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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

THE ALAOGLU THEOREM FOR TOPOLOGICAL $\mathbb{BC}$–MODULE

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Abstract. In this paper we prove the bicomplex version of the Alaoglu theorem for a topological $\mathbb{BC}$-module X. The concept of a $\mathbb{BC}$-dual pair and a product-type open cover in bicomplex is also introduced.

Доведено бікомплексну версію теореми Alaoglu у топологічному $\mathbb{BC}$-модулі X. Також наведено поняття про $\mathbb{BC}$-дуальну пару та відкрите покриття типу добутку в бікомплексі.

1. Introduction

Classical topological vector spaces are one of the very important structures that involves an algebraic structure with a topology investigated in functional analysis. Topological vector spaces has some interesting applications to various other branches of mathematics such as in differential calculus and measure theory in infinite-dimensional spaces. For details on topological vector spaces we refer to [4, 21, 24]. Some interesting work such as that on bicomplex Hilbert space was carried out by Gervais Lavoie, Marchildon and D. Rochon [6, 7, 8] that gave way to the research on various other bicomplex version of the classical spaces. The bicomplex version of topological vector spaces, called bicomplex module, and its various properties were studied in [11, 13, 14]. The main purpose of this paper is to study separation properties in bicomplex modules and the bicomplex version of the Alaoglu Theorem. Various separation results to be used in this theorem has already been proved in various other papers such as [11, 13, 14, 16].

In Section 2 we summarize the basics of bicomplex and hyperbolic numbers which may otherwise be found in details in [1, 17, 22]. Our main aim of this section is to give an overview of how real number system is different from the hyperbolic number system which is an affordable replacement for the set of real number system and the difference in the set of complex number and the set of bicomplex number and how the conjugations and modulus are constructed in the bicomplex number system. Section 3 deals with a $\mathbb{BC}$-dual pair and it idempotent decomposition.

Finally in Section 4 we introduce the Alaoglu Theorem for topological $\mathbb{BC}$-module. For further more details on bicomplex analysis and their applications one can refer to [1, 2, 3, 5, 6, 7, 8, 9, 10, 12, 15, 17, 20, 22] and references therein.

2. Preliminaries

We shall briefly summarize some basic properties of bicomplex numbers and hyperbolic numbers. We know that in case of complex numbers, we take only one imaginary unit, however for bicomplex numbers, we consider two imaginary units $\mathbf{i}$ and $\mathbf{j}$ satisfying $\mathbf{ij} = \mathbf{ji}$ with $\mathbf{i^2} = \mathbf{j^2} = -1$. Now let $\mathbb{C}(\mathbf{i})$ be the set of complex numbers with imaginary units $\mathbf{i}$ and let $\mathbb{C}(\mathbf{j})$ be the set of complex numbers with imaginary units $\mathbf{j}$. We define set of

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bicomplex numbers denoted by $\mathbb{BC}$ as

$$\mathbb{BC} = \{Z = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{ij}x_4 : x_1, x_2, x_3, x_4 \in \mathbb{R}\} = \{Z = z_1 + \mathbf{j}z_2 : z_1, z_2 \in \mathbb{C}(\mathbf{i})\}. \tag{2.1}$$

Then the set $\mathbb{BC}$ becomes a ring under the usual operations of addition and multiplication defined by

$$Z + W = (z_1 + \mathbf{j}z_2) + (w_1 + \mathbf{j}w_2) = (z_1 + w_1) + \mathbf{j}(z_2 + w_2)$$

$$Z \cdot W = (z_1 + \mathbf{j}z_2)(w_1 + \mathbf{j}w_2) = (z_1w_1 - z_2w_2) + \mathbf{j}(z_2w_1 + z_1w_2).$$

The set of complex numbers $\mathbb{C}(\mathbf{i})$ is a subring of $\mathbb{BC}$. Due to the fact that the set $\mathbb{BC}$ has two imaginary units whose square is -1 and one hyperbolic unit whose square is 1, one can define the following three conjugations for $\mathbb{BC}$:

(i) the bar-conjugation, $\overline{Z} = \overline{z}_1 + \mathbf{j}\overline{z}_2$,
(ii) the $\dagger$-conjugation, $Z^\dagger = z_1 - \mathbf{j}z_2$, and
(iii) the $*$-conjugation $Z^* = \overline{Z}^\dagger = \overline{z}_1 - \mathbf{j}\overline{z}_2$,

where $\overline{z}_1, \overline{z}_2$ are the usual conjugations of complex numbers z_1, z_2 in $\mathbb{C}(\mathbf{i})$.

All of the above conjugations satisfy additive, multiplicative and involutive operations on $\mathbb{BC}$ and is a ring automorphism of $\mathbb{BC}$. Three possible moduli arise according to the multiplication of a bicomplex numbers and its three different conjugations:

(i) $Z \cdot Z^\dagger = |Z|_{\mathbf{i}}{}^2 = z_1{}^2 + z_2{}^2 = (|\eta_1|^2 - |\eta_2|^2) + 2Re(\eta_1\eta_2^*)\mathbf{i}$,
(ii) $Z \cdot \overline{Z} = |Z|_{\mathbf{j}}{}^2 = \eta_1{}^2 + \eta_2{}^2 = (|z_1|^2 - |z_2|^2) + 2Re(z_1\overline{z}_2)\mathbf{j}$,
(iii) $Z \cdot Z^* = |Z|_{\mathbf{k}}{}^2 = (|z_1|^2 + |z_2|^2) - 2Im(z_1\overline{z}_2)\mathbf{k}$.

