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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

THE QUASI-ANTISYMMETRIC D_{-w}-LAGUERRE-HAHN ORTHOGONAL POLYNOMIALS OF CLASS ONE

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Abstract. In this work, we solve the system of Laguerre-Freud equations for the recurrence coefficients $\zeta_n, \theta_{n+1}, n \geq 0$, of the D_w-Laguerre-Hahn orthogonal sequences of polynomials of class one in the case when $\zeta_0 = -\alpha_0$, $\zeta_{n+1} = \alpha_n - \alpha_{n+1}$ and $\theta_{n+1} = -\alpha_n^2$ with $\alpha_n \neq 0$ $n \geq 0$, where D_w is the divided difference operator. There are essentially six canonical cases.

В роботі розв'язано систему рівнянь Лагерра-Фрейда для рекурентних коефіцієнтів ζ_n, $\theta_{n+1}, n \geq 0$, послідовностей ортогональних D_w-многочленів Лагерра-Хана першого роду у випадку, коли $\zeta_0 = -\alpha_0$, $\zeta_{n+1} = \alpha_n - \alpha_{n+1}$ і $\theta_{n+1} = -\alpha_n^2$ з $\alpha_n \neq 0$, $n \geq 0$, де D_w є оператором розділеної різниці. Встановлено шість канонічних випадків.

1. Introduction and preliminary results

There are many papers whose interest is $D_{-w}-$Laguerre-Hahn orthogonal polynomials, the polynomials that are related to the divided difference operator D_w [1, 5, 12, 14]. In [14], the authors have established a system satisfied by coefficients of the recurrence relation of D_{-w}- Laguerre-Hahn orthogonal sequences of class one. This system is not linear and it was solved only in the symmetric case (see [14]). So, the aim of this paper is to solve the system in a special nonsymmetrical case. Indeed, we exhaustively describe the family of D_{-w}-Laguerre-Hahn sequences $\{Z_n\}_{n\geq 0}$, of class $s = 1$, verifying the following three-term recurrence relation:

$$Z_{n+2}(x) = \Big(x - (\alpha_n - \alpha_{n+1})\Big) Z_{n+1}(x) + \alpha_n^2 Z_n(x) \ , \quad n \geq 0 \ ,$$
$$Z_1(x) = x + \alpha_0 \ , \qquad Z_0(x) = 1 \ ,$$

with $\alpha_n \neq 0\,,\ n \geq 0$. This family has been the subject of some works: for instance, Maroni [10, 11] characterized such sequences by a particular quadratic decomposition and by a perturbation of a symmetric form.

The structure of the manuscript is as follows. The first section is devoted to preliminary results and notations used in the sequel. In the second section, first we give some properties of the regular form associated with the sequence $\{Z_n\}_{n\geq 0}$. Especially, we focus our attention to the case where it is D_{-w}-Laguerre-Hahn of class one. Second, using these properties and the Laguerre-Freud equations for the recurrence coefficients $\zeta_n\,,\ \theta_{n+1}\,,\ n \geq 0$, of orthogonal polynomials with respect to a D_w-Laguerre-Hahn form of class one, we obtain all the sequences which we are looking for. Finally, we show that there are essentially six canonical cases.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called forms (linear functionals). We denote by $\langle \vartheta, f \rangle$ the action of $\vartheta \in \mathcal{P}'$ on $f \in \mathcal{P}$. For $n \geq 0$, $(\vartheta)_n = \langle \vartheta, x^n \rangle$ are the moments of ϑ. In particular, a form ϑ is called symmetric if all its moments of odd order are zero [3].

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For any linear form ϑ, any polynomial h, any $a \in \mathbb{C} - \{0\}$, $b, c \in \mathbb{C}$, let $D\vartheta = \vartheta'$, $h\vartheta$, $h_a\vartheta$, $t_b\vartheta$, δ_c, and $(x-c)^{-1}\vartheta$ be the linear forms defined by:

$$\langle \vartheta', f \rangle := -\langle \vartheta, f' \rangle, \quad \langle h\vartheta, f \rangle := \langle \vartheta, hf \rangle, \quad \langle h_a\vartheta, f \rangle := \langle \vartheta, h_a f \rangle = \langle \vartheta, f(ax) \rangle,$$
$$\langle t_b\vartheta, f \rangle := \langle \vartheta, t_{-b} f \rangle = \langle \vartheta, f(x+b) \rangle, \quad \langle \delta_c, f \rangle := f(c), \quad \left\langle (x-c)^{-1}\vartheta, f \right\rangle := \langle \vartheta, \Theta_c f \rangle,$$

with $\big(\Theta_c f\big)(x) := \dfrac{f(x)-f(c)}{x-c}$, $f \in \mathcal{P}$.

It is straightforward to prove that for a, $b \in \mathbb{C}$, $\vartheta \in \mathcal{P}'$, and $f \in \mathcal{P}$, we have [9]

$$(b-a)\Theta_a(\Theta_b f) = \Theta_b f - \Theta_a f, \tag{1.1}$$
$$f(x^{-1}\vartheta) = x^{-1}(f\vartheta) + \langle \vartheta, \Theta_0 f \rangle \delta_0, \tag{1.2}$$
$$f(t_b\vartheta) = t_b\big((t_{-b}f)\vartheta\big). \tag{1.3}$$

We also define the right-multiplication of a form ϑ by a polynomial h by

$$(\vartheta h)(x) := \left\langle \vartheta, \frac{xh(x)-\xi h(\xi)}{x-\xi} \right\rangle = \sum_{i=0}^{n} \Big(\sum_{j=i}^{n} a_j (v)_{j-i} \Big) x^i, \ h(x) = \sum_{i=0}^{n} a_i x^i.$$

Next, it is possible to define the product of two forms as

$$\langle \vartheta v, f \rangle := \langle \vartheta, vf \rangle, \ \ \vartheta, v \in \mathcal{P}', \ f \in \mathcal{P}.$$

For $\vartheta, v \in \mathcal{P}'$ and $f, g \in \mathcal{P}$, we have the following results [8]:

$$(fg)(\vartheta v) = (f\vartheta)(gv) + x(\vartheta\Theta_0 f)(gv) + x(v\Theta_0 g)(f\vartheta). \tag{1.4}$$

Let us recall that a form ϑ is said to be regular (quasi-definite) if there exists a sequence $\{Z_n\}_{n\geq 0}$ of polynomials, with $\deg Z_n = n$, $n \geq 0$, such that

$$\langle \vartheta, Z_n Z_m \rangle = r_n \delta_{n,m} \quad , \quad r_n \neq 0, \quad n \geq 0,$$

where $\delta_{n,m}$ denotes the Kronecker symbol.

We can always assume that each Z_n is monic, i.e., $Z_n(x) = x^n +$ lower degree terms. Then the sequence $\{Z_n\}_{n\geq 0}$ is said to be orthogonal with respect to ϑ (monic orthogonal polynomial sequence (MOPS) in short).

It is well-known that the sequence $\{Z_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (see, for instance, the monograph by Chihara [3])

$$\begin{aligned} &Z_{n+2}(x) = (x-\zeta_{n+1})Z_{n+1}(x) - \theta_{n+1}Z_n(x), \quad n \geq 0, \\ &Z_1(x) = x - \zeta_0, \qquad Z_0(x) = 1, \end{aligned} \tag{1.5}$$

with $\big(\zeta_n, \theta_{n+1}\big) \in \mathbb{C} \times (\mathbb{C} - \{0\})$, $n \geq 0$. By convention we set $\theta_0 = (\vartheta)_0$.

The form ϑ is said to be normalized if $(\vartheta)_0 = 1$. In this paper, we suppose that any form will be normalized.

Now, let us introduce the divided difference operator [6]

$$(D_w f)(x) := \frac{f(x+w)-f(x)}{w}, \qquad f \in \mathcal{P}, \ w \neq 0.$$

We have $D_w = \frac{1}{w}(t_{-w} - I_{\mathcal{P}})$, where $I_{\mathcal{P}}$ is the identity operator on $\mathcal{P}$. The transposed D_w^T of D_w is

$$D_w^T = \frac{1}{w}(t_w - I_{\mathcal{P}'}) = -D_{-w}. \tag{1.6}$$

Thus, we have

$$\langle D_w\vartheta, f \rangle = -\langle \vartheta, D_{-w} f \rangle, \qquad \vartheta \in \mathcal{P}', \ f \in \mathcal{P}.$$

For $f \in \mathcal{P}$ and $\vartheta \in \mathcal{P}'$ we have [2]

$$D_w(f\vartheta) = (t_{-w}f)(D_w\vartheta) + (D_w f)\vartheta. \tag{1.7}$$

Now, let us recall some features about the D_w-Laguerre-Hahn character [1].

Definition 1.1 ([1])**.** The regular form ϑ is called a D_w-Laguerre-Hahn form if it is regular and there exist three polynomials ϕ (monic), ψ and B, $\deg(\phi) = t \geq 0$, $\deg(\psi) = p \geq 1$, $\deg(B) = q \geq 0$, such that

$$D_w(\phi\vartheta) + \psi\vartheta + B\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0. \tag{1.8}$$

The corresponding MOPS $\{Z_n\}_{n\geq 0}$ is said to be D_{-w}-Laguerre-Hahn.

Remark 1.2. (1) If $B = 0$, the form ϑ is said to be D_w-semiclassical.

(2) The form ϑ is called a D_w-Laguerre-Hahn form if there exist four polynomials ϕ (monic), ψ_1, ψ_2, and B, $\deg(\phi) = t \geq 0$, $\deg(\psi_1 + \psi_2) = p \geq 1$, $\deg(B) = q \geq 0$, such that [5]

$$D_w(\phi\vartheta) + \psi_1\vartheta + t_{-w}(\psi_2\vartheta) + B\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0.$$

Proposition 1.3. *We define $d = \max(t, q)$. The D_w-Laguerre-Hahn form ϑ satisfying (1.8) is of class $s = \max(d-2, p-1)$ if and only if*

$$\prod_{c\in\mathcal{Z}_{(\phi)}} \{|(\Theta_c\phi+\psi)(c-w)| + |B(c-w)| + |\langle\vartheta, \Theta_{c-w}(\Theta_c\phi+\psi) + (t_{-w}\vartheta)(\Theta_0\Theta_{c-w}B)\rangle|\} \neq 0, \tag{1.9}$$

where $\mathcal{Z}_{(\phi)} := \{z \in \mathbb{C},\ \phi(z) = 0\}$.

The D_w-Laguerre-Hahn character is shift invariant. Indeed, the shifted form $\tilde{\vartheta} = (h_{a^{-1}} \circ t_{-b})\vartheta,\ a \in \mathbb{C} - \{0\},\ b \in \mathbb{C}$, satisfies

$$D_{wa^{-1}}(\tilde{\phi}\tilde{\vartheta}) + \tilde{\psi}\tilde{\vartheta} + \tilde{B}\big(x^{-1}(\tilde{\vartheta}t_{-wa^{-1}}\tilde{\vartheta})\big) = 0,$$

with

$$\tilde{\phi}(x) = a^{-t}\phi(ax+b), \qquad \tilde{B}(x) = a^{-t}B(ax+b), \qquad \tilde{\psi}(x) = a^{1-t}\psi(ax+b).$$

The sequence $\{\tilde{Z}_n(x) = a^{-n}Z_n(ax+b)\}_{n\geq 0}$ is orthogonal with respect to $\tilde{\vartheta}$ and fulfils (1.5) with

$$\tilde{\zeta}_n = \frac{\zeta_n - b}{a}, \qquad \tilde{\theta}_{n+1} = \frac{\theta_{n+1}}{a^2}, \quad n \geq 0.$$

The next result [14] characterizes elements of the functional equation satisfied by any symmetric D_w-Laguerre-Hahn form.

