rceil$.
(ii) *There is at least one number* $k_0 \in \mathbb{N}$ *such that* $F_{k_0+\lceil p\rceil-1}(a) = 1$ *and* $F_{k_0+\lceil p\rceil-1}(y) \neq 0$.

Then $\mathcal{Q}_y \colon g(x) \to g(a \diamond x \bullet y)$, $g \in H_{bs}(\ell_p)$ *is hypercyclic on* $H_{bs}(\ell_p)$.

Proof. Let us show that $\mathcal{Q}_y$ is topologically transitive on $H_{bs}(\ell_p)$. Let U and V be disjoint open subsets in $H_{bs}(\ell_p)$. Since the linear space

$$\bigcup_{n=1}^{\infty} H_s^n(\ell_p) \subset H_{bs}(\ell_p)$$

is dense in $H_{bs}(\ell_p)$, it has nonempty intersections with U and V. Thus, there are integers $r, j > k_0$ such that $H_s^r(\ell_p) \cap U \neq \varnothing$ and $H_s^j(\ell_p) \cap V \neq \varnothing$. Let $m = \max(r, j)$. Then $H_s^m(\ell_p)$ has nonempty intersections with both U and V. Since the topology of $H_s^n(\ell_p)$ is induced by the topology of $H_{bs}(\ell_p)$, the intersections $U_m = H_s^m(\ell_p) \cap U$ and $V_m = H_s^m(\ell_p) \cap V$ are open in $H_{bs}(\ell_p)$. By Theorem 3.1 the restriction of $\mathcal{Q}_y$ to $H_s^m(\ell_p)$ is hypercyclic and so topologically transitive. That is, there is a number n such that $\mathcal{Q}_y^n(U_m) \subset V_m$. Thus, $\mathcal{Q}_y^n(U) \subset V$ and so $\mathcal{Q}_y$ is topologically transitive. Hence, $\mathcal{Q}_y$ is hypercyclic. □

If we consider another algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$ in $H_s^n(\ell_p)$, then $\mathcal{F}_n^{\mathbf{P}}$ will give us another isomorphism from $H(\mathbb{C}^n)$ to $H_s^n(\ell_p)$. Clearly, the mapping $(\mathcal{F}_n^{\mathbf{P}})^{-1}\mathcal{Q}_y\mathcal{F}_n^{\mathbf{P}}$ is hypercyclic on $H(\mathbb{C}^n)$ if and only if $\mathcal{Q}_y$ is hypercyclic on $H_s^n(\ell_p)$.

Let us consider the case of symmetric analytic functions on ℓ_1. There is another natural algebraic basis in $\mathcal{P}_s(\ell_1)$, so-called the basis *elementary* symmetric polynomials $(G_k)_{k=1}^{\infty}$, $k \in \mathbb{N}$,

$$G_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

It is easy to check that $G_k(x \bullet y) = \sum_{i=0}^{k} G_i(x) G_{k-i}(y)$, where $G_0 \equiv 1$.

It is known (see e.g. [15, 29]) that

$$\sum_{n=0}^{\infty} G_n(x) = \prod_{i=1}^{\infty}(1 + x_i).$$

Thus,

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_n(a \diamond x) &= \prod_{i,j=1}^{\infty} (1 + a_j x_i) \\
&= \prod_{i=1}^{\infty}(1 + a_1 x_i) \prod_{i=1}^{\infty}(1 + a_2 x_i) \cdots \prod_{i=1}^{\infty}(1 + a_j x_i) \cdots \\
&= \sum_{n=0}^{\infty} a_1^n G_n(x) \sum_{n=0}^{\infty} a_2^n G_n(x) \cdots \sum_{n=0}^{\infty} a_j^n G_n(x) \cdots \\
&= \sum_{n=0}^{\infty} \sum_{k_1+\cdots+k_n=n} \sum_{j_1<\cdots<j_n} a_{j_1}^{k_1} a_{j_2}^{k_2} \cdots a_{j_n}^{k_n} G_{k_1}(x) \cdots G_{k_n}(x).
\end{aligned}
$$

So,

$$
G_n(a \diamond x) = \sum_{k_1+\cdots+k_n=n} \sum_{j_1<\cdots<j_n} a_{j_1}^{k_1} a_{j_2}^{k_2} \cdots a_{j_n}^{k_n} G_{k_1}(x) \cdots G_{k_n}(x).
$$