It should be noted that these modulus are $\mathbb{C}(\mathbf{i})-$, $\mathbb{C}(\mathbf{j})-$ and $\mathbb{D}$-valued. For details of conjucations on the set of bicomplex numbers see [1, 17, 22]. However, the $\dagger$-conjugation defined by $Z^\dagger = z_1 - \mathbf{j}z_2$, where $Z = z_1 + \mathbf{j}z_2; z_1, z_2 \in \mathbb{C}(\mathbf{i})$ is important in the sense that it paved us a way to define the invertiblity of a bicomplex number. A bicomplex number Z is said to be invertible if $Z \cdot Z^\dagger \neq 0$ and its inverse is given by

$$Z^{-1} = \frac{Z^\dagger}{Z \cdot Z^\dagger} = \frac{Z^\dagger}{|Z|_{\mathbf{i}}^2}.$$

Further, if $Z \neq 0$, but $Z \cdot Z^\dagger = |Z|_{\mathbf{i}}^2 = 0$, then Z is said to be a zero-divisor. We denote the set of all zero-divisors by

$$\mathcal{NC} = \left\{Z = z_1 + \mathbf{j}z_2 : Z \neq 0, Z \cdot Z^\dagger = z_1^2 + z_2^2 = 0\right\}$$

and is called the null cone of the set of bicomplex number $\mathbb{BC}$.

Now there are two special zero divisors and we call them idempotent elements. They are defined by

$$\mathbf{e_1} = \frac{1}{2}(1 + \mathbf{ij}) \quad \text{and} \quad \mathbf{e_2} = \frac{1}{2}(1 - \mathbf{ij}),$$

where $z_1 = \frac{1}{2}$ and $z_2 = \mathbf{i}\frac{1}{2}$, considering $\mathbf{e_1}$ and $\mathbf{e_2}$ as bicomplex numbers, although these are hyperbolic numbers that is a subring of $\mathbb{BC}$. We can see that $\mathbf{e_1}$ and $\mathbf{e_2}$ are zero divisors and are mutually complementary idempotent elements.

The sets $\mathbb{BC}_{\mathbf{e_1}} = \mathbf{e_1}\mathbb{BC}$ and $\mathbb{BC}_{\mathbf{e_2}} = \mathbf{e_2}\mathbb{BC}$ are (principal) ideals in the ring $\mathbb{BC}$ and have the property that

$$\mathbb{BC}_{\mathbf{e_1}} \cap \mathbb{BC}_{\mathbf{e_2}} = \{0\}$$

and

$$\mathbb{BC} = \mathbb{BC}_{\mathbf{e_1}} + \mathbb{BC}_{\mathbf{e_2}}. \tag{2.2}$$

Equation (2.2) is called the idempotent decomposition of the ring of bicomplex numbers $\mathbb{BC}$. The Euclidean norm $|\cdot|$ of a bicomplex number $Z = z_1 + \mathbf{j}z_2 = \alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2}$, where $\alpha_1 := z_1 - \mathbf{i}z_2$ and $\alpha_2 := z_1 + \mathbf{i}z_2$ are complex numbers in $\mathbb{C}(\mathbf{i})$. Note also that

$\alpha_1\mathbf{e_1} \in \mathbb{BC}_{\mathbf{e_1}}$ and $\alpha_2\mathbf{e_2} \in \mathbb{BC}_{\mathbf{e_2}}$ is defined as $|Z| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} = \sqrt{|z_1|^2 + |z_2|^2}$ and it is easy check that for any Z and W in $\mathbb{BC}$, we have

$$|Z \cdot W| \leq \sqrt{2}|Z||W|.$$

The $\mathbb{D}$-valued norm of the bicomplex number $Z = \alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2}$ denoted by $|Z|_{\mathbf{k}}$ is defined as $|Z|_{\mathbf{k}} = |\alpha_1|\mathbf{e_1} + |\alpha_2|\mathbf{e_2}$, where $|\alpha_1|$ and $|\alpha_2|$ are the usual modulus of complex numbers α_1 and α_2. Further $|Z \cdot W|_{\mathbf{k}} = |Z|_{\mathbf{k}} \cdot |W|_{\mathbf{k}}$ and the Euclidean norm and the hyperbolic norm of a bicomplex number is related by

$$||Z|_{\mathbf{k}}| = |Z|.$$

For the above discussion one can refer to [1] and [17].

The hyperbolic numbers, denoted by $\mathbb{D}$, is a ring of all numbers of the form $Z = a + \mathbf{k}b$, where a,b $\in \mathbb{R}$, with $\mathbf{k}$ satisfying $\mathbf{k^2} = 1$, i.e.,

$$\mathbb{D} = \left\{a + \mathbf{k}b : a, b \in \mathbb{R},\ \mathbf{k^2} = 1,\ \mathbf{k} \notin \mathbb{R}\right\}.$$

The set $\left\{x + \mathbf{ij}y : x, y \in \mathbb{R},\ \mathbf{i^2} = \mathbf{j^2} = -1\right\}$ is a subset of the set of bicomplex numbers which is isomorphic to $\mathbb{D}$ as a real algebra. It is well known that we can also decompose the set of hyperbolic as

$$\mathbb{D} = \mathbb{D}\mathbf{e_1} + \mathbb{D}\mathbf{e_2}. \tag{2.3}$$

We call equation (2.3) the idempotent decomposition of $\mathbb{D}$. Thus the idempotent representation of any hyperbolic number $x = x_1 + \mathbf{k}x_2$ is given by

$$x = \beta_1\mathbf{e_1} + \beta_2\mathbf{e_2}, \qquad \beta_1, \beta_2 \in \mathbb{R}$$

with $\beta_1 = x_1 + x_2$, $\beta_2 = x_1 - x_2$. Also the set of positive hyperbolic numbers, denoted by $\mathbb{D}^+$, is the set of all those hyperbolic numbers whose idempotent components are non-negative, i.e.,

$$\mathbb{D}^+ = \{\beta_1\mathbf{e_1} + \beta_2\mathbf{e_2} : \beta_1, \beta_2 \geq 0\}.$$