Proposition 1.4. *Let s be the class of ϑ. Then we have the following.*

(1) *If s is odd, then $2\phi - w\psi$ is odd, ψ is even and $B(x) = -B(-x-w)$.*
(2) *If s is even, then $2\phi - w\psi$ is even, ψ is odd and $B(x) = B(-x-w)$.*

In the sequel, we assume that $\{Z_n\}_{n\geq 0}$ is D_{-w}-Laguerre-Hahn MOPS of class one. This means that

$$\begin{aligned} \phi(x) = c_3x^3 + c_2x^2 + c_1x + c_0, &\qquad \psi(x) = d_2x^2 + d_1x + d_0,\\ B(x) = b_3x^3 + b_2x^2 + b_1x + b_0, &\qquad |c_3| + |b_3| + |d_2| \neq 0,\\ |c_3| + |c_2| + |c_1| + |c_0| \neq 0, &\qquad |d_2| + |d_1| \neq 0. \end{aligned}$$

Furthermore, the Laguerre-Freud equations giving recursively the recurrence coefficients for D_{-w}-Laguerre-Hahn orthogonal polynomial of class one are [14]

$$r\theta_1 = wb_2 - b_1 - d_0 - w^2 b_3 - (r + b_3)\zeta_0^2 - (2b_2 + d_1 - 3wb_3)\zeta_0, \tag{1.10}$$

$$\begin{aligned}(r - 2c_3)(\theta_2 + \theta_1) &= \Theta_{\zeta_1}(2\phi - w\psi)(\zeta_0) + w(2b_2 - 3wb_3) + 3wb_3(\zeta_0 + \zeta_1)\\ &\quad - b_3\theta_1 - \psi(\zeta_1) - 2(\Theta_{\zeta_1}B)(\zeta_0),\end{aligned} \tag{1.11}$$

$$\begin{aligned}(r - 2nc_3)(\theta_n + \theta_{n+1}) - 4c_3\sum_{\nu=0}^{n-2}\theta_{\nu+1} &= \sum_{\nu=0}^{n-1}\big(\Theta_{\zeta_n}(2\phi - w\psi)\big)(\zeta_\nu)\\ &\quad + 3wb_3(\zeta_0 + \zeta_n) + 2b_2 w + \frac{1}{3}n(n-1)(n-2)w^2 c_3 - 2b_3\theta_1 - \psi(\zeta_n)\\ &\quad - 2(\Theta_{\zeta_n}B)(\zeta_0) - (n+2)w^2 b_3 - \frac{1}{2}n(n-1)w^2 r, \quad n \geq 2,\end{aligned} \tag{1.12}$$

$$\begin{aligned}&\{2(r - c_3)\zeta_1 - 2(2c_3 - r - 2b_3)\zeta_0 + 2d_1 + wd_2 - 2c_2 + 4b_2 - 6wb_3\}\theta_1\\ &\quad = (2\phi - w\psi)(\zeta_0) - B(\zeta_0) - B(\zeta_0 - w),\end{aligned} \tag{1.13}$$

$$\begin{aligned}&\Xi_n\theta_{n+1} - 2(2c_2 - wd_2)\sum_{\nu=0}^{n-1}\theta_{\nu+1} - 6c_3\sum_{\nu=0}^{n-1}\theta_{\nu+1}(\zeta_\nu + \zeta_{\nu+1})\\ &\quad = \sum_{\nu=0}^{n}(2\phi - w\psi)(\zeta_\nu) - B(\zeta_0) - B(\zeta_0 - w) - [2b_3(2\zeta_0 + \zeta_1)\\ &\qquad + 2b_2 - 3wb_3]\theta_1 + w^2\{n[(n-1)c_3 - r] - b_3\}\sum_{\nu=0}^{n}\zeta_\nu\\ &\qquad - \frac{1}{2}n(n+1)w^2(d_1 + 2b_2 - 3wb_3) + \frac{1}{6}(n^2-1)nw^2(2c_2 - wd_2)\\ &\qquad - (n^2-1)w^2\zeta_0 b_3, \quad n \geq 1,\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}r &= d_2 + 2b_3,\\ \Xi_n &= 2[r - (2n+1)c_3]\zeta_{n+1} + 2(r - 2nc_3)\zeta_n\\ &\quad - 4c_3\sum_{\nu=0}^{n}\zeta_\nu - (2n+1)(2c_2 - wd_2) + 2d_1 + 4b_2 + 2b_3(2\zeta_0 - 3w), \quad n \geq 1.\end{aligned} \tag{1.15}$$

2. D_w-Laguerre-Hahn forms of class one: special case

From now on, let ϑ be a D_w-Laguerre-Hahn form of class s satisfing (1.8) and its associated MOPS $\{Z_n\}_{n\geq 0}$ satisfy

$$\begin{aligned}&Z_{n+2}(x) = \Big(x - (\alpha_n - \alpha_{n+1})\Big)Z_{n+1}(x) + \alpha_n^2 Z_n(x), \quad n \geq 0,\\ &Z_1(x) = x + \alpha_0, \qquad Z_0(x) = 1,\end{aligned} \tag{2.16}$$

where $\alpha_n \neq 0$, $n \geq 0$.

The next lemma will play an important role in the sequel.

Lemma 2.1. [11] *The following statements are equivalent:*

(1) *the MOPS $\{Z_n\}_{n\geq 0}$ satisfies (2.16);*
(2) *$(\vartheta)_{2n} = 0$, $n \geq 1$, and the form $x\vartheta$ is regular;*
(3) *there exists a regular symmetric form u such that $\vartheta = (\vartheta)_1 x^{-1}u + \delta_0$.*

Remark 2.2. The form ϑ is said to be quasi-antisymmetric (i.e $(\vartheta)_{2n} = 0$, $n \geq 1$). For more information about these forms see [10, 11].

2.1. **Class and functional equation of the form** $u = (\vartheta)_1^{-1} x\vartheta$**.** In the sequel, our aim is to characterize the structure of the polynomial elements of the functional equation (1.8) satisfied by the form ϑ, for which its corresponding MOPS $\{Z_n\}_{n\geq 0}$ fulfills (2.16). This is possible through the study of the D_w-Laguerre-Hahn character of the symmetric form u defined by

$$\lambda u = x\vartheta, \qquad \lambda = (\vartheta)_1 \neq 0. \tag{2.17}$$

Consequently, according to [4], the form u is regular if and only if $Z_n(0) \neq 0,\ n \geq 0$. Now, multiplying equation (1.8) by $x(x+w)$ and taking into account (1.2)–(1.4) and (1.6)–(1.7), we get after some calculations that

$$D_w(Uu) + Vu + W\big(x^{-1}(ut_{-w}u)\big) = 0,$$

where

$$\begin{aligned} U(x) &= k\{(x-w)\phi(x) + w(t_w B)(x)\},\\ V(x) &= k\{(x+w)\psi(x) - 2\phi(x) + B(x) + (t_w B)(x)\},\\ W(x) &= \lambda k B(x), \end{aligned} \tag{2.18}$$

with k chosen so that U is a monic polynomial.

Theorem 2.3. *The form u is D_w-Laguerre-Hahn of class $\tilde{s}$ satisfying*

$$D_w(\tilde{U}u) + \tilde{V}u + \tilde{W}\big(x^{-1}(ut_{-w}u)\big) = 0. \tag{2.19}$$

Moreover,

(1) *If $(\phi(0), B(-w)) \neq (0,0)$ and $\big((w\psi - \phi)(0), B(0)\big) \neq (0,0)$, then*

$$\tilde{U}(x) = U(x), \quad \tilde{V}(x) = V(x), \quad \tilde{W}(x) = W(x),$$

and $\tilde{s} = s+1$.

(2) *If $(\phi(0), B(-w)) = (0,0)$ and $\big(\psi(0), B(0)\big) \neq (0,0)$, then*

$$\begin{aligned} \tilde{U}(x) &= k\{(x-w)(\Theta_0\phi)(x) + w(\Theta_0 t_w B)(x)\},\\ \tilde{V}(x) &= k\{\psi(x) - (\Theta_0\phi)(x) + (\Theta_{-w}B)(x) + (\Theta_0 t_w B)(x)\},\\ \tilde{W}(x) &= \lambda k(\Theta_{-w}B)(x), \end{aligned}$$

and $\tilde{s} = s$.

(3) *If $(\phi(0), B(-w)) \neq (0,0)$ and $\big((w\psi - \phi)(0), B(0)\big) = (0,0)$, then*

$$\begin{aligned} \tilde{U}(x) &= k\{\phi(x) + w(\Theta_w t_w B)(x)\},\\ \tilde{V}(x) &= k\{\psi(x) + \big(\Theta_0(w\psi - \phi)\big)(x) + (\Theta_0 B)(x) + (\Theta_w t_w B)(x)\},\\ \tilde{W}(x) &= \lambda k(\Theta_0 B)(x), \end{aligned}$$

and $\tilde{s} = s$.

(4) *If $(\phi(0), B(-w)) = (0,0)$ and $\big(\psi(0), B(0)\big) = (0,0)$, then*

$$\begin{aligned} \tilde{U}(x) &= k\{(\Theta_0\phi)(x) + w(\Theta_0\Theta_w t_w B)(x)\},\\ \tilde{V}(x) &= k\{(\Theta_0\psi)(x) + (\Theta_0\Theta_{-w}B)(x) + (\Theta_0\Theta_w t_w B)(x)\},\\ \tilde{W}(x) &= \lambda k(\Theta_0\Theta_{-w}B)(x), \end{aligned}$$

and $\tilde{s} = s-1$.

For the proof, we need the following lemma.

Lemma 2.4. (1) *For all roots c of ϕ, we have*

$$\langle u, \Theta_{c-w}(V+\Theta_c U) + (t_{-w}u)\Theta_0\Theta_{c-w}W\rangle = \frac{k}{\lambda}\{c(c-w)\langle \vartheta, \Theta_{c-w}(\psi+\Theta_c\phi) + t_{-w}\vartheta\Theta_0\Theta_{c-w}B\rangle - c(\Theta_c\phi+\psi)(c-w) - B(c-w)\}, \tag{2.20}$$

$$(\Theta_c U + V)(c-w) = k\{c(\Theta_c\phi+\psi)(c-w) + 2B(c-w)\}, \tag{2.21}$$

$$W(c-w) = \lambda k B(c-w). \tag{2.22}$$

(2) *The class of the form u depends only on the zeros $x=0$ and $x=w$ of U.*

Proof. (1) Let c be a root of ϕ. We have [13]

$$\langle u, \Theta_{c-w}\Theta_c\big((\xi-w)\phi\big)\rangle = \frac{1}{\lambda}\{(c-2w)(c-w)\langle\vartheta,\Theta_{c-w}\Theta_c\phi\rangle + 2(c-w)\langle\vartheta,\Theta_c\phi\rangle + \langle\vartheta,\phi\rangle - (c-2w)(\Theta_c\phi)(c-w)\}, \tag{2.23}$$

$$\langle u, \Theta_{c-w}\big((\xi+w)\psi - 2\phi\big)\rangle = \frac{1}{\lambda}\{c(c-w)\langle\vartheta,\Theta_{c-w}\psi\rangle + \langle\vartheta,(x+c)\psi\rangle - 2c(c-w)\langle\vartheta,\Theta_{c-w}\phi\rangle - 2\langle\vartheta,\phi\rangle - c\psi(c-w) + 2\phi(c-w)\}. \tag{2.24}$$

Taking into account (1.1), we obtain

$$w\big(\Theta_{c-w}\Theta_c(t_w B)\big)(x) = \big(\Theta_c(t_w B)\big)(x) - \big(\Theta_{c-w}(t_w B)\big)(x). \tag{2.25}$$

Then, from (2.25) we get

$$\langle u, w\big(\Theta_{c-w}\Theta_c(t_w B)\big)(x)\rangle = \frac{1}{\lambda}\{c\langle\vartheta,\Theta_c(t_w B)\rangle - (c-w)\langle\vartheta,\Theta_{c-w}(t_w B)\rangle + B(c-2w) - B(c-w)\}, \tag{2.26}$$

since $x = (x-c)+c = (x-c+w)+c-w$.

Proceeding as in (2.26), we can easily prove that

$$\langle u, \Theta_{c-w}\big(B + t_w B\big)\rangle = \frac{1}{\lambda}\{\langle\vartheta, B+t_w B\rangle + (c-w)\langle\vartheta,\Theta_{c-w}B + \Theta_{c-w}(t_w B)\rangle - B(c-w) - B(c-2w)\}. \tag{2.27}$$

Now, taking into account to (1.3), (1.4) and (2.17), we obtain

$$u(t_{-w}u) = \frac{1}{\lambda^2}\{x(x+w)[\vartheta(t_{-w}\vartheta)] - x(x+w)(t_{-w}\vartheta) - x^2\vartheta\}.$$

Hence, employing the identical procedure used to derive (2.26), we obtain through a few simple calculations

$$\langle ut_{-w}u, \Theta_0\Theta_{c-w}W\rangle = \frac{k}{\lambda}\{c(c-w)\langle\vartheta t_{-w}\vartheta, \Theta_0\Theta_{c-w}B\rangle + \langle B\big(x^{-1}(\vartheta t_{-w}\vartheta)\big), x+c\rangle - (c-w)\langle\vartheta,\Theta_{c-w}B\rangle - c\langle\vartheta,\Theta_c(t_w B)\rangle - \langle\vartheta, B+t_w B\rangle + B(c-w)\}. \tag{2.28}$$

Using (2.18), (2.23)–(2.24) and (2.26)-(2.28), we can deduce that

$$\begin{aligned}\langle u, \Theta_{c-w}(\Theta_c U + V) + t_{-w}u\Theta_0\Theta_{c-w}W\rangle &= \frac{k}{\lambda}\{c(c-w)\langle\vartheta,\Theta_{c-w}(\Theta_c\phi+\psi) + t_{-w}\vartheta\Theta_0\Theta_{c-w}B\rangle \\ &\quad - c(\psi+\Theta_c\phi)(c-w) - B(c-w)\} \\ &\quad + \frac{k}{\lambda}\langle D_w(\phi\vartheta) + \psi\vartheta + B(x^{-1}\vartheta t_{-w}\vartheta), x+c\rangle.\end{aligned}$$

This yields (2.20), since $\langle D_w(\phi\vartheta) + \psi\vartheta + B(x^{-1}\vartheta t_{-w}\vartheta), x+c\rangle = 0$, by (1.8). Next, it is easy to get (2.21) and (2.22) from (2.18).