Taking into account that

$$
G_{k_j}(x \bullet y) = \sum_{i=0}^{k_j} G_i(x) G_{k_j - i}(y),
$$

we can write

$$
\begin{aligned}
G_n(a \diamond x \bullet y) &= \sum_{i=0}^{n} G_i(a \diamond x) G_{n-i}(y) \\
&= \sum_{i=0}^{n} G_{n-i}(y) \sum_{k_1+\cdots+k_i=i} \sum_{j_1<\cdots<j_n} a_{j_1}^{k_1} a_{j_2}^{k_2} \cdots a_{j_i}^{k_i} G_{k_1}(x) \cdots G_{k_i}(x).
\end{aligned} \tag{3.2}
$$

In particular, for example, we will have

$$
\begin{aligned}
G_1(a \diamond x \bullet y) &= G_1(x) \sum_{j=1}^{\infty} a_j + G_1(y); \\
G_2(a \diamond x \bullet y) &= G_2(a \diamond x) + G_1(a \diamond x \bullet y) G_1(y) + G_2(y) \\
&= \sum_{k_1+k_2=2} \sum_{j_1<j_2} a_{j_1}^{k_1} a_{j_2}^{k_2} G_{k_1}(x) G_{k_2}(x) + G_1(a \diamond x) G_1(y) + G_2(y) \\
&= 2G_2(x) \sum_{j=1}^{\infty} a_j^2 + \big(G_1(x)\big)^2 \sum_{j_1<j_2} a_{j_1} a_{j_2} + G_1(x) G_1(y) \sum_{j=1}^{\infty} a_j + G_2(y).
\end{aligned}
$$

For every $f(t_1, \ldots, t_n) \in H(\mathbb{C}^n)$,

$$
\mathcal{Q}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \ldots, t_n) = \mathcal{Q}_y f(G_1(x), \ldots, G_n(x)) = f(G_1(a \diamond x \bullet y), \ldots, G_n(a \diamond x \bullet y)).
$$

Let $G_k(y) = c_k$ and $c_0 = 1$. To find $\big(\mathcal{F}_n^{\mathbf{G}}\big)^{(-1)} \mathcal{Q}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \ldots, t_n)$ substitute (3.2) into the formula for $\mathcal{Q}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \ldots, t_n)$ and apply $\big(\mathcal{F}_n^{\mathbf{G}}\big)^{(-1)}$ taking $G(x) \rightsquigarrow t_n$ and $G(y) \rightsquigarrow c_n$. Thus, we have

$$
\begin{aligned}
&\big(\mathcal{F}_n^{\mathbf{G}}\big)^{(-1)} \mathcal{Q}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \ldots, t_n) \\
&= f\Big(t_1 \sum_{j=1}^{\infty} a_j + c_1, \ldots, \sum_{i=0}^{n} c_{n-i} \sum_{k_1+\cdots+k_i=i} \sum_{j_1<\cdots<j_n} a_{j_1}^{k_1} a_{j_2}^{k_2} \cdots a_{j_i}^{k_i} t_{k_1} \cdots t_{k_i}\Big).
\end{aligned} \tag{3.3}
$$

As we observed, $\left(\mathcal{F}_n^{\mathbf{G}}\right)^{(-1)}\mathcal{Q}_y\mathcal{F}_n^{\mathbf{G}}$ is hypercyclic if and only if $\mathcal{Q}_y$ is hypercyclic. Recall that the bases $\mathbf{F}$ and $\mathbf{G}$ are connected by the Newton recurrent formulas,

$$F_1 = G_1;$$

$$F_n = G_1F_{n-1} - G_2F_{n-2} + \cdots + (-1)^n G_{n-1}F_1 + (-1)^{n+1}nG_n \quad n > 1.$$

Therefore, we have the following result.

Theorem 3.3. *Let numbers $a_1, \ldots, a_n$ and $c_1, \ldots, c_n$ be such that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfy conditions of Theorem 3.1 for the case $p = 1$, where $b_1 = c_1$ and*

$$b_k = c_1b_{k-1} - c_2b_{k-2} + \cdots + (-1)^k c_{k-1}b_1 + (-1)^{k+1}kc_k$$

for $1 < k \le n$. Then the composition operator (3.3) is hereditarily hypercyclic on $H(\mathbb{C}^n)$.

Proof. Let $y = (y_1, \ldots, y_n)$ be a vector in $\mathbb{C}^n$ such that $F_1(y) = b_1, \ldots, F_n(y) = b_n$. Note that such a vector always exists and is unique up to permutations of coordinates. Moreover, from the construction of $b_1, \ldots, b_n$ and the Newton's formulas, we have $G_1(y) = c_1, \ldots, G_n(y) = c_n$. By Theorem 3.1, the operator $\mathcal{Q}_y$ is hereditarily hypercyclic. Thus, the composition operator (3.3) is hypercyclic as well. $\square$

Let us compute (3.3) in $H(\mathbb{C}^2)$.