A partial order relation on $\mathbb{D}$ is defined as follows. Given $x, y \in \mathbb{D}$, we write $x \preceq y$ if $y - x \in \mathbb{D}^+$. It is easy to see that this relation is reflexive, symmetric and antisymmetric and so it defines a partial order relation on $\mathbb{D}$. Also for $x, y \in \mathbb{D}$, if $x \preceq y$, then we say that y is $\mathbb{D}$-larger than x, and x is $\mathbb{D}$-smaller than y. The notions of upper and lower bounds also exist in the context of the hyperbolic plane. Given a subset S of $\mathbb{D}$ we can define $\mathbb{D}$-upper bounds and $\mathbb{D}$-lower bounds of this set S. Using these bounds, this set can be made $\mathbb{D}$-bounded from above and $\mathbb{D}$-bounded from below if they exist. Now, if the set is $\mathbb{D}$-bounded from above as well as from below, then we say that the set is $\mathbb{D}$-bounded. For the above discussions we refer to [1] and [17].

Consider the mappings

$$\pi_{1,\mathbf{i}}, \pi_{2,\mathbf{i}} : \mathbb{BC} \longrightarrow \mathbb{C}(\mathbf{i})$$

given by

$$\pi_{l,\mathbf{i}}(z) = \pi_{l,\mathbf{i}}(\alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2}) := \alpha_l \in \mathbb{C}(\mathbf{i}), \qquad \text{where } l = 1, 2.$$

These maps are just the projections onto the coordinate axis in $\mathbb{C}^2(\mathbf{i})$ with the basis $\{\mathbf{e_1}, \mathbf{e_2}\}$.

Definition 2.1 ([2])**.** A set $B \subset \mathbb{BC}$ is said to be a product-type if $B = B_1\mathbf{e_1} + B_2\mathbf{e_2}$. In this case, we will denote $B_l := \pi_{l,\mathbf{i}}(B)$, where $l = 1, 2$.

3. $\mathbb{BC}$-MODULE VALUED HOLOMORPHIC FUNCTIONS

In this section we study the $\mathbb{BC}$-module valued Liouville's Theorem and the Cauchy Integral Theorem.

Definition 3.1. ([23, Section 3]). The set of bicomplex numbers forms a commutative ring. A module X defined over the ring $\mathbb{BC}$ of bicomplex numbers is called a $\mathbb{BC}$-module.

Let X be a $\mathbb{BC}$-module. Then we can write (see [6, 23])

$$X = \mathbf{e_1}X_1 + \mathbf{e_2}X_2, \tag{3.4}$$

where $X_1 = \mathbf{e_1}X$ and $X_2 = \mathbf{e_2}X$ are complex-linear spaces as well as $\mathbb{BC}$-modules. Equation (3.4) is known as the idempotent decomposition of X. Therefore, any $x \in X$ can be uniquely expressed as $x = \mathbf{e_1}x + \mathbf{e_2}x = \mathbf{e_1}x_1 + \mathbf{e_2}x_2$.

Definition 3.2. ([11, Definition 2]). Let X be a $\mathbb{BC}$-module and τ be a topology on X such that the operation

(i) $+ : X \times X \to X$ and
(ii) $\cdot : \mathbb{BC} \times X \to X$ are continuous.
(iii) For every $x \in X$, the set $\{x\}$ which has x as its only member is a closed set.

Then τ is a topology on the bicomplex module X and (X, τ) is called a topological bicomplex module or a topological $\mathbb{BC}$-module.

Definition 3.3. ([11, Definition 9]). A topological $\mathbb{BC}$-module X is called a $\mathbb{BC}$-normed module space if there exists a map $\|\cdot\| : X \to \mathbb{R}^+ = [0, \infty)$ known as a $\mathbb{BC}$-norm on X that satisfies the following conditions:

(i) The map $\|\cdot\| : X \to \mathbb{R}^+$ is a norm over the field $\mathbb{C}(\mathbf{i})$ or the field $\mathbb{C}(\mathbf{j})$.
(ii) For each $\alpha \in \mathbb{BC}$ and $x \in X$, the inequality $\|\alpha x\| \le \sqrt{2}|\alpha|\|x\|$ holds.

Note that X is a topological vector space over the field $\mathbb{C}(\mathbf{i})$ or the field $\mathbb{C}(\mathbf{j})$. A complete $\mathbb{BC}$-normed module space is called a $\mathbb{BC}$-Banach module. For further details, refer to [6] and [8].

Definition 3.4. Let X be a topological $\mathbb{BC}$-module. Then the dual space of X consisting of all continuous $\mathbb{BC}$-linear functionals $f : \mathbb{BC} \to \mathbb{BC}$ is denoted $X^*_{\mathbb{BC}}$ is again a $\mathbb{BC}$-module over the ring of bicomplex numbers with the addition and scalar multiplication defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \quad \text{and} \quad (\alpha f)(x) = \alpha f(x) \tag{3.5}$$

for $f_1, f_2, f \in X^*_{\mathbb{BC}}$, $x \in X$ and $\alpha \in \mathbb{BC}$.