(2) Let c be a root of E such that $c \notin \{0, w\}$. Using (2.18) we obtain

$$(c-w)\phi(c) + wB(c-w) = 0. \tag{2.29}$$

We suppose $|(V+\Theta_c U)(c-w)| + |W(c-w)| = 0$. In this case, we have

$$\begin{cases} B(c-w) = 0, \\ c\{(\Theta_c\phi + \psi)(c-w) + 2B(c-w) - \phi(c)\} = 0. \end{cases} \tag{2.30}$$

Therefore, from (2.29) and (2.30), we have

$$\phi(c) = B(c-w) = (\psi + \Theta_c\phi)(c-w) = 0.$$

Hence, from (2.20) we obtain

$$\langle u, \Theta_{c-w}(V+\Theta_c U) + t_{-w}u\Theta_0\Theta_{c-w}W\rangle = \frac{kc(c-w)}{\lambda}\langle \vartheta, \Theta_{c-w}(\psi+\Theta_c\phi) + t_{-w}\vartheta\Theta_0\Theta_{c-w}B\rangle \neq 0,$$

since ϑ is D_w-Laguerre-Hahn and so satisfies (1.9). □

Proof of Theorem 2.3. We can write $(\Theta_0 U + V)(-w) = k\{2B(-w) - \phi(0)\}$, $W(-w) = \lambda k B(-w)$, $(\Theta_w U + V)(0) = k\{(w\psi - \phi)(0) + 2B(0)\}$ and $W(0) = \lambda k B(0)$.

(1) If $\big(\phi(0), B(-w)\big) \neq (0,0)$ and $\big((w\psi-\phi)(0), B(0)\big) \neq (0,0)$, then $|(\Theta_0 U+V)(-w)| + |W(-w)| \neq 0$ and $|(\theta_w U + V)(0)| + |W(0)| \neq 0$. Thus, (2.19) cannot be simplified and so the form u is of class

$$\tilde{s} = \max\big(\deg(U)-2, \deg(V)-1, \deg(W)-2\big) = \max\big(\deg(\phi)-1, \deg(\psi), \deg(B)-1\big).$$

Hence, $\tilde{s} = s+1$.

(2) If $\big(\phi(0), B(-w)\big) = (0,0)$ and $\big(\psi(0), B(0)\big) \neq (0,0)$, then

$$|(\Theta_0 U + V)(-w)| + |W(-w)| = 0,$$
$$\langle u, \Theta_{-w}(\Theta_0 U + V) + t_{-w}u\Theta_0\Theta_{-w}W\rangle = 0,$$

according to (2.20), (2.21), and (2.22). So, (2.19) can be simplified by the polynomial $x+w$ and becomes

$$D_w(\tilde{U}u) + \tilde{V}u + \tilde{W}\big(x^{-1}ut_{-w}u\big) = 0, \tag{2.31}$$

where

$$\tilde{U}(x) = k\{(x-w)(\Theta_0\phi)(x) + w(\Theta_0 t_w B)(x)\},$$
$$\tilde{V}(x) = k\{\psi(x) - (\Theta_0\phi)(x) + (\Theta_{-w}B)(x) + (\Theta_0 t_w B)(x)\},$$
$$\tilde{W}(x) = \lambda k(\Theta_{-w}B)(x).$$

If 0 is a root of $\tilde{U}$, then a simple calculation gives

$$\phi'(0) - B'(-w) = 0, \tag{2.32}$$

$$\begin{aligned} &(\tilde{V}+\Theta_0\tilde{U})(-w) = k\{\big(\psi+\Theta_0\phi\big)(-w) + 2B'(-w) - \phi'(0)\}, \\ &\tilde{W}(-w) = \lambda k B'(-w), \\ &\langle u, \Theta_{-w}(\tilde{V}+\Theta_0\tilde{U}) + t_{-w}u\theta_0\Theta_{-w}\tilde{W}\rangle = \tfrac{k}{\lambda}\big(\phi'(0) - B'(-w)\big) \\ &\qquad -\tfrac{k}{\lambda}(\psi+\Theta_0\phi)(-w) - \tfrac{kw}{\lambda}\langle\vartheta, \Theta_{-w}(\psi+\Theta_0\phi) + t_{-w}\vartheta\Theta_0\Theta_{-w}B\rangle. \end{aligned} \tag{2.33}$$

Assuming that $|(\tilde{V}+\Theta_0\tilde{U})(-w)| + |\tilde{W}(-w)| = 0$, from (2.32) and (2.33) we have that $\langle u, \Theta_{-w}(\tilde{V}+\Theta_0\tilde{U})\rangle + t_{-w}u\Theta_0\Theta_{-w}\tilde{W}\rangle \neq 0$, since ϑ is D_w-Laguerre-Hahn and satisfies (1.9). Hence, (2.31) cannot be simplified by $x+w$ and so, $\tilde{s} = s$.

(3) If $\big(w\psi(0)-\phi(0), B(0)\big) = (0,0)$ and $\big(\phi(0), B(-w)\big) \neq (0,0)$, then $|(\Theta_w U+V)(0)| + |W(0)| = 0$ and $\langle u, \Theta_0(V+\Theta_w U) + t_{-w}u\Theta_0^2 W\rangle = 0$. So, (2.19) is simplified by the polynomial x and it becomes

$$D_w(\hat{U}u) + \hat{V}u + \hat{W}\big(x^{-1}ut_{-w}u\big) = 0, \tag{2.34}$$

where

$$\begin{aligned}
&\hat{U}(x) = k\{(\phi(x) + w(\Theta_w t_w B)(x)\},\\
&\hat{V}(x) = k\{\psi + \big(\Theta_0(w\psi - \phi)\big)(x) + (\Theta_0 B)(x) + (\Theta_w t_w B)(x)\},\\
&\hat{W}(x) = \lambda k(\Theta_0 B)(x).
\end{aligned}$$

If w is a root of $\hat{E}$, then

$$\phi(w) + wB^{'}(0) = 0, \tag{2.35}$$

$$\begin{aligned}
&\hat{W}(0) = \lambda k B^{'}(0),\\
&(\hat{V} + \Theta_w\hat{U})(0) = k\{(w\psi^{'} - \phi^{'})(0) + 2B^{'}(0) + (\Theta_w\phi + \psi)(0)\},\\
&\langle u, \Theta_0(\hat{V} + \Theta_w\hat{U}) + t_{-w}u\Theta_0^2\hat{W}\rangle = \tfrac{wk}{\lambda}\langle\vartheta, \Theta_0\big(\Theta_w\phi + \psi\big) + (t_{-w}\vartheta)\Theta_0^2 B\rangle\\
&\qquad\qquad -\tfrac{k}{\lambda}\big(w\psi^{'} - \phi^{'}\big)(0) - \tfrac{k}{\lambda w}\big(\phi(w) + wB^{'}(0)\big).
\end{aligned} \tag{2.36}$$

Assuming that $|(\Theta_w\hat{U} + \hat{V})(0)| + |\hat{W}(0)| = 0$, from (2.35) and (2.36) we can deduce that $\langle u, \Theta_0(\hat{V} + \Theta_w\hat{U}) + t_{-w}u\Theta_0^2\hat{W}\rangle \neq 0$, since ϑ is a D_w-Laguerre-Hahn and satisfies (1.9). Hence, (2.34) cannot be simplified by x and so, $\tilde{s} = s$.

(4) If $(\phi(0), \psi(0)) = (0,0)$ and $\big(B(0), B(-w)\big) = (0,0)$, then (2.34) is simplified by $x + w$ and becomes

$$D_w(\breve{U}u) + \breve{V}u + \breve{W}\big(x^{-1}ut_{-w}u\big) = 0,$$

where

$$\begin{aligned}
&\breve{U}(x) = k\{(\Theta_0\phi)(x) + w(\Theta_0\Theta_w t_w B)(x)\},\\
&\breve{V}(x) = k\{(\Theta_0\psi)(x) + (\Theta_0\Theta_{-w}B)(x) + (\Theta_0\Theta_w t_w B)(x)\},\\
&\breve{W}(x) = \lambda k(\Theta_{-w}\Theta_0 B)(x),
\end{aligned}$$

and so, $\tilde{s} = s - 1$.

□

2.2. **Structure of the polynomials ϕ, ψ and B.** Let us spilt up each polynomial form ϕ, ψ, $\Theta_0\phi$, $\Theta_0\psi$, B, $t_w B$, $\Theta_0 B$, $\Theta_{-w}B$, $\Theta_0 t_w B$, $\Theta_w t_w B$, $\Theta_0\Theta_w B$ and $\Theta_0\Theta_{-w}t_w B$ according to its odd and even parts that is,

$$\begin{aligned}
&\phi(x) = \phi^e(x^2) + x\phi^o(x^2),\ \psi(x) = \psi^e(x^2) + x\psi^o(x^2),\\
&(\Theta_0\phi)(x) = \phi_1^e(x^2) + x\phi_1^o(x^2),\ (\Theta_0\psi)(x) = \psi_1^e(x^2) + x\psi_1^o(x^2),\\
&B(x) = B^e(x^2) + xB^o(x^2),\ (t_w B)(x) = (t_w B)^e(x^2) + x(t_w B)^o(x^2),\\
&(\Theta_0 B)(x) = B_1^e(x^2) + xB_1^o(x^2),\ (\Theta_{-w}B)(x) = B_2^e(x^2) + xB_2^o(x^2),\\
&(\Theta_0 t_w B)(x) = (t_w B)_1^e(x^2) + x(t_w B)_1^o(x^2),\\
&(\Theta_w t_w B)(x) = (t_w B)_2^e(x^2) + x(t_w B)_2^o(x^2),\\
&(\Theta_0\Theta_{-w}B(x) = B_3^e(x^2) + xB_3^o(x^2),\\
&(\Theta_0\Theta_w t_w B)(x) = (t_w B)_3^e(x^2) + x(t_w B)_3^o(x^2).
\end{aligned} \tag{2.37}$$

Proposition 2.5. *Let ϑ be a D_w-Laguerre-Hahn form of class $s = 1$ satisfying (1.8). The following statements hold:*

(1) *If* $(\phi(0), B(-w)) \neq (0,0)$ *and* $((w\psi - \phi)(0), B(0)) \neq (0,0)$, *then*

$$\begin{aligned} 2\phi^e(x) &= w\psi^e(x) + w^2\psi^o(x) + 2wB^o(x), \\ (x-w^2)\psi^o(x) &= 2wB^o(x) - 2B^e(x), \\ B^e(x) &= (t_wB)^e(x), \quad B^o(x) = -(t_wB)^o(x). \end{aligned} \tag{2.38}$$

(2) *If* $(\phi(0), B(-w)) = (0,0)$ *and* $(\psi(0), B(0)) \neq (0,0)$, *then*

$$\begin{aligned} &(x-w^2)[\phi^e(x) - x\psi^o(x)] = 2x(B^e - wB^o)(x), \\ &(x-w^2)[w\phi^o(x) - 2x\psi^o(x) + w\psi^e(x)] = 2(2x+w^2)B^e(x) - 6wxB^o(x), \\ &B^e(x) = (t_wB)^e(x) - w(t_wB)^o(x), \\ &xB^o(x) = w(t_wB)^e(x) - x(t_wB)^o(x). \end{aligned} \tag{2.39}$$

(3) *If* $(\phi(0), B(-w)) \neq (0,0)$ *and* $((w\psi - \phi)(0), B(0)) = (0,0)$, *then*

$$\begin{aligned} &x\psi^o(x) + w\psi^e(x) + 2B^e(x) = \phi^e(x), \\ &(w^2 - 2x)\psi^o(x) - w\psi^e(x) - 4B^e(x) + 2wB^o(x) = w\phi^o(x), \\ &w(t_wB)^e(x) + x(t_wB)^o(x) = -(x-w^2)B^o(x), \\ &x[(t_wB)^e(x) + w(t_wB)^o(x)] = (x-w^2)B^e(x). \end{aligned} \tag{2.40}$$

(4) *If* $(\phi(0), B(-w)) = (0,0)$ *and* $(\psi(0), B(0)) = (0,0)$, *then*

$$\begin{aligned} &(x-w^2)\psi^o(x) + 2(B^e(x) - wB^o(x)) = 0, \\ &(x-w^2)[2\phi^e - w\psi^e(x)] = 2w(xB^o(x) - wB^e(x)), \\ &B^e(x) = (t_wB)^e(x), \quad B^0(x) = -(t_wB)^o(x). \end{aligned} \tag{2.41}$$

For the proof, we need the following lemma.

