Methods of Functional Analysis and Topology
Vol. 29 (2023), no. 3-4, pp. 101–110
doi.org/10.31392/MFAT-npu26_3-4.2023.03

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

THE NUMERICAL RADIUS POINTS OF $\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$

SUNG GUEN KIM AND CHANG YEOL LEE

Abstract. For $n \geq 2$ and a Banach space E we let

$$\Pi(E) = \{[x^*, x_1, \dots, x_n] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \dots, n\ \},$$

$\mathcal{L}(^nE : E)$ denote the space of all continuous n-linear mappings from E to itself. An element $[x^*, x_1, \dots, x_n] \in \Pi(E)$ is called a numerical radius point of $T \in \mathcal{L}(^nE : E)$ if

$$|x^*(T(x_1, \dots, x_n))| = v(T),$$

where $v(T)$ is the numerical radius of T. By $\mathrm{Nradius}(T)$ we denote the set of all numerical radius points of T.

Let $0 \leq \theta \leq \frac{\pi}{2}$ and $\ell^2_{(\infty,\theta)} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x, y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}.$$

In this paper, we show that the numerical radius of $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ equals to its norm $\|T\|$. Using this, we classify $\mathrm{Nradius}(T)$ for every $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ in connection with the norming points of the bilinear mapping associated with T. Let

$$\mathrm{NA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is norm attaining}\}$$

and

$$\mathrm{NRA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is numerical radius attaining}\}.$$

We also show that $\mathrm{NA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})) = \mathrm{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$, which generalizes some results in [12].

Для $n \geq 2$ і банахова простору E покладемо

$$\Pi(E) = \{[x^*, x_1, \dots, x_n] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ для } j = 1, \dots, n\ \},$$

де $\mathcal{L}(^nE : E)$ позначає простір усіх неперервних n-лінійних відображень E на себе. Елемент $[x^*, x_1, \dots, x_n] \in \Pi(E)$ називається точкою чисельного радіусу $T \in \mathcal{L}(^nE : E)$, якщо

$$|x^*(T(x_1, \dots, x_n))| = v(T),$$

де $v(T)$ — чисельний радіус T. За $\mathrm{Nradius}(T)$ позначимо множину всіх точок чисельного радіусу T.

Нехай $0 \leq \theta \leq \frac{\pi}{2}$ і $\ell^2_{(\infty,\theta)} = \mathbb{R}^2$ із повернeною супремум нормою

$$\|(x, y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}.$$

Показано, що чисельний радіус $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ дорівнює своїй нормі $\|T\|$. Використовуючи це, ми класифікуємо $\mathrm{Nradius}(T)$ для кожного $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$, пов'язуючи з нормуючими точками білінійного відображення, відповідного T. Нехай

$$\mathrm{NA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ досягає норми}\}$$

і

$$\mathrm{NRA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ досягає чисельного радіусу}\}.$$

Ми також показуємо що $\mathrm{NA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})) = \mathrm{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$, що узагальнює деякі результати роботи [12].

2020 *Mathematics Subject Classification.* 46A22.

Keywords. Numerical radius, norm, numerical radius attaining bilinear mappings, numerical radius points.

Sung Guen Kim is the corresponding author.

1. Introduction

Let us sketch a brief history of norm or numerical radius attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm or numerical radius attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces. Choi, Domingo, Kim and Maestre [6] showed that for a scattered compact Hausdorff space K, every continuous n-homogeneous polynomial on $\mathcal{C}(K:\mathbb{C})$ can be approximated by norm attaining ones at extreme points and also that the set of all extreme points of the unit ball of $\mathcal{C}(K:\mathbb{C})$ is a norming set for every continuous complex polynomial. The authors obtained similar results if "norm" is replaced by "numerical radius."