Example 3.4. Making routine computations in (3.3) for $n = 2$, $y = (y_1, y_2)$, and $a = (a_1, a_2)$ we can see that

$$\left(\mathcal{F}_2^{\mathbf{G}}\right)^{(-1)}\mathcal{Q}_y\mathcal{F}_2^{\mathbf{G}}f(t_1,t_2) = f((a_1+a_2)t_1 + c_1, t_1^2a_1a_2 + 2t_2(a_1^2+a_2^2) + c_1t_1(a_1+a_2) + c_2).$$

Let $a_1 = 2$, $a_2 = -1$, $y_1 = 1$, $y_2 = 2$. Then $F_1(a) = 1$, $F_2(a) = 5$, $b_1 = F_1(y) = 3$, $b_2 = F_2(y) = 5$. By Theorem 3.1, $\mathcal{Q}_y$ is hypercyclic on $H_s^2(\ell_1)$ and so, for $c_1 = G_1(y) = 3$ and $c_2 = G_2(y) = 5$, the operator

$$\left(\mathcal{F}_2^{\mathbf{G}}\right)^{(-1)}\mathcal{Q}_y\mathcal{F}_2^{\mathbf{G}} \colon f(t_1,t_2) \mapsto f(t_1+3, -2t_1^2 + 10t_2 + 3t_1 + 5)$$

is hereditarily hypercyclic on $H(\mathbb{C}^2)$.

References

[1] R. Alencar, R. Aron, P. Galindo and A. Zagorodnyuk, *Algebra of symmetric holomorphic functions on ℓ_p*, Bull. Lond. Math. Soc. **35** (2003), 55-64. doi:10.1112/S0024609302001431.

[2] R. Aron and J. Bès, *Hypercyclic differentiation operators*, Contemporary Mathematics, **232** (1999), 39-46. doi:10.48550/arXiv.math/9906199.

[3] R. Aron, P. Galindo, D. Pinasco, I. Zalduendo, *Group-symmetric holomorphic functions on a Banach space*, Bull. Lond. Math. Soc. **48** (2016), 779–796. doi:10.1112/blms/bdw043.

[4] F. Bayart, E. Matheron, *Dynamics of linear operators*, Cambridge University Press, New York, 2009. doi:10.1017/CBO9780511581113.

[5] N. Baziv, A. Zagorodnyuk, *Analytic Invariants of Semidirect Products of Symmetric Groups on Banach Spaces*, Symmetry, **15** (2023), 2117. doi:10.3390/sym15122117.

[6] L. Bernal-González, *Backward Φ-shifts and universality*, J. Math. Anal. Appl. **306** (2005), 180–196. doi:10.1016/j.jmaa.2004.12.038.

[7] L. Bernal-González, *Universal entire functions for affine endomorphisms of $\mathbb{C}^N$*, J. Math. Anal. Appl. **305** (2005), 690–697. doi:10.1016/j.jmaa.2004.12.031.

[8] L. Bernal-González, J. A. Prado-Tendero *Sequences of differential operators: exponentials, hyperciclicity and equicontinuity*, Ann. Polon. Math. **77** (2001), 169–187.

[9] J. Bès, *Three problems on hypercyclic operators*, Ph. D. Thesis. Kent State University, 1998.

[10] G.D. Birkhoff, *Démonstration d'un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris, **189** (1929), 473–475.

[11] I. Burtnyak, Y. Chopyuk, S. Vasylyshyn, T. Vasylyshyn, *Algebras of Weakly Symmetric Functions on Spaces of Lebesgue Measurable Functions*, Carpathian Math. Publ. **15** (2023), 411–419. doi:10.15330/cmp.15.2.411--419.

[12] K.C. Chan and J.H. Shapiro, *The cyclic behavior of translation operators on Hilbert spaces of entire functions,* Indiana Univ. Math. J. **40** (1991), 1421–1449.

[13] I. V. Chernega, *A semiring in the spectrum of the algebra of symmetric analytic functions in the space ℓ_1,* J. Math. Sci. **212** (2016), 38–45. doi:10.1007/s10958-015-2647-3. Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, **57** (2014), 35–40.

[14] I. Chernega, P. Galindo and A. Zagorodnyuk, *Some algebras of symmetric analytic functions and their spectra,* Proc. Edinburgh Math. Soc. **55** (2012), 125–142. doi:10.1017/S0013091509001655.

[15] I. Chernega, P. Galindo and A. Zagorodnyuk, *The convolution operation on the spectra of algebras of symmetric analytic functions* J. Math. Anal. Appl. **395 (2)** (2012), 569–577. doi:10.1016/j.jmaa.2012.04.087.