Lemma 2.6. *Let* $f \in \mathcal{P}$ *and* a, $b \in \mathbb{C}$ *such that* $a \neq b$ *and* $f(a) = f(b) = 0$. *We have*

$$\begin{aligned} (x-a^2)(\Theta_a f)^e(x) &= af^e(x) + xf^o(x), \\ (x-a^2)(\Theta_a f)^o(x) &= f^e(x) + af^o(x), \end{aligned} \tag{2.42}$$

$$\begin{aligned} (x-a^2)(x-b^2)(\Theta_a\Theta_b f)^e(x) &= (ab+x)f^e(x) + (a+b)xf^o(x), \\ (x-a^2)(x-b^2)(\Theta_a\Theta_b f)^o(x) &= (a+b)f^e(x) + (x+ab)f^o(x). \end{aligned} \tag{2.43}$$

Proof. Taking into account the definition of the operator Θ_a and that $f(a) = 0$, we get $(x-a)(\Theta_a f)(x) = f(x)$. Which gives

$$f^o(x) = (\Theta_a f)^e(x) - a(\Theta_a f)^o(x), \quad f^e(x) = x(\Theta_a f)^o(x) - a(\Theta_a f)^e(x).$$

Hence, the desired result (2.42). Finally, from (2.42) we can deduce (2.43) since $(\Theta_a f)(b) = 0$. □

Proof. (of Proposition 2.5) Writing

$$\tilde{U}(x) = \tilde{U}^e(x^2) + x\tilde{U}^o(x^2), \qquad \tilde{V}(x) = \tilde{V}^e(x^2) + x\tilde{V}^o(x^2),$$
$$\tilde{W}(x) = \tilde{W}^e(x^2) + x\tilde{W}^o(x^2), \qquad (t_w\tilde{W})(x) = (t_w\tilde{W})^e(x^2) + x(t_w\tilde{W})^o(x^2).$$

We have to examine the following situations:

(1) $(\phi(0), B(-w)) \neq (0,0)$ and $((w\psi - \phi)(0), B(0)) \neq (0,0)$. According to (2.37) and from the expression of polynomials $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ given in Theorem 2.3, we get

$$\begin{aligned} &\tilde{U}^e(x) = k\{-w\phi^e(x) + x\phi^o(x) + w(t_wB)^e(x)\}, \\ &\tilde{U}^o(x) = k\{-w\phi^o(x) + \phi^e(x) + w(t_wB)^o(x)\}, \\ &\tilde{V}^e(x) = k\{w\psi^e(x) + x\psi^o(x) - 2\phi^e(x) + B^e(x) + (t_wB)^e(x)\}, \\ &\tilde{V}^o(x) = k\{w\psi^o(x) + \psi^e(x) - 2\phi^o(x) + B^o(x) + (t_wB)^o(x)\}, \\ &\tilde{W}(x)(x) - (t_w\tilde{W})(-x) = \lambda k\{B^e(x^2) + xB^o(x^2) - (t_wB)^e(x^2) + x(t_wB)^o(x^2)\}. \end{aligned}$$

Then, $(2\tilde{U}^o - w\tilde{V}^o)(x) = \tilde{V}^e(x) = (B^e - (t_wB)^e)(x) = (B^o + (t_wB)^o)(x) = 0$, from Proposition 1.4, since $\tilde{s} = 2$. This gives (2.38).

(2) $(\phi(0), B(-w)) = (0,0)$ and $(\psi(0), B(0)) \neq (0,0)$. Similarly as above,

$$\tilde{U}^e(x) = k\{-w\phi_1^e(x) + x\phi_1^o(x) + w(t_wB)_1^e(x)\},$$
$$\tilde{U}^o(x) = k\{\phi_1^e(x) - w\phi_1^o(x) + w(t_wB)_1^o(x)\},$$
$$\tilde{V}^e(x) = k\{\psi^e(x) - \phi_1^e(x) + B_2^e(x) + (t_wB)_1^e(x)\},$$
$$\tilde{V}^o(x) = k\{\psi^o(x) - \phi_1^o(x) + B_2^o(x) + (t_wB)_1^o(x)\},$$
$$\tilde{W}(x) + (t_w\tilde{W})(-x) = \lambda k\{B_2^e(x^2) + xB_2^o(x^2) + (t_wB)_1^e(x^2) - x(t_wB)_1^o(x^2)\}.$$

If $\tilde{s} = 1$, then $(2\tilde{U}^e - w\tilde{V}^e)(x) = \tilde{V}^o(x) = (B_2^e + (t_wB)_1^e)(x) = (B_2^o - (t_wB)_1^o)(x) = 0$. This leads to result (2.39) from (2.42).

(3) $(\phi(0), B(-w)) \neq (0,0)$ and $((w\psi - \phi)(0), B(0)) = (0,0)$. In this case, we have

$$\tilde{U}^e(x) = k\{\phi^e(x) + w(t_wB)_2^e(x)\}, \quad \tilde{U}^o(x) = k\{\phi^o(x) + w(t_wB)_2^o(x)\},$$
$$\tilde{V}^e(x) = k\{\psi^e(x) + w\psi_1^e(x) - \phi_1^e(x) + B_1^e(x) + (t_wB)_2^e(x)\},$$
$$\tilde{V}^o(x) = k\{\psi^o(x) + w\psi_1^o(x) - \phi_1^o(x) + B_1^o(x) + (t_wB)_2^o(x)\},$$
$$\tilde{W}(x) + (t_w\tilde{W})(-x) = \lambda k\{B_1^e(x^2) + xB_1^o(x^2) + (t_wB)_2^e(x^2) - x(t_wB)_2^o(x^2)\}.$$

Since u is of odd class, $(2\tilde{U}^e - w\tilde{V}^e)(x) = \tilde{V}^o(x) = (B_1^e + (t_wB)_2^e)(x) = (B_1^o - (t_wB)_2^o)(x) = 0$. This gives the desired result (2.40) from (2.42).

(4) $(\phi(0), B(-w)) = (0,0)$ and $(\psi(0), B(0)) = (0,0)$. In this case, we obtain

$$\tilde{U}^e(x) = k\{\phi_1^e(x) + w(t_wB)_3^e(x)\}, \quad \tilde{U}^o(x) = k\{\phi_1^o(x) + w(t_wB)_3^o(x)\},$$
$$\tilde{V}^e(x) = k\{\psi_1^e(x) + B_3^e(x) + (t_wB)_3^e(x)\},$$
$$\tilde{V}^o(x) = k\{\psi_1^o(x) + B_3^o(x) + (t_wB)_3^o(x)\},$$
$$\tilde{W}(x) - (t_w\tilde{W})(-x) = \lambda k\{B_3^e(x^2) + xB_3^o(x^2) + x(t_wB)_3^o(x^2) - (t_wB)_3^e(x^2)\}.$$

Since u is of even class, $(2\tilde{U}^o - w\tilde{V}^o)(x) = \tilde{V}^e(x) = (B_3^e - (t_wB)_3^e)(x) = (B_3^o + (t_wB)_3^o)(x) = 0$. This gives (2.41) from (2.42) and (2.43).

□

Theorem 2.7. *If ϑ is a D_w-Laguerre-Hahn form of class one satisfying (1.8) and its corresponding MOPS fulfilling (2.16), then*

$$\begin{aligned} &\phi(x) = c_3x^3 + c_2x^2 + c_1x, \quad \psi(x) = d_2x^2 + d_1x, \quad B(x) = b_2x^2 + b_1x, \\ &(c_3, c_2) \neq (0,0), \quad (c_3, d_2, b_3) \neq (0,0,0), \quad (b_2, c_2) \neq (0,0), \end{aligned} \tag{2.44}$$

with

$$d_1 = -2b_2, \quad b_1 = wb_2, \quad 2c_2 = wd_2, \tag{2.45}$$

or

$$\begin{aligned} &\phi(x) = c_3x^3 + c_2x^2 + c_1x + c_0, \quad \psi(x) = d_2x^2 + d_1x + d_0, \quad B(x) = b_2x^2 + b_1x, \\ &(c_3, c_2) \neq (0,0), \quad (c_3, d_2, b_3) \neq (0,0,0), \quad (c_3, d_1 + 2b_2) \neq (0,0), \\ &(c_0, b_2) \neq (0,0), \quad (b_2, c_2) \neq (0,0), \end{aligned} \tag{2.46}$$

with

$$\begin{aligned} &c_0 = wd_0, \quad c_1 = wd_1 + wb_2 - d_0, \quad c_2 = wd_2 + d_1 + 2b_2, \\ &wc_3 = -wd_2 - 2(d_1 + 2b_2), \quad 2b_1 = wb_2. \end{aligned} \tag{2.47}$$

For the proof, we need the following lemma.

Lemma 2.8. [14] *The functional equation (1.8) is equivalent to*

$$D_{-w}\big((\phi - w\Psi)\vartheta\big) + \psi\vartheta + (t_wB)\big(x^{-1}(\vartheta t_w\vartheta)\big) = 0.$$

Proof. (of Theorem 2.7) We have to consider two cases:

- **A.** $\deg(\phi) \leq 1$. We have $\phi^e(x) = c_0$, $\phi^o(x) = c_1$, $\psi^e(x) = d_2 x + d_0$, $\psi^o(x) = d_1$, $B^e(x) = b_2 x + b_0$, $B^o(x) = b_3 x + b_1$, $(t_w B)^e(x) = (b_2 - 3wb_3)x + b_0 + w^2 b_2 - w^3 b_3 - wb_1$ and $(t_w B)^o(x) = b_3 x + b_1 + 3w^2 b_3 - 2wb_2$ with $|b_3| + |d_2| \neq 0$.. By virtue of Proposition 2.5, such an assumption leads to a contradiction.
- **B.** $2 \leq \deg(\phi) \leq 3$. In this case, we get $\phi^e(x) = c_2 x + c_0$, $\phi^o(x) = c_3 x + c_1$, $\psi^e(x) = d_2 x + d_0$, $\psi^o(x) = d_1$, $B^e(x) = b_2 x + b_0$, $B^o(x) = b_3 x + b_1$, $(t_w B)^e(x) = (b_2 - 3wb_3)x + w^2 b_2 + b_0 - w^3 b_3 - b_1 w$ and $(t_w B)^o(x) = b_3 x + 3w^2 b_3 + b_1 - 2wb_2$ with $|c_3| + |b_3| + |d_2| \neq 0$. Following Proposition 2.5, there are four cases to consider.
  - **B$_1$.** $(\phi(0), B(-w)) = (0, 0)$ and $(\psi(0), B(0)) = (0, 0)$. Then

$$\begin{aligned} &d_0 = 0, \quad b_3 = 0, \quad b_0 = 0, \\ &c_0 = 0, \quad d_1 = -2b_2, \quad 2c_2 = wd_2, \quad b_1 = wb_2. \end{aligned} \tag{2.48}$$

Now, we assume that $(b_2, c_2) = (0, 0)$. Here, from (2.48) we have $d_2 = d_1 = d_0 = 0$, which is a contradiction. This leads to result (2.44)–(2.45).

  - **B$_2$.** $(\phi(0), B(-w)) \neq (0, 0)$ and $\big((w\psi - \phi)(0), B(0)\big) = (0, 0)$. Hence

$$\begin{gathered} b_0 = 0, \quad b_3 = 0, \quad c_0 = wd_0, \quad c_1 = wd_1 + wb_2 - d_0, \\ c_2 = wd_2 + d_1 + 2b_2, \quad wc_3 = -wd_2 - 2(d_1 + 2b_2), \quad 2b_1 = wb_2. \end{gathered} \tag{2.49}$$

Now, we assume that $(c_3, d_1 + 2b_2) = (0, 0)$ or $(c_2, b_2) = (0, 0)$.
If $(c_3, d_1 + 2b_2) = (0, 0)$, then from (2.49), we get $d_2 = 0$, which yields to contradiction.
If $(c_2, b_2) = (0, 0)$, thus from (2.49), we have

$$\begin{gathered} b_3 = b_2 = b_1 = b_0 = 0, \quad c_0 = wd_0, \quad c_1 = wd_1 - d_0, \\ wc_3 = -3d_1, \quad wd_2 = -d_1. \end{gathered} \tag{2.50}$$

On the other hand, from (1.8), we have

$$\langle D_w(\phi\vartheta) + \psi\vartheta + B\big(x^{-1}(\vartheta t_{-w}\vartheta)\big), x^n \rangle = 0, \qquad n \geq 0. \tag{2.51}$$

Therefore, from (2.50) and (2.51) we can deduce for $n = 0, 1$ that

$$d_1(\vartheta)_1 + d_0 = 0, \tag{2.52}$$

$$2d_0(\vartheta)_1 - w\big(d_1(\vartheta)_1 + d_0\big) = 0. \tag{2.53}$$

Taking into account (2.52), we obtain from (2.53) that $a_0 = 0$. It is a contradiction. Thus, we deduce (2.46) and (2.47).

  - **B$_3$.** $(\phi(0), B(-w)) = (0, 0)$ and $(\psi(0), B(0)) \neq (0, 0)$, thus

$$\begin{gathered} b_3 = 0, \quad 2b_1 = 3wb_2, \quad 2b_0 = w^2 b_2, \quad c_0 = 0, \quad c_1 = -d_0 - wb_2, \\ c_2 = d_1 + 2b_2, \quad wc_3 = 2(d_1 + 2b_2) - wd_2. \end{gathered}$$

Equivalently,

$$\begin{gathered} b_3 = 0, \quad b_0 + w^2 b_2 - wb_1 = 0, \quad 2(b_1 - 2wb_2) = -wb_2, \\ c_0 - wd_0 = -wd_0, c_1 - wd_1 = -wd_1 - d_0 - wb_2, \\ c_2 - wd_2 = d_1 + 2b_2 - wd_2, \quad -wc_3 = wd_2 - 2(d_1 + 2b_2). \end{gathered}$$

This leads to results (50) and (51) from Lemma 2.8 since $\big(\phi(0), (t_w B))(0)\big) = (0, 0)$ and $\big(\psi(0), (t_w B)(w)\big) \neq (0, 0)$.