Now since f is a continuous $\mathbb{BC}$-linear functional on the $\mathbb{BC}$-module X, it implies that $f = f_1\mathbf{e_1} + f_2\mathbf{e_2}$, where f_1 and f_2 are continuous linear functionals on $X_1 = X\mathbf{e_1}$ and $X_2 = X\mathbf{e_2}$, respectively. Thus we see that

$$X^*_{\mathbb{BC}} = X^*_1\mathbf{e_1} + X^*_2\mathbf{e_2}.$$

Definition 3.5. Let X be a topological $\mathbb{BC}$-module and Ω be a subset of $\mathbb{BC}$. Then a function f is called a bounded function if $f(\Omega)$ is a bounded subset of X.

Definition 3.6. Let $\Omega \subseteq \mathbb{BC}$ and X be a topological $\mathbb{BC}$-module. Then a mapping $F : \Omega \to X$ is called bicomplex analytic (or bicomplex holomorphic) in Ω if the limit

$$\begin{aligned} F^{'}(z_o) &= \lim_{\substack{H \to 0 \\ H = Z - Z_0 \notin \mathcal{NC} \cup \{0\}}} \frac{F(Z) - F(Z_0)}{Z - Z_o} \\ &= \lim_{\substack{H \to 0 \\ H = Z - Z_0 \notin \mathcal{NC} \cup \{0\}}} \frac{F(Z_o + H) - F(Z_o)}{H} \end{aligned}$$

exists for every $Z \in \Omega$ such that $H = Z - Z_o$ is a bicomplex number.

If F is $\mathbb{BC}$-holomorphic in $\mathbb{BC}$, then F is called $\mathbb{BC}$-entire in $\mathbb{BC}$.

Definition 3.7. ([11, Definition 8]). A subset M of $\mathbb{BC}$-linear functionals on a $\mathbb{BC}$-module X is called total if $F(x) = 0$ for each $F \in M$ implies $x = 0$.

Theorem 3.8. *If x^* is a continuous $\mathbb{BC}$-linear functional on a topological $\mathbb{BC}$-module X and $F : \Omega \to X$ is $\mathbb{BC}$-holomorphic in $\mathbb{BC}$, then the function $x^*oF : \Omega \to \mathbb{BC}$ is $\mathbb{BC}$-holomorphic in Ω.*

Proof. Now,

$$\begin{aligned}\lim_{\substack{H\to 0\\ H=Z-Z_0\notin\mathcal{NC}\cup\{0\}}} \frac{(x^*oF)(Z)-(x^*oF)(Z_o)}{Z-Z_o} &= x^*\left(\lim_{\substack{H\to 0\\ H=Z-Z_0\notin\mathcal{NC}\cup\{0\}}} \frac{F(Z)-F(Z_o)}{Z-Z_o}\right)\\ &= x^*(F^{'}(Z_o))\end{aligned}$$

exists for each $Z_o \in \Omega$. □

Remark 3.9. If $F : \Omega \subset \mathbb{BC} \to X$ is a $\mathbb{BC}$-module valued holomorphic function with $X = X_1\mathbf{e_1} + X_2\mathbf{e_2}$ and $Z = \alpha_1\mathbf{e_1} + \alpha_2\mathbf{e_2} \in \Omega$, then

$$F(Z) = F_1(\alpha_1)\mathbf{e_1} + F_2(\alpha_2)\mathbf{e_2},$$

where $F_1 : \Omega_1 \longrightarrow X_1$ and $F_2 : \Omega_2 \longrightarrow X_2$ are holomorphic function, and we can write

$$(x^*oF)(Z) = (x_1^*oF_1)(\alpha_1)\mathbf{e_1} + (x_2^*oF_2)(\alpha_2)\mathbf{e_2}$$

where

$$x_1^*oF_1 : \Omega_1 \to \mathbb{C}$$

and

$$x_2^*oF_2 : \Omega_2 \to \mathbb{C}$$

are holomorphic functions. Then x^*oF is $\mathbb{BC}$-holomorphic if and only if $x_1^*oF_1$ and $x_2^*oF_2$ are holomorphic.

By using decomposition of a $\mathbb{BC}$-linear functional, one can easily prove the following theorem:

Theorem 3.10. *Let M be a subset of continuous $\mathbb{BC}$-linear functionals on a topological $\mathbb{BC}$-module X such that*

$$M = M_1\mathbf{e_1} + M_2\mathbf{e_2}$$

where $M_1 \subseteq X_1^*$ and $M_2 \subseteq X_2^*$. Then M is total in X^* if and only if M_1 and M_2 are total in X_1^* and X_2^*, respectively.

Now we will prove the $\mathbb{BC}$ module-valued Liouville's theorem:

Theorem 3.11. *Let X be a topological $\mathbb{BC}$-module with a total $\mathbb{BC}$-dual and let $M \subset X^*$, which is total in X^*. Then a bounded $\mathbb{BC}$-entire function $F : \mathbb{BC} \to X$ is always constant.*

Proof. We know that continuous image of a bounded set is bounded [11, Corollary 2.10]. Since F is bounded and so by Theorem 3.8, x^*oF is bounded $\mathbb{BC}$-entire for every $x^* \in X^*_{\mathbb{BC}}$. Further by Remark 3.9

$$(x^*oF)(Z) = (x_1^*oF_1)(\alpha_1)\mathbf{e_1} + (x_2^*oF_2)(\alpha_2)\mathbf{e_2}.$$

By applying Liouville's theorem for standard $\mathbb{C}$-entire functions [21, Theorem 6.6.5], we conclude that (x^*oF) is constant for each $x^* \in X^*_{\mathbb{BC}}$.