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere in a Banach space E. $\mathcal{L}(^nE:E)$ is usually endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E}\|T(x_1,\cdots,x_n)\|$. $\mathcal{L}_s(^nE:E)$ denotes the closed subspace of all continuous symmetric n-linear mappings on E. We let

$$\Pi(E) = \Big\{[x^*, x_1, \ldots, x_n] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n\Big\}.$$

An element $[x^*, x_1, \ldots, x_n] \in \Pi(E)$ is called a *numerical radius point* of $T \in \mathcal{L}(^nE:E)$ if $|x^*(T(x_1,\ldots,x_n))| = v(T)$, where the numerical radius is defined by

$$v(T) = \sup_{[y^*,y_1,\ldots,y_n]\in\Pi(E)}\Big|y^*\Big(T(y_1,\ldots,y_n)\Big)\Big|.$$

Nradius(T) denotes the set of all numerical radius points of T. Note that

$$[x^*, x_1, \ldots, x_n] \in \text{Nradius}(T)$$

if and only if $[-x^*, -x_1, \ldots, -x_n] \in \text{Nradius}(T)$.

Let $\ell_p^m = \mathbb{R}^m$ with the ℓ_p-norm for $m \geq 2$ and $1 \leq p \leq \infty$. Kim [11] also studied Nradius(T) for every $T \in \mathcal{L}(^n\ell_\infty^m : \ell_\infty^m)$ $(m \in \mathbb{N})$ and classified Nradius(T) for every $T \in \mathcal{L}(^2\ell_\infty^2 : \ell_\infty^2)$.

An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ or $\mathcal{L}(^nE:E)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $\|T\| = \|T(x_1,\ldots,x_n)\|$. We denote the set of all norming points of T by Norm(T).

Kim [7, 9, 10] classified Norm(T) for every $T \in \mathcal{L}_s(^2\ell_\infty^2)$, $\mathcal{L}(^2\ell_\infty^2)$ or $\mathcal{L}_s(^3\ell_1^2)$, respectively.

A mapping $P : E \to \mathbb{C}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1}|P(x)|$.

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

An element $[x^*, x] \in \Pi(E)$ is called a *numerical radius point* of $P \in \mathcal{P}(^nE:E)$ if $|x^*(P(x))| = v(P)$, where the numerical radius is

$$v(P) = \sup_{[y^*,y]\in\Pi(E)}\Big|y^*(P(y))\Big|.$$

We denote the set of all numerical radius points of P by Nradius(P). Notice that $[x^*, x] \in$ Nradius(P) if and only if $[-x^*, -x] \in$ Nradius(P).

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ if $\|x\| = 1$ and $\|P\| = \|P(x)\|$. We denote the set of all norming points of P by Norm(P).

Kim [8] classified Norm(P) for every $\mathcal{P}(^2l^2_\infty)$. If $T \in \mathcal{L}(^nE)$ or $\mathcal{L}(^nE : E)$ and Norm$(T) \neq \emptyset$, T is called a *norm attaining* and if $T \in \mathcal{L}(^nE : E)$ and Nradius$(T) \neq \emptyset$, T is called a *numerical radius attaining.* Similarly, if $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and Norm$(P) \neq \emptyset$, P is called a *norm attaining* and if $P \in \mathcal{P}(^nE : E)$ and Nradius$(P) \neq \emptyset$, P is called a *numerical radius attaining* (see [3]).

Choi, Domingo, Kim and Maestre [6] showed that for a scattered compact Hausdorff space K and $n \in \mathbb{N}$, $P \in \mathcal{P}(^n\mathcal{C}(K : \mathbb{C}) : \mathcal{C}(K : \mathbb{C}))$ is norm attaining if and only if it is numerical radius attaining.

Let

$$\mathrm{NA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is norm attaining}\}$$

and

$$\mathrm{NRA}(\mathcal{L}(^nE : E)) = \{T \in \mathcal{L}(^nE : E) : T \text{ is numerical radius attaining}\}.$$

It seems to be interesting to characterize a Banach space E such that $\mathrm{NA}(\mathcal{L}(^nE : E)) = \mathrm{NRA}(\mathcal{L}(^nE : E))$. Kim [12] showed that for every $n \geq 2$, $\mathrm{NA}(\mathcal{L}(^n\ell_1 : \ell_1)) = \mathrm{NRA}(\mathcal{L}(^n\ell_1 : \ell_1))$ and also characterized $\mathrm{NA}(\mathcal{L}(^n\ell_1 : \ell_1))$.