  - **B$_4$.** $(\phi(0), B(-w)) \neq (0, 0)$ and $((w\psi - \phi)(0), B(0)) \neq (0, 0)$, then

$$\begin{gathered} b_0 = 0, \quad b_3 = 0, \quad b_1 = wb_2, \quad d_1 = -2b_2, \\ 2c_0 = wd_0, \quad 2c_2 = wd_2. \end{gathered} \tag{2.54}$$

Taking into account (2.54) and (2.51), we get for $n = 0$, $d_0 = 0$. It is a contradiction.

□

Remark 2.9. (1) If $c_3 = 1$, then from (2.47) and (2.51), for $n = 0$, we get

$$\begin{aligned} &2c_0 = w^2[b_2 - (1 + d_2)\alpha_0], \quad 2c_1 = w[(1 + d_2)(\alpha_0 - w) - 3b_2], \\ &2c_2 = w(d_2 - 1), \quad 2d_1 = -w(1 + d_2) - 4b_2, \quad 2d_0 = w[b_2 - (1 + d_2)\alpha_0]. \end{aligned} \tag{2.55}$$

(2) If $c_3 = 0$ and $c_2 = 1$, hence from (2.47), we obtain

$$\begin{aligned} &2c_0 = w(wb_2 - 2\alpha_0), \quad 2c_1 = 2\alpha_0 - 2w - wb_2, \\ &wd_2 = 2, \quad d_1 = -1 - 2b_2, \quad 2d_0 = wb_2 - 2\alpha_0. \end{aligned} \tag{2.56}$$

2.3. **Recurrence coefficients of $\{Z_n\}_{n\geq 0}$.** In the sequel, we assume that $\{Z_n\}_{n\geq 0}$ is a D_{-w}-Laguerre-Hahn sequence of class $s = 1$ satisfying (2.16). By virtue of the Theorem 2.7, it follows that

$$D_w(\phi\vartheta) + \psi\vartheta + B(x^{-1}(\vartheta t_{-w}\vartheta)) = 0,$$

with

$$\phi(x) = c_3x^3 + c_2x^2 + c_1x + c_0, \quad \psi(x) = d_2x^2 + d_1x + d_0, \quad B(x) = b_2x^2 + b_1x.$$

In this case, the system (1.10)–(1.15) becomes

$$(d_1 + 2b_2)\alpha_0 + wb_2 - b_1 - d_0 = 0, \tag{2.57}$$

$$\begin{aligned} &4c_3 \sum_{\nu=0}^{n-1} \alpha_\nu\alpha_{\nu+1} + 2\{(2n-3)c_3 - d_2\}\alpha_n\alpha_{n-1} + \{(2c_2 - wd_2)n - d_1 - 2b_2\}\alpha_n \\ &\quad - \{(2c_2 - wd_2)(n-1) - d_1 - 2b_2\}\alpha_{n-1} + (wd_1 - 2c_1)n + \frac{1}{2}n(n-1)w^2d_2 \\ &\quad - \frac{1}{3}n(n-1)(n-2)w^2c_3 - 2b_2\alpha_0 + d_0 + 2b_1 - 2wb_2 = 0, \ n \geq 1, \end{aligned} \tag{2.58}$$

$$\begin{aligned} &\{2(d_2 - c_3)\alpha_0\alpha_1 - 2(d_1 + b_2)\alpha_0 + 2(c_1 + wb_2) - wd_1 - 2b_1\}\alpha_0 \\ &\quad + wd_0 + b_2w^2 - 2c_0 - wb_1 = 0, \end{aligned} \tag{2.59}$$

$$\begin{aligned} &2(2c_2 - wd_2) \sum_{\nu=0}^{n-1} \alpha_\nu\alpha_{\nu+1} + \{n(n-1)w^2c_3 + 2c_1 - nw^2d_2 - wd_1\}\alpha_n \\ &\quad + (wd_0 - 2c_0)(n+1) \\ &\quad + 2\{[(2n-3)c_3 - d_2]\alpha_{n-1} - [(2n+1)c_3 - d_2]\alpha_{n+1} + (2c_2 - wd_2)n - d_1 - 2b_2\}\alpha_n^2 \\ &\quad + \frac{1}{2}n(n+1)w^2(d_1 + 2b_2) - \frac{1}{6}n(n^2-1)(2c_2 - wd_2)w^2 \\ &\quad + 2(wb_2 - b_1)\alpha_0 + b_2w^2 - wb_1 = 0, \ n \geq 1. \end{aligned} \tag{2.60}$$

According to Theorem 2.7 and its proof, we are going to consider the following two cases: $(\phi(0), B(-w)) = (0,0)$ and $(\phi(0), B(-w)) \neq (0,0)$.

2.3.1. *Case* $(\phi(0), B(-w)) = (0, 0)$. In this case, taking into account (2.44)–(2.45), system (2.57)–(2.60) can be written as

$$4c_3 \sum_{\nu=0}^{n-1} \alpha_\nu \alpha_{\nu+1} + 2\{(2n-3)c_3 - d_2\}\alpha_n \alpha_{n-1} - 2(c_1 + wb_2)n + \frac{1}{2}n(n-1)w^2 d_2 - \frac{1}{3}n(n-1)(n-2)w^2 c_3 - 2b_2\alpha_0 = 0, \ n \geq 1, \tag{2.61}$$

$$2\{(2n-3)c_3 - d_2\}\alpha_{n-1}\alpha_n - 2\{(2n+1)c_3 - d_2\}\alpha_{n+1}\alpha_n + n(n-1)w^2 c_3 + 2(c_1 + wb_2) - nw^2 d_2 = 0, \quad n \geq 1. \tag{2.62}$$

Subtracting identities (2.61) and (2.62), we obtain for $n \geq 1$, that

$$4c_3 \sum_{\nu=0}^{n} \alpha_\nu \alpha_{\nu+1} + 2\{(2n-1)c_3 - d_2\}\alpha_n \alpha_{n+1} - 2(c_1 + wb_2)(n+1) + \frac{1}{2}n(n+1)w^2 d_2 - \frac{1}{3}n(n^2-1)w^2 c_3 - 2b_2\alpha_0 = 0. \tag{2.63}$$

Let

$$S_n = \sum_{\nu=0}^{n} \alpha_\nu \alpha_{\nu+1}, \qquad n \geq 0. \tag{2.64}$$

Then

$$S_n - S_{n-1} = \alpha_n \alpha_{n+1}, \quad n \geq 0, \qquad S_{-1} = 0. \tag{2.65}$$

Taking into account relations (2.64) and (2.65), (2.63) becomes for $n \geq 0$,

$$\{(2n+1)c_3 - d_2\}S_n - \{(2n-1)c_3 - d_2\}S_{n-1} - (n+1)(c_1 + wb_2) + \frac{1}{4}n(n+1)w^2 d_2 - \frac{1}{6}n(n^2-1)w^2 c_3 - b_2\alpha_0 = 0. \tag{2.66}$$

Remark 2.10. (1) If $c_3 = -d_2 = 1$ and $c_1 + wb_2 = 0$, then necessarily we have that $b_2 \neq 0$. Indeed, if $c_3 = -d_2 = 1$ and $c_1 + wb_2 = 0$, then, from (2.66) for $n = 0$, we have $2\alpha_1 = b_2 \neq 0$.

(2) If we take $n = 1$ in (2.66) and taking into account the previous result, we get

$$\alpha_2 = \frac{w^2}{4b_2}. \tag{2.67}$$

Proposition 2.11. *We have*

$$S_n = (n+1)\frac{(n+2)\{n(n-1)w^2 c_3 - 2nw^2 d_2 + 12(c_1 + wb_2)\} + 24b_2\alpha_0}{24\{(2n+1)c_3 - d_2\}}, \ n \geq 0. \tag{2.68}$$

For the proof, we need the following lemma.

Lemma 2.12 ([7]). *Let*

$$Q_n(m) = \sum_{\nu=0}^{n} \nu^m, \qquad n \geq 0, \quad m \geq 1.$$

Then

$$Q_n(1) = \frac{1}{2}n(n+1), \ Q_n(2) = \frac{1}{6}n(n+1)(2n+1), \ Q_n(3) = \frac{1}{4}n^2(n+1)^2, \quad n \geq 0.$$

Proof of Proposition 2.11. Equation (2.66) can be written as

$$\{(2n+1)c_3 - d_2\}S_n - \{((2n-1)c_3 - d_2\}S_{n-1} = (n+1)(c_1 + wb_2) - \frac{1}{4}n(n+1)w^2 d_2 + \frac{1}{6}n(n^2-1)w^2 c_3 + b_2\alpha_0, \qquad n \geq 0.$$

So, we obtain

$$\begin{aligned}\{(2n+1)c_3-d_2\}S_n &= \sum_{\nu=0}^{n}\Big\{(\nu+1)(c_1+wb_2)-\frac{1}{4}\nu(\nu+1)w^2d_2\\ &\qquad +\frac{1}{6}\nu(\nu^2-1)w^2c_3+b_2\alpha_0\Big\}\\ &= (n+1)(c_1+wb_2)+Q_n(1)(c_1+wb_2)+\frac{1}{6}\big(Q_n(3)-Q_n(1)\big)w^2c_3\\ &\qquad -\frac{1}{4}\big(Q_n(2)+Q_n(1)\big)w^2d_2+(n+1)b_2\alpha_0, \quad n\geq 0.\end{aligned}$$

Taking into account Lemma 2.12, we get (2.68). □

Corollary 2.13. *The sequences $\{\zeta_n\}_{n\geq 0}$ and $\{\theta_{n+1}\}_{n\geq 0}$ are defined by*

$$\zeta_0=-\alpha_0,\quad \zeta_{2n+2}=\alpha_{2n+1}-\alpha_{2n+2},\quad \zeta_{2n+1}=\alpha_{2n}-\alpha_{2n+1},\qquad n\geq 0,$$
$$\theta_{2n+1}=-\alpha_{2n}^2,\quad \theta_{2n+2}=-\alpha_{2n+1}^2,\qquad n\geq 0,$$

where, for $n\geq 0$,

$$\begin{aligned}\alpha_{2n}&=\alpha_0\Lambda_n,\\ \alpha_{2n+1}&=\alpha_0^{-1}\Omega_n,\end{aligned}\tag{2.69}$$

with

$$\Lambda_n=\begin{cases}\frac{\Gamma(-\mu_3)\Gamma(-\mu_4)\Gamma(n-\mu_1)\Gamma(n-\mu_2)}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(n-\mu_3)\Gamma(n-\mu_4)}, & \textit{if } (c_3,c_2)=(0,1),\\ -\frac{1+d_2}{4n-1-d_2}\frac{\Gamma(-\rho_5)\Gamma(-\rho_6)\Gamma(-\rho_7)\Gamma(-\rho_8)}{\Gamma(-\rho_1)\Gamma(-\rho_2)\Gamma(-\rho_3)\Gamma(-\rho_4)}\frac{\Gamma(n-\rho_1)\Gamma(n-\rho_2)\Gamma(n-\rho_3)\Gamma(n-\rho_4)}{\Gamma(n-\rho_5)\Gamma(n-\rho_6)\Gamma(n-\rho_7)\Gamma(n-\rho_8)}, & \\ & \textit{if } c_3=1,\ 1+d_2\neq 0,\\ \frac{\Gamma(-\nu_3)\Gamma(-\nu_4)\Gamma(n-\nu_1)\Gamma(n-\nu_2)}{\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(n-\nu_3)\Gamma(n-\nu_4)}, & \textit{if } c_3=1,\ d_2=-1,\ c_1+wb_2\neq 0,\\ \delta_{0,n}+\frac{nw^2}{4b_2\alpha_0}, & \textit{if } c_3=1,\ d_2=-1,\ c_1+wb_2=0,\end{cases}$$

and

$$\Omega_n=\begin{cases}w^2\frac{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(n+1-\mu_3)\Gamma(n+1-\mu_4)}{\Gamma(-\mu_3)\Gamma(-\mu_4)\Gamma(n-\mu_1)\Gamma(n-\mu_2)}, & \textit{if } (c_3,c_2)=(0,1),\\ \frac{4w^2}{(1+d_2)(4n+1-d_2)}\frac{\Gamma(-\rho_1)\Gamma(-\rho_2)\Gamma(-\rho_3)\Gamma(-\rho_4)}{\Gamma(-\rho_5)\Gamma(-\rho_6)\Gamma(-\rho_7)\Gamma(-\rho_8)}\frac{\Gamma(n+1-\rho_5)\Gamma(n+1-\rho_6)\Gamma(n+1-\rho_7)\Gamma(n+1-\rho_8)}{\Gamma(n-\rho_1)\Gamma(n-\rho_2)\Gamma(n-\rho_3)\Gamma(n-\rho_4)}, & \\ & \textit{if } c_3=1,\ 1+d_2\neq 0,\\ \frac{w^2}{4}\frac{\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(n+1-\nu_3)\Gamma(n+1-\nu_4)}{\Gamma(-\nu_3)\Gamma(-\nu_4)\Gamma(n-\nu_1)\Gamma(n-\nu_2)}, & \textit{if } c_3=1,\ d_2=-1,\ c_1+wb_2\neq 0,\\ \frac{1}{2}(2n+1)b_2\alpha_0, & \textit{if } c_3=1,\ d_2=-1,\ c_1+wb_2=0.\end{cases}$$