Thus for any $a, b \in \mathbb{BC}$

$$(x^*oF)(a) = (x^*oF)(b)$$

for all $x^* \in X^*_{\mathbb{BC}}$ in the total set M, i.e.,

$$\begin{aligned}(x^*oF)(a-b) &= (x_1^*oF_1)(a_1-b_1)\mathbf{e_1} + (x_2^*oF_2)(a_2-b_2)\mathbf{e_2} \\ &= 0.\end{aligned}$$

So by using Remark 3.9,

$$x_1^*oF_1(a_1-b_1) = 0 \ \ \forall x_1^* \in M,$$

and

$$x_2^*oF_2(a_1-b_1) = 0 \ \forall x_2^* \in M.$$

Hence F is constant. □

If γ_1 and γ_2 are two curves in the complex plane with parametrization $\phi_1 : [0,1] \to \gamma_1$ and $\phi_2 : [0,1] \to \gamma_2$, then the hyperbolic curve Γ can be written as $\Gamma = \gamma_1\mathbf{e_1} + \gamma_2\mathbf{e_2}$. In the sequel, a hyperbolic curve Γ is called a $\mathbb{BC}$-rectifiable Jordan curve if and only if γ_1 and γ_2 are rectifiable Jordan curves in the complex plane. Also $\gamma_1 \times \gamma_2$ is a rectifiable set, see [2, Page-2544-2547] for more details.

Now we will prove the Cauchy Integral Theorem for a $\mathbb{BC}$-Banach module. For the Cauchy Integral Theorem for bicomplex valued holomorphic functions see [22].

Theorem 3.12. *Let X be a $\mathbb{BC}$-Banach module with total $\mathbb{BC}$-dual and F be a function from a region $\Omega \subset \mathbb{BC}$ bounded by a $\mathbb{BC}$-rectifiable Jordan curve Γ into a $\mathbb{BC}$-Banach module $(X, ||\cdot||)$ and F be $\mathbb{BC}$-holomorphic that maps on a region Ω and continuous on Γ. Then*

$$\int_\Gamma x^*F(Z)dZ \wedge dZ^\dagger = 0 \qquad (\textit{line integral limit in the } \mathbb{BC} - \textit{norm } ||\cdot||).$$

Proof. Let x^* be a continuous $\mathbb{BC}$-linear functional on the $\mathbb{BC}$-Banach module X with

$$G = \int_\Gamma F(Z)dZ \wedge dZ^\dagger.$$

Then

$$\begin{aligned}x^*(G) &= \int_\Gamma x^*(F(Z))dZ \wedge dZ^\dagger \\ &= \int_{\gamma_1} x_1^*F_1(\alpha_1)d\alpha_1\mathbf{e_1} + \int_{\gamma_2} x_2^*F_2(\alpha_2)d\alpha_2\mathbf{e_2},\end{aligned}$$

where $F_i : \Omega_i \subset \mathbb{C}(\mathbf{i}) \longrightarrow (X_i, ||\cdot||_i)$ for $i = 1, 2$.

Since F_i is analytic in Ω_i and continuous on γ_i, x_i^* is analytic in Ω_i and continuous on γ_i for $i = 1, 2$. Therefore by the Cauchy Integral theorem, $x_i^*(G_i) = 0$ for $i = 1, 2$. Since x_i^*, for $i = 1, 2$, is an arbitrary continuous linear functionals, $G = 0$ by [21, Theorem 7.7.7(b)]. □

Remark 3.13. In a recent paper [19] various results on identity theorem for $\mathbb{BC}$-holomorphic function and $\mathbb{D}$-holomorplic function are proved. By using idempotent decomposition for bicomplex holomorphic functions, the Cauchy Integral Formula and Identity Theorems can be easily proved for $\mathbb{BC}$-Banach module-valued holomorphic functions. For Cauchy Integral formula for bicomplex holomorphic function we refer to [3]. Due to the presence of bicomplex version of Hahn-Banach thoerems for $\mathbb{BC}$-Banach module and F-$\mathbb{BC}$-module the theory of $\mathbb{BC}$-Banach valued and F-$\mathbb{BC}$-valued holomorphic function can be established easily.

Theorem 3.14. *Let $X_{\mathbb{D}}^*$ be a dual space of a topological $\mathbb{D}$-module X, $R \subset X$ be a $\mathbb{D}$-convex, $\mathbb{D}$-absorbing set, and $S \subset X$ be a subspace such that $R\mathbf{e_1} \cap S\mathbf{e_1} = R\mathbf{e_2} \cap S\mathbf{e_2} = \emptyset$. Then there exists $F \in X_{\mathbb{D}}^*$ such that $F(Z) = 0 \ \forall \ Z \in S$.*

Proof. Let $F \in X_{\mathbb{D}}^*$. Then we can define

$$F(Z) = G_1\mathbf{e_1}(Z\mathbf{e_1}) + G_2\mathbf{e_2}(Z\mathbf{e_2}) \ \ \forall \ Z \in X,$$

where G_1 and G_2 are real linear function from X to $\mathbb{R}$. Since S is a subspace, by [14, Theorem 8.7] there is a maximal subspace $H \supset S$ which does not meet R. Hence

$$\begin{aligned} Z \in H \quad &\Leftrightarrow \quad F(Z) = 0 \\ &\Leftrightarrow \quad G_1\mathbf{e_1}(Z\mathbf{e_1}) = 0 \ \text{ and } \ G_2\mathbf{e_2}(Z\mathbf{e_2}) = 0 \\ &\Leftrightarrow \quad Z \in N(G_1)\mathbf{e_1} \cap N(G_2)\mathbf{e_2} \ \ \{\text{where } N(G_1) \text{ and } N(G_2) \text{ is the null space}\} \\ &\Leftrightarrow \quad Z \in N(F). \end{aligned}$$