Let $0 \leq \theta \leq \frac{\pi}{2}$ and $\ell^2_{(\infty,\theta)} = \mathbb{R}^2$ with the *rotated supremum norm*

$$\|(x,y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}.$$

In this paper, we show that the numerical radius of $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ equals to its norm $\|T\|$. Using this, we classify Nradius(T) for every $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ in connection with the norming points of the bilinear mapping associated with T. We also show that $\mathrm{NA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})) = \mathrm{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$, which generalizes some results in [12].

2. Results

Let $0 \leq \theta \leq \frac{\pi}{2}$ and $\ell^2_{(\infty,\theta)} = \mathbb{R}^2$ with the rotated supremum norm

$$\|(x,y)\|_{(\infty,\theta)} = \max\Big\{|x\cos\theta + y\sin\theta|,\ |x\sin\theta - y\cos\theta|\Big\}.$$

Note that $\|(x,y)\|_{(\infty,0)} = \|(x,y)\|_\infty$ and $\|(x,y)\|_{(\infty,\frac{\pi}{4})} = \frac{1}{\sqrt{2}}\|(x,y)\|_1$.

Lemma 2.1. *Let* $0 \leq \theta \leq \frac{\pi}{2}$. *Let*

$$W_1 = (-\sin\theta + \cos\theta,\ \sin\theta + \cos\theta) \text{ and } W_2 = (\sin\theta + \cos\theta,\ \sin\theta - \cos\theta).$$

Then, ℓ^2_1 *and* $\ell^2_{(\infty,\theta)}$ *are isometric via the mapping* $\phi : \ell^2_1 \to \ell^2_{(\infty,\theta)}$ *such that*

$$\phi(t_1,t_2) := t_1W_1 + t_2W_2 = \big((t_1+t_2)\cos\theta + (t_2-t_1)\sin\theta,\ (t_1-t_2)\cos\theta + (t_1+t_2)\sin\theta\big).$$

Proof. It follows that for $(t_1,t_2) \in \ell^2_1$ that

$$\begin{aligned}
&\|\phi(t_1,t_2)\|_{(\infty,\theta)} \\
&= \|\big((t_1+t_2)\cos\theta + (t_2-t_1)\sin\theta,\ (t_1-t_2)\cos\theta + (t_1+t_2)\sin\theta\big)\|_{(\infty,\theta)} \\
&= \max\Big\{\big|\cos\theta\big((t_1+t_2)\cos\theta + (t_2-t_1)\sin\theta\big) + \sin\theta\big((t_1-t_2)\cos\theta + (t_1+t_2)\sin\theta\big)\big|, \\
&\qquad \big|\sin\theta\big((t_1+t_2)\cos\theta + (t_2-t_1)\sin\theta\big) - \cos\theta\big((t_1-t_2)\cos\theta + (t_1+t_2)\sin\theta\big)\big|\Big\} \\
&= \max\Big\{|t_1+t_2|,\ |t_1-t_2|\Big\} = |t_1| + |t_2| = \|(t_1,t_2)\|_1.
\end{aligned}$$

□

Note that $\{W_1, W_2\}$ is a basis for $\ell^2_{(\infty,\theta)}$. Let $(t,s) \in \ell^2_\infty$. We define $f_{(t,s)} \in {\ell^2_{(\infty,\theta)}}^*$ by

$$f_{(t,s)}(xW_1 + yW_2) := tx + sy.$$

The following characterizes the dual space of $\ell^2_{(\infty,\theta)}$.

Lemma 2.2. *Let $0 \le \theta \le \frac{\pi}{2}$. Then the following hold:*
(1) $\|f_{(t,s)}\| = \|(t,s)\|_\infty$;
(2) ${\ell^2_{(\infty,\theta)}}^* = \{f_{(t,s)} : (t,s) \in \ell^2_\infty\}$.