Here ρ_1, ρ_2, ρ_3, ρ_4 are roots of the fourth degree equation $A_n(1)=0$; ρ_5, ρ_6, ρ_7, ρ_8 are roots of the fourth degree equation $B_n(1)=0$; μ_1, μ_2 are roots of the second degree equation $A_n(0)=0$; μ_3, μ_4 are roots of the equation $B_n(0)=0$; ν_1, ν_2 are roots of the second degree equation $(n+1)(2n+1)w^2+2(c_1+wb_2)=0$; and ν_3, ν_4 are roots of the equation $(2n+1)nw^2+2(c_1+wb_2)=0$ for

$$A_n(c_3)=(n+1)(2nc_3-d_2)\{(2n+1)^2w^2c_3-(2n+1)w^2d_2+4(c_1+wb_2)\}-2b_2\alpha_0(c_3+d_2),$$

$$B_n(c_3)=(2n+1)\{(2n-1)c_3-d_2\}\{2n^2w^2c_3-nw^2d_2+2(c_1+wb_2)\}-2b_2\alpha_0(c_3+d_2),$$

$$\rho_1=\frac{1}{4}\{d_2-2+\sqrt{q-2\sqrt{\Delta}}\},\quad \rho_2=\frac{1}{4}\{d_2-2-\sqrt{q-2\sqrt{\Delta}}\},$$

$$\rho_3 = \frac{1}{4}\{d_2 - 2 + \sqrt{q + 2\sqrt{\Delta}}\}, \quad \rho_4 = \frac{1}{4}\{d_2 - 2 - \sqrt{q + 2\sqrt{\Delta}}\},$$
$$\rho_5 = \frac{1}{4}\{d_2 + \sqrt{q - 2\sqrt{\Delta}}\}, \quad \rho_6 = \frac{1}{4}\{d_2 - \sqrt{q - 2\sqrt{\Delta}}\},$$
$$\rho_7 = \frac{1}{4}\{d_2 + \sqrt{q + 2\sqrt{\Delta}}\}, \quad \rho_8 = \frac{1}{4}\{d_2 - \sqrt{q + 2\sqrt{\Delta}}\},$$
$$\mu_1 = \frac{2(c_1 + wb_2) - 3w - \sqrt{\tilde{\Theta}}}{4w}, \quad \mu_2 = \frac{2(c_1 + wb_2) - 3w + \sqrt{\tilde{\Theta}}}{4w},$$
$$\mu_3 = \frac{2(c_1 + wb_2) - w - \sqrt{\tilde{\Theta}}}{4w}, \quad \mu_4 = \frac{2(c_1 + wb_2) - w + \sqrt{\tilde{\Theta}}}{4w},$$
$$\nu_1 = -\frac{3w + \sqrt{w^2 - 16(c_1 + wb_2)}}{4w}, \quad \nu_2 = -\frac{3w - \sqrt{w^2 - 16(c_1 + wb_2)}}{4w},$$
$$\nu_3 = -\frac{w + \sqrt{w^2 - 16(c_1 + wb_2)}}{4w}, \quad \nu_4 = -\frac{w - \sqrt{w^2 - 16(c_1 + wb_2)}}{4w},$$

where

$$\Delta = \{4w^{-2}(c_1 + wb_2) + 1 + d_2\}^2 + 16w^{-2}b_2\alpha_0(1 + d_2),$$
$$q = 2(1 + d_2) + d_2^2 - 8w^{-2}(c_1 + wb_2),$$
$$\tilde{\Theta} = \{w + 2(c_1 + wb_2)\}^2 + 8b_2\alpha_0 d_2.$$

Proof. From (2.65), we get

$$\alpha_{2n}\alpha_{2n+1} = S_{2n} - S_{2n-1}, \ \alpha_{2n+1}\alpha_{2n+2} = S_{2n+1} - S_{2n}, \ n \geq 0.$$

From the Proposition 2.11, we obtain

$$\alpha_{2n}\alpha_{2n+1} = \frac{B_n(c_3)}{2\{(4n+1)c_3 - d_2\}\{(4n-1)c_3 - d_2\}}, \quad n \geq 0,$$

and

$$\alpha_{2n+1}\alpha_{2n+2} = \frac{A_n(c_3)}{2\{(4n+3)c_3 - d_2\}\{(4n+1)c_3 - d_2\}}, \quad n \geq 0. \tag{2.70}$$

This leads to

$$\frac{\alpha_{2n+2}}{\alpha_{2n}} = \frac{\{(4n-1)c_3 - d_2\}}{\{(4n+3)c_3 - d_2\}}\frac{A_n(c_3)}{B_n(c_3)}, \quad n \geq 0.$$

Here,

$$\alpha_{2n+2} = \alpha_0\Lambda_{n+1}, \quad n \geq 0.$$

On account of (2.70) and the above equation, we get

$$\alpha_{2n+1} = -\frac{A_n(c_3)}{2\alpha_0\{(4n+1)c_3 - d_2\}\{(4n+3)c_3 - d_2\}\Lambda_{n+1}}.$$

Hence, from (2.67), we can deduce (2.69). □

2.3.2. *The case* $(\phi(0), B(-w)) \neq (0,0)$. In this case, taking into account (2.46)–(2.47) and (2.64)–(2.65), system (2.57)–(2.60) can be written as

$$w(c_3 + d_2)\alpha_0 + 2d_0 - wb_2 = 0, \tag{2.71}$$

$$\begin{aligned} &2\{(2n-1)c_3 - d_2\}S_{n-1} - 2\{(2n-3)c_3 - d_2\}S_{n-2} - \tfrac{w}{2}\{(2n-1)c_3 - d_2\}\alpha_n \\ &\quad + \tfrac{w}{2}\{(2n-3)c_3 - d_2\}\alpha_{n-1} + (2n+1)d_0 + \tfrac{1}{2}n^2w^2d_2 \\ &\quad - \tfrac{1}{6}n(2n^2 - 6n + 1)w^2c_3 - 2b_2\alpha_0 - wb_2 = 0, \ n \geq 1, \end{aligned} \tag{2.72}$$

$$\begin{aligned} &\{2(d_2 - c_3)\alpha_0\alpha_1 + (wd_2 + wc_3 + 2b_2)\alpha_0 + wb_2 - 2d_0 - \tfrac{1}{2}w^2(c_3 + d_2)\}\alpha_0 \\ &\quad + \tfrac{1}{2}w^2b_2 - wd_0 = 0, \end{aligned} \tag{2.73}$$

$$\begin{aligned}&-2wc_3S_{n-1}+\{\tfrac{1}{2}(2n^2-2n-1)w^2c_3-\tfrac{1}{2}(2n+1)w^2d_2-2d_0\}\alpha_n-wd_0(n+1)\\&\quad+2\{2[(2n-1)c_3-d_2]S_{n-1}-[(2n-3)c_3-d_2]S_{n-2}-[(2n+1)c_3-d_2]S_n\\&\qquad-\tfrac{w}{2}[(2n-1)c_3-d_2]\alpha_n\}\alpha_n-\tfrac{1}{4}n(n+1)w^3d_2+\tfrac{1}{12}n(n+1)(2n-5)w^3c_3\\&\qquad\quad+wb_2\alpha_0+\tfrac{1}{2}w^2b_2=0,\ n\geq 1.\end{aligned} \tag{2.74}$$

Proposition 2.14. *We have*

$$(c_3-d_2)\alpha_1=w(c_3+d_2)+b_2, \tag{2.75}$$

$$\{(2n+1)c_3-d_2\}\alpha_{n+1}-\{(2n-3)c_3-d_2\}\alpha_{n-1}=(1-2n)wc_3+wd_2,\ n\geq 1. \tag{2.76}$$

Proof. Equation (2.72) takes the form

$$\begin{aligned}&2\{(2n-1)c_3-d_2\}S_{n-1}-2\{(2n-3)c_3-d_2\}S_{n-2}=\tfrac{w}{2}\{(2n-1)c_3-d_2\}\alpha_n\\&\quad-\tfrac{w}{2}\{(2n-3)c_3-d_2\}\alpha_{n-1}-(2n+1)d_0-\tfrac{1}{2}n^2w^2d_2\\&\qquad+\tfrac{1}{6}n(2n^2-6n+1)w^2c_3+2b_2\alpha_0+wb_2=0,\ n\geq 1.\end{aligned}$$

So, we get

$$\begin{aligned}&2\{(2n-1)c_3-d_2\}S_{n-1}=\tfrac{w}{2}\{(2n-1)c_3-d_2\}\alpha_n+\tfrac{w}{2}(c_3+d_2)\alpha_0\\&\quad+\sum_{\nu=1}^{n}\Big\{\frac{1}{6}\nu(2\nu^2-6\nu+1)w^2c_3-\frac{1}{2}\nu^2w^2d_2-(2\nu+1)d_0+2b_2\alpha_0+wb_2\Big\},\ n\geq 1.\end{aligned}$$

Taking into account Lemma 2.12 and (2.71), we obtain

$$\begin{aligned}&2\{(2n-1)c_3-d_2\}S_{n-1}-\tfrac{w}{2}\{(2n-1)c_3-d_2\}\alpha_n=\tfrac{1}{12}n(n+1)(n^2-3n-1)w^2c_3\\&\quad-\tfrac{1}{12}n(n+1)(2n+1)w^2d_2-(n+1)^2d_0+\tfrac{1}{2}(2n+1)wb_2+2b_2\alpha_0n,\ n\geq 1.\end{aligned}$$

According to the above equation and (2.65), we get from (2.73) and (2.74) the desired result (2.75) and (2.76), after some calculations. □

Corollary 2.15. *The sequences* $\{\zeta_n\}_{n\geq 0}$ *and* $\{\theta_{n+1}\}_{n\geq 0}$ *are defined by*

$$\zeta_0=-\alpha_0,\quad \zeta_{2n+2}=\alpha_{2n+1}-\alpha_{2n+2},\quad \zeta_{2n+1}=\alpha_{2n}-\alpha_{2n+1},\qquad n\geq 0,$$
$$\theta_{2n+1}=-\alpha_{2n}^2,\quad \theta_{2n+2}=-\alpha_{2n+1}^2,\ n\geq 0,$$

where, for $n\geq 0$,

$$\alpha_{2n}=\frac{1}{(4n-1)c_3-d_2}\big\{wd_2n-wc_3n(2n-1)-(c_3+d_2)\alpha_0\big\}, \tag{2.77}$$

$$\alpha_{2n+1}=\frac{1}{(4n+1)c_3-d_2}\big\{(n+1)[wd_2-wc_3(2n-1)]+b_2\big\}. \tag{2.78}$$

Proof. From the equation (2.76), we get

$$\{(4n+1)c_3-d_2\}\alpha_{2n+1}-\{(4n-3)c_3-d_2\}\alpha_{2n-1}=wd_2-wc_3(4n-1),\quad n\geq 1.$$

Taking into account Lemma 2.12 and (2.75), we obtain (2.78). Proceeding as in (2.78), we can easily prove (2.77). □

2.4. **The canonical cases.** Before quoting the different canonical situations, let us proceed to the general transformation

$$\tilde{Z}_n(x)=a^{-n}Z_n(ax),\quad n\geq 0,$$
$$\tilde{\alpha}_n=a^{-1}\alpha_n,\ \tilde{\zeta}_n=a^{-1}\zeta_n,\ \tilde{\theta}_{n+1}=a^{-2}\theta\gamma_{n+1},\quad n\geq 0.$$

Then, the form $\tilde{\vartheta}=h_{a^{-1}}\vartheta$ fulfils

$$D_{wa^{-1}}\big(a^{-t}\phi(ax)\tilde{\vartheta}\big)+a^{1-t}\psi(ax)\tilde{\vartheta}+a^{-t}B(ax)\big(x^{-1}(\tilde{\vartheta}t_{-wa^{-1}}\tilde{\vartheta})\big)=0,\ t=\deg(\Phi).$$

Any so-called canonical case will be denoted by $\tilde{\alpha}_n$, $\tilde{\zeta}_n$, $\tilde{\theta}_{n+1}$, $\tilde{\vartheta}$.