Therefore, $S \subset H = N(F)$, implies $S \subset N(F)$. Hence, $F(Z) = 0 \ \ \forall \ Z \in S$. □

Theorem 3.15. *Let $X_{\mathbb{D}}^*$ be a continuous dual space of locally $\mathbb{D}$-convex module X, S be a closed subspace of X and $Z \notin S$. Then there exists $F \in X_{\mathbb{D}}^*$ such that $F(Y) = 0 \ \ \forall \ Y \in S$ and $F(X) \neq 0$.*

Proof. Since X is locally $\mathbb{D}$-convex module and $Z \notin S$, there is an open $\mathbb{D}$-convex neighborhood R of Z which does not meet S. By Theorem 3.14, there exist $F \in X_{\mathbb{D}}^*$ such that

$$F(Y) = 0 \ \ \forall \ Y \in S \quad \text{and} \quad F(Z) \neq 0 \ \ \forall Z \in R.$$

Hence, $F(Z) \neq 0$. □

Theorem 3.16. *Let $X_{\mathbb{D}}^*$ be a continuous dual space of a locally $\mathbb{D}$-convex module X and $Z \notin cl\{0\}$. Then there exists $F \in X_{\mathbb{D}}^*$ such that $F(Z) \neq 0$.*

Proof. Since $cl\{0\}$ is a closed subspace of X and $Z \notin cl\{0\}$, by Theorem 3.15 there exists $F \in X_{\mathbb{D}}^*$ such that

$$F(Y) = 0 \ \ \forall \ Y \in cl\{0\} \ \text{ and } \ F(Z) \neq 0.$$ □

4. Alouge Theorem with bicomplex scalars

In this section we are going to establish a bicomplex version of the Alouge theorem. We start with a definition of a $\mathbb{BC}$-bilinear functional.

Definition 4.1. Let X be a $\mathbb{BC}$-module. A function

$$(\cdot,\cdot)_{\mathbb{BC}} : X \times X \longrightarrow \mathbb{BC}$$

is called $\mathbb{BC}$-bilinear functional if the following conditions are satisfied
i) $(x+y,z)_{\mathbb{BC}} = (x,z)_{\mathbb{BC}} + (y,z)_{\mathbb{BC}} \ \forall x,y,z \in X.$
ii) $(\alpha x,y)_{\mathbb{BC}} = \alpha(x,y)_{\mathbb{BC}} \ \forall \alpha \in \mathbb{BC}$ *and* $\forall \ x,y \in X.$
iii) $(x,y+z)_{\mathbb{BC}} = (x,y)_{\mathbb{BC}} + (x,z)_{\mathbb{BC}} \ \forall \ x,y,z \in X.$
iv) $(x,\alpha y)_{\mathbb{BC}} = \alpha(x,y)_{\mathbb{BC}} \ \forall \ x,y \in X$ *and* $\alpha \in \mathbb{BC}.$

A $\mathbb{BC}$-bilinear functional can be decomposed uniquely as

$$(x,y)_{\mathbb{BC}} = (x_1,y_1)\mathbf{e_1} + (x_2,y_2)\mathbf{e_2},$$

where $x = x_1\mathbf{e_1} + x_2\mathbf{e_2}, y = y_1\mathbf{e_1} + y_2\mathbf{e_2} \in X$ and $(\cdot,\cdot)$ is a bilinear functional on topological vector spaces.

Suppose X and Y be a $\mathbb{BC}$-modules. Then a $\mathbb{BC}$-bilinear functional $(\cdot,\cdot)_{\mathbb{BC}} : X \times Y \longrightarrow \mathbb{BC}$ is a map which is $\mathbb{BC}$-linear in either argument if the other one is fixed. We usually omit the explicit reference to a bilinear functional, $(\cdot,\cdot)_{\mathbb{BC}}$, and just refer to (X,Y) as a

$\mathbb{BC}$-paired spaces. If for every non-zero $x \in X$ there exists $y \in Y$ such that $(x, y)_{\mathbb{BC}} \neq 0$, then we say that Y distinguishes points of X. If every $\mathbb{BC}$-module of X distinguishes points of Y, then we say that (X, Y) is a $\mathbb{BC}$-dual pair.

Let us suppose that X and Y are $\mathbb{BC}$-module with $\mathbb{BC}$-bilinear function $(\cdot, \cdot)_{\mathbb{BC}}$. For any $x = x_1\mathbf{e_1} + x_2\mathbf{e_2} \in X$, $y = y_1\mathbf{e_1} + y_2\mathbf{e_2} \in Y$,

$$(x, y)_{\mathbb{BC}} = \beta_1(x_1, y_1)\mathbf{e_1} + \beta_2(x_2, y_2)\mathbf{e_2}, \tag{4.6}$$

where $\beta_l(x_l, y_l) = \pi_{l,\mathbf{i}}\{(x_l, y_l)\} \in \mathbb{C}(\mathbf{i})$, for $l = 1, 2$.

Suppose (X, Y) is a $\mathbb{BC}$-dual pair. Then by the definition of a $\mathbb{BC}$-dual pair, $\forall x = x_1\mathbf{e_1} + x_2\mathbf{e_2} \in X$ and $\forall y = y_1\mathbf{e_1} + y_2\mathbf{e_2} \in Y$ we have $(x, y)_{\mathbb{BC}} \neq 0$. From equation (4.6)

$$\begin{aligned}(x, y)_{\mathbb{BC}} \neq 0 \quad &\Leftrightarrow \quad \beta_1(x_1, y_1)\mathbf{e}_1 + \beta_2(x_2, y_2)\mathbf{e}_2 \neq 0 = 0\mathbf{e}_1 + 0\mathbf{e}_1 \\ &\Leftrightarrow \quad \beta_1(x_1, y_1) \neq 0 \text{ and } \beta_2(x_2, y_2) \neq 0.\end{aligned}$$

Hence, we also write

$$(X, Y) = (X_1, Y_1)\mathbf{e}_1 + (X_2, Y_2)\mathbf{e}_2$$

where (X_l, Y) is a dual pair for $l = 1, 2$.