[16] I. Chernega, P. Galindo and A. Zagorodnyuk, *A multiplicative convolution on the spectra of algebras of symmetric analytic functions*, Rev Mat Complut **27** (2014), 575–585. doi:110.1007/s13163-013-0128-0.

[17] I. Chernega, O. Holubchak, Z. Novosad, A. Zagorodnyuk, *Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces*, Eur. J. Math. **6** (2020), 153–163. doi:10.1007/s40879-019-00390-z.

[18] I. Chernega, A. Zagorodnyuk, *Supersymmetric Polynomials and a Ring of Multisets of a Banach Algebra*, Axioms, **11** (2022), 511. doi:10.3390/axioms11100511.

[19] Y. Chopyuk, T. Vasylyshyn, A. Zagorodnyuk, *Rings of Multisets and Integer Multinumbers*, Mathematics, **10** (2022), 778. doi:10.3390/math10050778.

[20] S. Dineen, *Complex Analysis in Infinite Dimensional Spaces.* London: Springer, 1999. doi:10.1007/978-1-4471-0869-6.

[21] R.M. Gethner, J.H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. **100 (2)** (1987), 281–288.

[22] G. Godefroy, J.H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), 229-269. doi:10.1016/0022-1236(91)90078-J.

[23] M. Gonzaléz, R. Gonzalo and J. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. London Math. Soc. **59** (1999), 681–697. doi:10.1112/S0024610799007164.

[24] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. **36** (1999), 345-381. doi:10.1090/S0273-0979-99-00788-0.

[25] K. G. Grosse-Erdmann, A. Peris Manguillot, *Linear chaos*, Springer-Verlag, London, 2011. doi:10.1007/978-1-4471-2170-1.

[26] S. Grivaux, *Topologically transitive extensions of bounded operators*, Math Z. **249** (2005), 85–96. doi:10.1007/s00209-004-0690-8.

[27] C. Kitai, *Invariant closed sets for linear operators*, Ph.D. Thesis. University of Toronto, 1982.

[28] V. Kravtsiv, *Block-Supersymmetric Polynomials on Spaces of Absolutely Convergent Series. Symmetry*, **16 (2)** (2024), 179. doi:10.3390/sym16020179.

[29] I.G. Macdonald, *Symmetric Functions and Orthogonal Polynomials*, University Lecture Serie; AMS: Providence, RI, USA, vol. 12, 1997.

[30] G.R. MacLane, *Sequences of derivatives and normal families*, J. Analyse Math. **2** (1952), 72–78.

[31] Z.H. Mozhyrovska, A.V. Zagorodnyuk, *Hypercyclic behavior of translation operators on spaces of analytic functions on Hilbert spaces*, J. Funct. Spaces. (2015), ID 139289, 8 pages. doi:10.1155/2015/139289.

[32] Z. Novosad, *Topological Transitivity of Translation Operators in a Non-Separable Hilbert Space*, Carpathian Math. Publ. **15** (2023), 278–285. doi:10.15330/cmp.15.1.278-285.

[33] Z. Novosad, A. Zagorodnyuk, *Analytic Automorphisms and Transitivity of Analytic Mappings*, Mathematics, **8** (2020), 2179. doi:10.3390/math8122179.

[34] Z. Novosad, A. Zagorodnyuk, *Polynomial automorphisms and hypercyclic operators on spaces of analytic functions*, Archiv der Mathematik, **89** (2007), no. 2, 157–166. doi:10.1007/s00013-007-2043-4.

[35] Z. Novosad, A. Zagorodnyuk, *The Backward Shift and Two Infinite-Dimension Phenomena in Banach Spaces,* Symmetry, **15** (2023), 1855. doi:10.3390/sym15101855.

[36] T. Vasylyshyn, *Symmetric analytic functions on the Cartesian power of the complex Banach space of Lebesgue measurable essentially bounded functions on [0, 1],* J. Math. Anal. Appl. **509** (2022), 125977. doi:10.1016/j.jmaa.2021.125977.

[37] A. Zagorodnyuk, Z. Novosad, *Topological Transitivity of Shift Similar Operators on Nonseparable Hilbert Spaces,* J. Funct. Spaces, **2021** (2021), 9306342. doi:10.1155/2021/9306342.

Zoriana Novosad: https://zoriana.maths@gmail.com, zoryana@lute.lviv.ua

Department of High Mathematics and Quantitative Methods, Lviv University of Trade and Economics, 10, Tuhan-Baranovsky Str., Lviv 79005, Ukraine