2.4.1. *The case* $(\phi(0), B(-w)) = (0,0)$. On account of (2.44)–(2.45) and Corollary 2.13, we get the general situation

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \quad \alpha_{-1} = 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big(x(c_3x^2 + c_2x + c_1)\vartheta\big) + \big(d_2x^2 - 2b_2x\big)\vartheta \\ \qquad\qquad +b_2x(x+w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases} \tag{2.79}$$

with, for $n \geq 0$,

$$\alpha_{2n} = \alpha_0 \Lambda_n, \qquad \alpha_{2n+1} = \alpha_0^{-1}\Omega_n. \tag{2.80}$$

Theorem 2.16. *The following canonical cases arise:*

(1) *If* $\phi(x) = x^2 + c_1x$ *and* $B(x) = b_2x(x+w)$, *we have*

$$\begin{cases} \tilde{\zeta}_0 = -\tilde{\alpha}_0, \\ \tilde{\zeta}_{2n+2} = \tilde{\alpha}_{2n+1} - \tilde{\alpha}_{2n+2}, \quad \tilde{\zeta}_{2n+1} = \tilde{\alpha}_{2n} - \tilde{\alpha}_{2n+1}, \quad n \geq 0, \\ \tilde{\theta}_{2n+1} = -\tilde{\alpha}_{2n}^2, \quad \tilde{\theta}_{2n+2} = -\tilde{\alpha}_{2n+1}^2, \quad n \geq 0, \\ D_{-i}\big(\tilde{\phi}(x)\tilde{\vartheta}\big) + \tilde{\psi}(x)\tilde{\vartheta} + \tilde{B}(x)\big(x^{-1}(\tilde{\vartheta}t_i\tilde{\vartheta})\big) = 0, \end{cases} \tag{2.81}$$

with

$$\begin{cases} \tilde{\phi}(x) = x^2 + \big((\beta+2\tau)i + \frac{\tau}{\lambda}(2\tau+2\beta-1)\big)x, \\ \tilde{\psi}(x) = 2ix^2 + \frac{2\tau}{\lambda}(2\tau+2\beta-1)ix, \\ \tilde{B}(x) = -\frac{\tau}{\lambda}(2\tau+2\beta-1)ix(x-i), \end{cases} \tag{2.82}$$

and

$$\begin{cases} \tilde{\alpha}_{2n} = -\lambda\frac{\Gamma(\tau+\beta)\Gamma(\tau+\frac{1}{2})\Gamma(n+\tau+1)\Gamma(n+\tau+\beta+\frac{1}{2})}{\Gamma(\tau+1)\Gamma(\tau+\beta+\frac{1}{2})\Gamma(n+\tau+\frac{1}{2})\Gamma(n+\tau+\beta)}, \quad n \geq 0, \\ \tilde{\alpha}_{2n+1} = \frac{1}{\lambda}\frac{\Gamma(\tau+1)\Gamma(\tau+\beta+\frac{1}{2})\Gamma(n+\tau+\frac{3}{2})\Gamma((n+\tau+\beta+1)}{\Gamma(\tau+\beta)\Gamma(\tau+\frac{1}{2})\Gamma(n+\tau+1)\Gamma(n+\tau+\beta+\frac{1}{2})}, \quad n \geq 0. \end{cases} \tag{2.83}$$

(2) *If* $\phi(x) = x^3 + c_2x^2 + c_1x$ *and* $B(x) = b_2x(x+w)$, *we have the following subcases:*

(a) $d_2 + 1 \neq 0$*:*

$$\begin{cases} \tilde{\zeta}_0 = -\tilde{\alpha}_0, \\ \tilde{\zeta}_{2n+2} = \tilde{\alpha}_{2n+1} - \tilde{\alpha}_{2n+2}, \quad \tilde{\zeta}_{2n+1} = \tilde{\alpha}_{2n} - \tilde{\alpha}_{2n+1}, \quad n \geq 0, \\ \tilde{\theta}_{2n+1} = -\tilde{\alpha}_{2n}^2, \quad \tilde{\theta}_{2n+2} = -\tilde{\alpha}_{2n+1}^2, \quad n \geq 0, \\ D_i\big(\tilde{\phi}(x)\tilde{\vartheta}\big) + \tilde{\psi}(x)\tilde{\vartheta} + \tilde{B}(x)\big(x^{-1}(\tilde{\vartheta}t_{-i}\tilde{\vartheta})\big) = 0, \end{cases} \tag{2.84}$$

with

$$\begin{cases} \tilde{\phi}(x) = x^3 - (2\tau+\alpha+\beta)ix^2 - \{2\tau^2 + 2\tau\alpha + 2\tau\beta + \alpha\beta - \tau \\ \qquad + \frac{1}{\lambda(4\tau+2\alpha+2\beta-1)}[(2\tau+1)(2\tau+2\alpha+2\beta-1)(\tau+\alpha)(\tau+\beta) \\ \qquad + (4\tau+2\alpha+2\beta-1)(-2\tau^2-2\tau\alpha-2\tau\beta-\alpha\beta+\tau)]i\}x, \\ \tilde{\psi}(x) = -2(2\tau+\alpha+\beta)x^2 \\ \qquad - \frac{2}{\lambda(4\tau+2\alpha+2\beta-1)}[2\tau+1)(2\tau+2\alpha+2\beta-1)(\tau+\alpha)(\tau+\beta) \\ \qquad + (4\tau+2\alpha+2\beta-1)(-2\tau^2-2\tau\alpha-2\tau\beta-\alpha\beta+\tau)], \\ \tilde{B}(x) = \frac{1}{\lambda(4\tau+2\alpha+2\beta-1)}[(2\tau+1)(2\tau+2\alpha+2\beta-1)(\tau+\alpha)(\tau+\beta) \\ \qquad + (4\tau+2\alpha+2\beta-1)(-2\tau^2-2\tau\alpha-2\tau\beta-\alpha\beta+\tau)]x(x+i), \end{cases} \tag{2.85}$$

and, for $n \geq 0$,

$$\begin{cases} \tilde{\alpha}_{2n} = -\lambda \frac{4\tau+2\alpha+2\beta-1}{4n+4\tau+2\alpha+2\beta-1} \frac{\Gamma(\tau+\alpha)\Gamma(\tau+\beta)\Gamma(\tau+\frac{1}{2})\Gamma(\tau+\alpha+\beta+\frac{1}{2})}{\Gamma(\tau+\alpha+\frac{1}{2})\Gamma(\tau+\beta+\frac{1}{2})\Gamma(\tau+1)\Gamma(\tau+\alpha+\beta)} \\ \qquad \times \frac{\Gamma(n+\tau+1)\Gamma(n+\tau+\alpha+\beta)\Gamma(n+\tau+\alpha+\frac{1}{2})\Gamma(n+\tau+\beta+\frac{1}{2})}{\Gamma(n+\tau+\alpha)\Gamma(n+\tau+\beta)\Gamma(n+\tau+\frac{1}{2})\Gamma(n+\tau+\alpha+\beta+\frac{1}{2})}, \\ \tilde{\alpha}_{2n+1} = \frac{4}{\lambda(4n+4\tau+2\alpha+2\beta+1)} \frac{\Gamma(\tau+\alpha+\frac{1}{2})\Gamma(\tau+\beta+\frac{1}{2})\Gamma(\tau+1)\Gamma(\tau+\alpha+\beta)}{\Gamma(\tau+\alpha)\Gamma(\tau+\beta)\Gamma(\tau+\frac{1}{2})\Gamma(\tau+\alpha+\beta+\frac{1}{2})} \\ \qquad \times \frac{\Gamma(n+\tau+\alpha+1)\Gamma(n+\tau+\beta+1)\Gamma(n+\tau+\frac{3}{2})\Gamma(n+\tau+\alpha+\beta+\frac{3}{2})}{\Gamma(n+\tau+1)\Gamma(n+\tau+\alpha+\beta)\Gamma(n+\tau+\alpha+\frac{1}{2})\Gamma(n+\tau+\beta+\frac{1}{2})}. \end{cases} \tag{2.86}$$

(b) *$d_2 + 1 = 0$ and $c_1 + wb_2 \neq 0$.*

$$\begin{cases} \tilde{\zeta}_0 = -\tilde{\alpha}_0, \quad \tilde{\zeta}_{2n+2} = \tilde{\alpha}_{2n+1} - \tilde{\alpha}_{2n+2}, \quad \tilde{\zeta}_{2n+1} = \tilde{\alpha}_{2n} - \tilde{\alpha}_{2n+1}, \quad n \geq 0, \\ \tilde{\theta}_{2n+1} = -\tilde{\alpha}_{2n}^2, \quad \tilde{\theta}_{2n+2} = -\tilde{\alpha}_{2n+1}^2, \quad n \geq 0, \\ D_{-i}\big(\tilde{\Phi}(x)\tilde{\vartheta}\big) + \tilde{\Psi}(x)\tilde{\vartheta} + \tilde{B}(x)\big(x^{-1}(\tilde{\vartheta}t_i\tilde{\vartheta})\big) = 0, \end{cases} \tag{2.87}$$

with

$$\begin{cases} \tilde{\phi}(x) = x^3 + \frac{1}{2}ix^2 + [(\beta^2 - \frac{1}{16})(1 - \frac{1}{\lambda}) - 2\frac{\rho}{\lambda}]x, \\ \tilde{\psi}(x) = -x^2 - \frac{2}{\lambda}(\beta^2 - \frac{1}{16} + 2\rho)x, \\ \tilde{B}(x) = \frac{i}{\lambda}(\beta^2 - \frac{1}{16} + 2\rho)x(x-i), \end{cases} \tag{2.88}$$

and for $n \geq 0$,

$$\begin{cases} \tilde{\alpha}_{2n} = -\lambda \frac{\Gamma(\frac{1}{2}-2\beta)\Gamma(2\beta+\frac{1}{2})\Gamma(n+\beta+\frac{3}{4})\Gamma(n-\beta+\frac{3}{4})}{\Gamma(\frac{3}{4}-\beta)\Gamma(\beta+\frac{3}{4})\Gamma(n-2\beta+\frac{1}{2})\Gamma(n+2\beta+\frac{1}{2})}, \\ \tilde{\alpha}_{2n+1} = \frac{1}{4\lambda} \frac{\Gamma(\frac{3}{4}-\beta)\Gamma(\beta+\frac{3}{4})\Gamma(n-2\beta+\frac{3}{2})\Gamma(n+2\beta+\frac{3}{2})}{\Gamma(\frac{1}{2}-2\beta)\Gamma(2\beta+\frac{1}{2})\Gamma(n+\beta+\frac{3}{4})\Gamma(n-\beta+\frac{3}{4})}. \end{cases} \tag{2.89}$$

(c) *$d_2 + 1 = 0$ and $c_1 + wb_2 = 0$.*

$$\begin{cases} \tilde{\zeta}_0 = -\tilde{\alpha}_0, \quad \tilde{\zeta}_{2n+2} = \tilde{\alpha}_{2n+1} - \tilde{\alpha}_{2n+2}, \quad \tilde{\zeta}_{2n+1} = \tilde{\alpha}_{2n} - \tilde{\alpha}_{2n+1}, \quad n \geq 0, \\ \tilde{\theta}_{2n+1} = -\tilde{\alpha}_{2n}^2, \quad \tilde{\theta}_{2n+2} = -\tilde{\alpha}_{2n+1}^2, \quad n \geq 0, \\ D_{i}\big(\tilde{\Phi}(x)\tilde{\vartheta}\big) + \tilde{\Psi}(x)\tilde{\vartheta} + \tilde{B}(x)\big(x^{-1}(\tilde{\vartheta}t_{-i}\tilde{\vartheta})\big) = 0, \end{cases} \tag{2.90}$$

with

$$\tilde{\phi}(x) = x^3 - \frac{1}{2}ix^2 - 2\frac{\rho}{\lambda}ix, \quad \tilde{\psi}(x) = -x^2 - 4\frac{\rho}{\lambda}x, \quad \tilde{B}(x) = 2\frac{\rho}{\lambda}x(x+i), \tag{2.91}$$

and, for $n \geq 0$,

$$\tilde{\alpha}_{2n} = -\lambda\big(\delta_{0,n} + \frac{n}{8\rho}\big), \quad \tilde{\alpha}_{2n+1} = \frac{\rho}{\lambda}(2n+1). \tag{2.92}$$

Proof. **(1)** In this case, (2.79)–(2.80) reduces to

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big(x^2 + c_1x\big)\vartheta + (\frac{2}{w}x^2 - 2b_2x)\vartheta + b_2x(x+w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases}$$

with

$$\begin{cases} \alpha_{2n} = \alpha_0 \frac{\Gamma(-\mu_3)\Gamma(-\mu_4)\Gamma(n-\mu_1)\Gamma(n-\mu_2)}{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(n-\mu_3)\Gamma(n-\mu_4)}, \\ \alpha_{2n+1} = \frac{w^2}{\alpha_0} \frac{\Gamma(-\mu_1)\Gamma(-\mu_2)\Gamma(n+1-\mu_3)\Gamma(n+1-\mu_4)}{\Gamma(-\mu_3)\Gamma(-\mu_4)\Gamma(n-\mu_1)\Gamma(n-\mu_2)}, \quad n \geq 0. \end{cases}$$

With the choice of $wa^{-1} = -i$ and putting $\tilde{\alpha}_0 = -\lambda, \quad a^{-1}c_1 = i(\beta + 2\tau) + \frac{\tau}{\lambda}(2\tau + 2\beta - 1), \quad b_2 = -\frac{\tau}{\lambda}(2\tau + 2\beta - 1)i$, we get (2.81)–(2.83).