If X and Y are paired $\mathbb{BC}$-modules and $y \in Y$, then the mapping

$$y^* : X \longrightarrow \mathbb{BC}$$

is a $\mathbb{BC}$-linear functional on X, that is, $y^* \in X^*_{\mathbb{BC}}$.

The map

$$D : Y \longrightarrow X^*_{\mathbb{BC}}\ , y \longrightarrow y^*$$

is generally not one-one. It is possible that $y^* \neq w^*$ even though $y = w$. For a function D to be injective, it is clearly necessary and sufficient that

$$(x, y)_{\mathbb{BC}} = 0 \text{ for each } x \in X \ \Rightarrow\ y = 0$$

or, equivalently,

$$y \neq 0 \ \Rightarrow\ \text{there is some } x \in X \text{ such that } (x, y)_{\mathbb{BC}} \neq 0.$$

Definition 4.2. If X and Y are paired $\mathbb{BC}$-modules, each over the ring of bicomplex numbers, then the map $P_y : X \to \mathbb{D}$ defined by

$$P_y(\cdot) = |\, \langle \cdot, y\rangle_{\mathbb{D}} \,|_{\mathbf{k}}$$

determines a seminorm on X for each $y \in Y$.

The weakest topology $\sigma(X, Y)$ for which the seminorms $\{P_y : y \in Y\}$ are continuous is called the weak topology on X for the pair (X, Y).

Theorem 4.3. *Let (X, Y) be a paired $\mathbb{BC}$-module. Then the following statements are equivalent:*
a) X distinguishes points of Y.
b) The map $D : Y \longrightarrow X^*_{\mathbb{BC}},\ y \longrightarrow y^*$ is injective.
c) $\sigma(Y, X)$ is Hausdroff.

Proof. a)⇒ b): Let X distinguish points of Y, i.e., for each $x_1 \in X_1\mathbf{e_1}$ and $x_2 \in X_2\mathbf{e_2}$ there exist $y_1 \in Y_1\mathbf{e_1}$ and $y_2 \in Y_2\mathbf{e_2}$ such that $\langle x_1, y_1\rangle \neq 0$ and $\langle x_2, y_2\rangle \neq 0$,respectively.

b)⇒ c): Let $D : Y \longrightarrow X^*_{\mathbb{BC}}$ is injective. Then for each non-zero $y_1 \in Y_1\mathbf{e_1}$ and $y_2 \in Y_2\mathbf{e_2}$, there is some $x_1 \in X_1\mathbf{e_1}$ and $x_2 \in X_2\mathbf{e_2}$ such that $0 \neq |\,\langle x_1, y_1\rangle\,| = p_{x_1}(y_1)$ and $0 \neq |\,\langle x_2, y_1\rangle\,| = p_{x_2}(y_2)$ respectively.

So, by [21, Theorem 5.5.1(a)] $\sigma(Y, X)$ is Hausdroff. □

Definition 4.4. A subset R of a $\mathbb{BC}$-module X is said to be $\mathbb{BC}$-compact if every open cover of R has a finite subcover.

Proposition 4.5. *Let $R = R_1\mathbf{e_1} + R_2\mathbf{e_2}$ be a product-type set in $\mathbb{BC}$-module X and let $G = \{Q_1, Q_2, Q_3, \dots\}$ be a collections of product-type open sets such that $\mathcal{G} = \bigcup_{i\in I} Q_i$ is product-type open cover of R indexed by a set I. Define*

$$G_1 = \{Q_{1,k} \subset X_1 \mid \exists\ Q_k \in G \text{ with } Q_k = Q_{1,k}\mathbf{e_1} + Q_{2,k}\mathbf{e_2}\}$$

and

$$G_2 = \{Q_{2,k} \subset X_2 \mid \exists\ Q_k \in G \text{ with } Q_k = Q_{1,k}\mathbf{e_1} + Q_{2,k}\mathbf{e_2}\}.$$

Then $\mathcal{G}_1 = \bigcup_{i\in I} Q_{1,i}$ will be an open cover of R_1 in X_1 and $\mathcal{G}_2 = \bigcup_{i\in I} Q_{2,i}$ will be an open cover of R_2 in X_2.

Proof. Let $Q_{1,i} \in G_1$. Then there exists $Q_i \in G$ with $Q_i \in G$ with $Q_i = Q_{1,i}\mathbf{e_1} + Q_{2,i}\mathbf{e_2}$, where $i \in I$. Since $\mathcal{G}$ is a product-type open cover of R, there exists $\bigcup_{i\in I} Q_{1,i}$ which is an open cover of R_1 in X_1. In a similar way one proves that $\bigcup_{i\in I} Q_{2,i}$ is an open cover of R_2 in X_2. □

Remark 4.6. Let $R = R_1\mathbf{e_1} + R_2\mathbf{e_2}$ be a product-type set in a $\mathbb{BC}$-module X. Then $\mathcal{G} = \bigcup_{i\in I} Q_i$ is a product-type open cover of R indexed by a set I if and only if $\mathcal{G}_1 = \bigcup_{i\in I} Q_{1,i}$ is an open cover of R_1 in X_1 and $\mathcal{G}_2 = \bigcup_{i\in I} Q_{2,i}$ is an open cover of R_2 in X_2.