(2) (a) In this case (2.79)–(2.80) can be written as

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big((x^3 + \frac{1}{2}wd_2x^2 + c_1x)\vartheta\big) + (d_2x^2 - 2b_2x)\vartheta + b_2x(x+w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases}$$

with, for $n \geq 0$,

$$\begin{cases} \alpha_{2n} = -\frac{\alpha_0(1+d_2)}{4n-1-d_2}\frac{\Gamma(-\rho_5)\Gamma(-\rho_6)\Gamma(-\rho_7)\Gamma(-\rho_8)\Gamma(n-\rho_1)\Gamma(n-\rho_2)\Gamma(n-\rho_3)\Gamma(n-\rho_4)}{\Gamma(-\rho_1)\Gamma(-\rho_2)\Gamma(-\rho_3)\Gamma(-\rho_4)\Gamma(n-\rho_5)\Gamma(n-\rho_6)\Gamma(n-\rho_7)\Gamma(n-\rho_8)}, \\ \alpha_{2n+1} = \frac{4w^2}{\alpha_0(4n+1-d_2}\frac{\Gamma(-\rho_1)\Gamma(-\rho_2)\Gamma(-\rho_3)\Gamma(-\rho_4)\Gamma(n+1-\rho_5)\Gamma(n+1-\rho_6)\Gamma(n+1-\rho_7)\Gamma(n+1-\rho_8)}{\Gamma(-\rho_5)\Gamma(-\rho_6)\Gamma(-\rho_7)\Gamma(-\rho_8)\Gamma(n-\rho_1)\Gamma(n-\rho_2)\Gamma(n-\rho_3)\Gamma(n-\rho_4)}. \end{cases}$$

The choice $wa^{-1} = i$ and putting $\tilde{\alpha}_0 = -\lambda$, $d_2 = -2(2\tau+\alpha+\beta)$, $a^{-2}(c_1 + wb_2) = -2\tau^2 - 2\tau\alpha - 2\tau\beta - \alpha\beta + \tau$, $a^{-1}b_2 = \frac{1}{\lambda(4\tau+2\alpha+2\beta-1)}[(2\tau+1)(2\tau+2\alpha+2\beta-1)(\tau+\alpha)(\tau+\beta) + (4\tau+2\alpha+2\beta-1)(-2\tau^2 - 2\tau\alpha - 2\tau\beta - \alpha\beta + \tau)]$ we obtain (2.84)-(2.86).

(b) In this case (2.79) and (2.80) become

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big((x^3 - \frac{1}{2}wx^2 + c_1x)\vartheta\big) - (x^2 + 2b_2x)\vartheta + b_2x(x+w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases}$$

with for $n \geq 0$,

$$\begin{cases} \alpha_{2n} = \alpha_0\frac{\Gamma(-\nu_3)\Gamma(-\nu_4)\Gamma(n-\nu_1)\Gamma(n-\nu_2)}{\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(n-\nu_3)\Gamma(n-\nu_4)}, \\ \alpha_{2n+1} = \frac{w^2}{4\alpha_0}\frac{\Gamma(-\nu_1)\Gamma(-\nu_2)\Gamma(n+1-\nu_3)\Gamma(n+1-\nu_4)}{\Gamma(-\nu_3)\Gamma(-\nu_4)\Gamma(n-\nu_1)\Gamma(n-\nu_2)}. \end{cases}$$

The choice $wa^{-1} = -i$ and putting $\tilde{\alpha}_0 = -\lambda$, $a^{-2}(c_1 + wb_2) = \beta^2 - \frac{1}{16}$, $a^{-1}b_2 = \frac{i}{\lambda}(\beta^2 - \frac{1}{16} + 2\rho)$, we get (2.87)-(2.89).

(c) In this case (2.79)–(2.80) becomes

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big((x^3 - \frac{1}{2}wx^2 - wb_2x)\vartheta\big) - (x^2 + 2b_2x)\vartheta + b_2x(x+w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases}$$

with for $n \geq 0$,

$$\alpha_{2n} = \alpha_0\big(\delta_{0,n} + \frac{nw^2}{4b_2\alpha_0}\big), \quad \alpha_{2n+1} = \frac{(2n+1)b_2}{2}.$$

By choosing $wa^{-1} = i$ and putting $\tilde{\alpha}_0 = -\lambda$, $a^{-1}b_2 = 2\frac{\rho}{\lambda}$, we obtain (2.90)–(2.92) □

Remark 2.17. (1) The form $\tilde{\vartheta}$ given by (2.84)–(2.85) is regular if and only if $4\tau + 2\alpha + 2\beta \pm 1 \neq -4n$, $n \geq 0$.

(2) The form $\tilde{\vartheta}$ given by (2.90)–(2.91) is regular if and only if $\rho \neq 0$.

2.4.2. *The case* $(\phi(0), B(-w)) \neq (0,0)$. On account of (2.46)–(2.47) and Corollary 2.15, we get the general situation,

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big((c_3x^3 + c_2x^2 + c_1x + c_0)\vartheta\big) + (d_2x^2 + d_1x + d_0)\vartheta \\ \qquad +b_2x(x + \frac{1}{2}w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases} \tag{2.93}$$

with, for $n \geq 0$,

$$\begin{aligned} \alpha_{2n} &= \tfrac{1}{(4n-1)c_3-d_2}\big\{wd_2n - wc_3n(2n-1) - (c_3+d_2)\alpha_0\big\}, \\ \alpha_{2n+1} &= \tfrac{1}{(4n+1)c_3-d_2}\big\{(n+1)[wd_2 - wc_3(2n-1)] + b_2\big\}. \end{aligned} \tag{2.94}$$

Theorem 2.18. *The following canonical cases arise:*

(1) *For* $\phi(x) = x^2 + c_1x + c_0$ *and* $B(x) = b_2x(x + \frac{1}{2}w)$, $(c_0, b_2) \neq (0,0)$, *we have*

$$\begin{cases} \tilde{\zeta}_0 = -\tilde{\alpha}_0, \quad \tilde{\zeta}_{2n+2} = \tilde{\alpha}_{2n+1} - \tilde{\alpha}_{2n+2}, \quad \tilde{\zeta}_{2n+1} = \tilde{\alpha}_{2n} - \tilde{\alpha}_{2n+1}, \quad n \geq 0, \\ \tilde{\theta}_{2n+1} = -\tilde{\alpha}_{2n}^2, \quad \tilde{\theta}_{2n+2} = -\tilde{\alpha}_{2n+1}^2, \quad n \geq 0, \\ D_{-i}\big(\tilde{\phi}(x)\tilde{\vartheta}\big) + \tilde{\psi}(x)\tilde{\vartheta} + \tilde{B}(x)\big(x^{-1}(\tilde{\vartheta}t_i\tilde{\vartheta})\big) = 0, \end{cases} \tag{2.95}$$

with

$$\begin{cases} \tilde{\phi}(x) = (x+i)(x + 2i\tau - \lambda), \\ \tilde{\psi}(x) = 2ix^2 - (1+2\tau)x - i(2\tau + i\lambda), \\ \tilde{B}(x) = 2\tau x(x - \frac{1}{2}i), \end{cases} \tag{2.96}$$

and

$$\tilde{\alpha}_{2n} = i(n + \tau + i\lambda), \quad \tilde{\alpha}_{2n+1} = (n+\tau+1)i, \quad n \geq 0. \tag{2.97}$$

(2) *In the case where* $\phi(x) = x^3 + c_2x^2 + c_1x + c_0$ *and* $B(x) = b_2x(x + \frac{1}{2}w)$, $(c_0, b_2) \neq (0,0)$, *we obtain the canonical case below:*

$$\begin{cases} \tilde{\zeta}_0 = -\tilde{\alpha}_0, \quad \tilde{\zeta}_{2n+2} = \tilde{\alpha}_{2n+1} - \tilde{\alpha}_{2n+2}, \quad \tilde{\zeta}_{2n+1} = \tilde{\alpha}_{2n} - \tilde{\alpha}_{2n+1}, \quad n \geq 0, \\ \tilde{\theta}_{2n+1} = -\tilde{\alpha}_{2n}^2, \quad \tilde{\theta}_{2n+2} = -\tilde{\alpha}_{2n+1}^2, \quad n \geq 0, \\ D_{i}\big(\tilde{\phi}(x)\tilde{\vartheta}\big) + \tilde{\psi}(x)\tilde{\vartheta}\tilde{B}(x)\big(x^{-1}(\tilde{\vartheta}t_{-i}\tilde{\vartheta})\big) = 0, \end{cases} \tag{2.98}$$

with

$$\begin{cases} \tilde{\phi}(x) = x^3 - (2\tau + 1 + \frac{\alpha^2}{\alpha+i\lambda})ix^2 + \frac{i}{\alpha+i\lambda}[\lambda\alpha(\tau+\alpha+1) + i\tau(\alpha+i\lambda)(4\tau+2\alpha-1) \\ \qquad +3\alpha^2\tau i]x + \frac{1}{\alpha+i\lambda}[i(\alpha+i\lambda)(2\tau^2+2\tau\alpha-\tau) + \lambda\alpha(2\tau+\alpha)], \\ \tilde{\psi}(x) = -(4\tau+1+2\frac{\alpha^2}{\alpha+i\lambda})x^2 + [2\tau(2\tau-1) + \frac{\alpha^2(4\tau+1)}{\alpha+i\lambda})]ix - \frac{i}{\alpha+i\lambda}[\lambda\alpha(2\tau+\alpha) \\ \qquad +i(\alpha+i\lambda)(2\tau^2+2\tau\alpha-\tau)], \\ \tilde{B}(x) = -2\tau(\tau - 1 + \frac{\alpha^2}{\alpha+i\lambda})ix(x + \frac{1}{2}i), \end{cases} \tag{2.99}$$

and, for $n \geq 0$,

$$\begin{cases} \tilde{\alpha}_{2n} = -\frac{(n+\tau+\alpha)[i(\alpha+i\lambda)(n+\tau)+\lambda\alpha]}{2(n+\tau)(\alpha+i\lambda)+\alpha^2}, \\ \tilde{\alpha}_{2n+1} = -i\frac{(n+\tau+1)[(\alpha-i\lambda)(n+\tau)+\alpha^2]}{(2n+2\tau+1)(\alpha+i\lambda)+\alpha^2}, \quad n \geq 0. \end{cases} \tag{2.100}$$

Proof. **(1)** In this case from (2.56), (2.93), and (2.94) are written as

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\big((x^2 + (\alpha_0 - w - \frac{w}{2}b_2)x + \frac{1}{2}w^2b_2 - w\alpha_0)\vartheta\big) + \big(\frac{2}{w}x^2 - (1+2b_2)x + \frac{1}{2}wb_2 - \alpha_0\big)\vartheta \\ \qquad +b_2x(x + \frac{1}{2}w)\big(x^{-1}(\vartheta t_{-w}\vartheta)\big) = 0, \end{cases}$$

with

$$\alpha_{2n} = \alpha_0 - nw, \quad \alpha_{2n+1} = -(n+1)w - \frac{w}{2}b_2, \quad n \geq 0.$$

By choosing $wa^{-1} = -i$ and putting $\tilde{\alpha}_0 = -(\lambda - i\tau)$, $\quad b_2 = 2\tau$, we get (2.95)–(2.97).

(2) In this case (2.93)–(2.94) can be written, using (2.55), as

$$\begin{cases} \zeta_n = \alpha_{n-1} - \alpha_n, \quad n \geq 0, \\ \theta_{n+1} = -\alpha_n^2, \quad n \geq 0, \\ D_w\Big(\Big(x^3 + \frac{1}{2}w(d_2-1)x^2 + \frac{1}{2}w[(1+d_2)(\alpha_0 - w) - 3b_2]x + \frac{1}{2}w^2[b_2 - (1+d_2)\alpha_0]\Big)\vartheta\Big) \\ \qquad +\Big(d_2x^2 - [\frac{w}{2}(1+d_2) + 2b_2]x + \frac{w}{2}[b_2 - (1+d_2)\alpha_0]\Big)\vartheta \\ \qquad +b_2x(x + \frac{1}{2}w)\Big(x^{-1}\big(\vartheta t_{-w}\vartheta\big)\Big) = 0, \end{cases}$$

with

$$\begin{cases} \alpha_{2n} = -\frac{1}{4n-1-d_2}\{(1+d_2)\alpha_0 + wn(2n-1) - wd_2n\}, \\ \alpha_{2n+1} = -\frac{1}{4n+1-d_2}\{(n+1)[w(2n-1) - wd_2] - b_2\}. \end{cases}$$

Taking $wa^{-1} = i$ and putting $\tilde{\alpha}_0 = -\frac{(\tau+\alpha)[i(\alpha+i\lambda)\tau+\lambda\alpha]}{\alpha^2+2\tau(\alpha+i\lambda)}$, $d_2 = -(4\tau+1+\frac{2\alpha^2}{\alpha+i\lambda})$, $a^{-1}b_2 = -2\tau(\tau-1+\frac{\alpha^2}{\alpha+i\lambda})i$, we obtain (2.98)–(2.100). □

Remark 2.19. (1) The form $\tilde{\vartheta}$ given by (2.95)–(2.96) is regular if and only if $\tau + i\lambda \neq -n, \quad \tau + 1 \neq -n, \quad n \geq 0.$

(2) The form $\tilde{\vartheta}$ given by (2.98)–(2.99) is regular if and only if $\tau + 1 \neq -n, \quad \tau + \alpha \neq -n, \quad \frac{\alpha^2}{\alpha+i\lambda} + 2\tau \neq -n, \quad \frac{\alpha^2}{\alpha+i\lambda} + \tau \neq -n, \quad \frac{\lambda\alpha}{\alpha+i\lambda} i \neq n, \quad n \geq 0.$

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