From the above proposition and remark, we have the following corollary:

Corollary 4.7. *A product-type subset R is $\mathbb{BC}$-compact in X if and only if $R\mathbf{e_1}$ and $R\mathbf{e_2}$ is compact in X_1 and X_2 respectively.*

The next definition is a particular case of Definition 4.2 above.

Definition 4.8. If $(X^{'}, X)$ are paired $\mathbb{BC}$-modules, each one considered over the bicomplex ring, then the mapping $P_{x^{'}} : X \longrightarrow \mathbb{D}$ defined by

$$P_{x^{'}}(x) = |x^{'}(x)|_{\mathbf{k}}$$

determines a seminorm on X for each $x \in X$ and $x^{'} \in X^{'}$. Since $\{P_{X^{'}}(x)\}$ are continuous so $\sigma(X^{'}, X)$ is a weak topology on X.

Proposition 4.9. *Let $(X, X^{'})$ be paired $\mathbb{BC}$-modules, each one is over the ring of bicomplex numbers, and $P_{X^{'}} : X \longrightarrow \mathbb{D}$ be a $\mathbb{D}$-seminorm on X with $\{P_{X^{'}}(x)\}$ being continuous. Then $\sigma(X^{'}, X)$ is the weak toplogy if and only if $\sigma(X_1^{'}, X_1)$ and $\sigma(X_2^{'}, X_2)$ are weak topologies on X_1 and X_2, respectively.*

Proof. Let $(X^{'}, X)$ be paired $\mathbb{BC}$-modules, each one is over the ring of bicomplex numbers. Suppose that $\sigma(X^{'}, X)$ is a weak topology on X. Then there exist continuous $\mathbb{D}$-seminorms $P_{x^{'}}(x) = x^{'}(x)\ \forall\ x^{'} \in X^{'}$. Since $x^{'}(x)$ is a continuous $\mathbb{BC}$-linear function, there exist continuous linear functions $x_1^{'}$ and $x_2^{'}$ such that

$$x^{'}(x) = x_1^{'}(x)\mathbf{e_1} + x_2^{'}(x)\mathbf{e_2}.$$

Writing each $x \in X$ as $x = x_1\mathbf{e_1} + x_2\mathbf{e_2}$ for some $x_1 \in X_1$ and $x_2 \in X_2$, we have

$$x^{'}(x) = x_1^{'}(x_1)\mathbf{e_1} + x_2^{'}(x_2)\mathbf{e_2}.$$

Therefore, $x_1^{'}(x_1)$ and $x_2^{'}(x_2)$ are continuous seminorms on X_1 and X_2, respectively. Thus $\sigma(X_1^{'}, X_1)$ and $\sigma(X_2^{'}, X_2)$ are weak topology on X_1 and X_2, respectively.

The converse follows by arguing in the reverse order. □

Definition 4.10. Let (X', X) be a $\mathbb{BC}$-pair. If $E = E_1\mathbf{e_1} + E_2\mathbf{e_2}$ is a product-type subset of X, then the polar E° of E is defined as

$$E^\circ = \left\{ f \in X^{'}\ :\ \sup_{u\in E} |f(u)|_{\mathbf{k}} \preceq 1 \right\}.$$

The idempotent decomposition of the polar E° is

$$\begin{aligned} E^\circ &= \left\{ f \in X^{'} \ : \ \sup_{u \in E} |f(u)|_{\mathbf{k}} \preceq 1 \right\} \\ &= \left\{ f_1 \in X_1^{'} \ : \ \sup_{u_1 \in E_1} |f_1(u_1)| \leq 1 \right\} \mathbf{e_1} + \left\{ f_2 \in X_2^{'} \ : \ \sup_{u_2 \in E_2} |f_2(u_2)| \leq 1 \right\} \mathbf{e_2} \\ &= E_1^\circ \mathbf{e_1} + E_2^\circ \mathbf{e_2}. \end{aligned}$$

Definition 4.11 ([18]). A product-type neighborhood $U \subset \mathbb{BC}$ of a point $Z_0 = \beta_1^0 \mathbf{e_1} + \beta_2^0 \mathbf{e_2} \in \mathbb{BC}$ is a set of the form $U = U_1 \mathbf{e_1} + U_2 \mathbf{e_2}$ such that U_1 and U_2 are neighborhoods of $\beta_1^0 \mathbf{e_1}$ and $\beta_2^0 \mathbf{e_2}$ in $\mathbb{BC}\mathbf{e_1}$ and $\mathbb{BC}\mathbf{e_2}$, respectively. Recall that a set U is a neighborhood of a point Z_0 if there exists a strictly positive $R \in \mathbb{D}^+$ such that $\mathbb{B}(Z_0, R) \subset U$.

Now we prove the Alouge theorem for topological $\mathbb{BC}$-modules.

Theorem 4.12. *Let U be a product-type neighborhood of 0 in a topological $\mathbb{BC}$-module X. Then its polar $U^\circ$is bicomplex $\sigma(X^{'}, X)$-compact if and only if U_1° and U_2° are $\sigma(X_1^{'}, X_1)$-compact and $\sigma(X_2^{'}, X_2)$-compact respectively.*

Proof. Suppose $U = U_1 \mathbf{e_1} + U_2 \mathbf{e_2}$ be a product-type neighborhood of 0 in a topological bicomplex module X. Firstly suppose that U° is $\sigma(X^{'}, X)$-compact. Then, by Corollary 4.7, U_1° and U_2° are $\sigma(X_1^{'}, X_1)$-compact and $\sigma(X_2^{'}, X_2)$-compact, respectively.
The converse part is also easily follows from Corollary 4.7. □

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