Proof. (1). It follows from Lemma 2.1 that

$$\begin{aligned}\|f_{(t,s)}\| &= \sup\{|f_{(t,s)}(xW_1 + yW_2)| : \|xW_1 + yW_2\|_{(\infty,\theta)} = 1\}\\ &= \sup\{|tx + sy| : \|xW_1 + yW_2\|_{(\infty,\theta)} = 1\}\\ &= \sup\{|tx + sy| : \|xe_1 + ye_2\|_1 = 1\}\\ &= \sup\{|t|,\ |s|\} = \|(t,s)\|_\infty.\end{aligned}$$

(2). ($\subseteq$). Let $f \in {\ell^2_{(\infty,\theta)}}^*$. Let $t = f(W_1)$ and $s = f(W_2)$. Thus, $f = f_{(t,s)}$.
($\supseteq$) is trivial. □

Let $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$. Let ϕ be the isometry in Lemma 2.1. Write

$$T(\phi(x_1,y_1),\ \phi(x_2,y_2)) = T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))W_1 + T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))W_2$$

for some $T_j \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)})$. For $j = 1, 2$, we define $S_j \in \mathcal{L}(^2\ \ell^2_1)$ by

$$S_j((x_1,y_1),\ (x_2,y_2)) = T_j(\phi(x_1,y_1),\ \phi(x_2,y_2)).$$

We also define $S_T \in \mathcal{L}(^2\ \ell^2_1 : \ell^2_1)$ by

$$S_T((x_1,y_1),\ (x_2,y_2)) = S_1((x_1,y_1),\ (x_2,y_2))e_1 + S_2((x_1,y_1),\ (x_2,y_2))e_2.$$

Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of the real or complex space ℓ_1 and $\{e_n^*\}_{n\in\mathbb{N}}$ the biorthogonal functionals associated with $\{e_n\}_{n\in\mathbb{N}}$. The following presents explicit formulas for the numerical radius and the norm of T for every $T \in \mathcal{L}(^n\ell_1 : \ell_1)$ and every $n \ge 2$.

Lemma 2.3. ([12]). *Let $n \ge 2$ and $T = \sum_{j\in\mathbb{N}} T_j e_j \in \mathcal{L}(^n\ell_1 : \ell_1)$ be such that*

$$T_j\Big(\sum_{i\in\mathbb{N}} x_i^{(1)} e_i, \cdots, \sum_{i\in\mathbb{N}} x_i^{(n)} e_i\Big) = \sum_{i_1,\ldots,i_n\in\mathbb{N}} a^{(j)}_{i_1\cdots i_n}\ x^{(1)}_{i_1}\cdots x^{(n)}_{i_n} \in \mathcal{L}(^n\ell_1)$$

for some $a^{(j)}_{i_1\cdots i_n} \in \mathbb{R}$. Then

$$\sup\Big\{\sum_{j\in\mathbb{N}} \Big|a^{(j)}_{i_1\cdots i_n}\Big| : (i_1,\ldots,i_n) \in \mathbb{N}^n\Big\} = v(T) = \|T\|.$$

Theorem 2.4. *Let $0 \le \theta \le \frac{\pi}{2}$. Let $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$. Then,*

$$\|T\|_{\mathcal{L}(^2\ \ell^2_{(\infty,\theta)}:\ell^2_{(\infty,\theta)})} = \|S_T\|_{\mathcal{L}(^2\ \ell^2_1:\ell^2_1)} = v(T) = v(S_T).$$

Proof. By Lemma 2.3, it suffices to show that $\|T\|_{\mathcal{L}(^2\ \ell^2_{(\infty,\theta)}:\ell^2_{(\infty,\theta)})} = \|S_T\|_{\mathcal{L}(^2\ \ell^2_1:\ell^2_1)}$ and $v(T) = v(S_T)$.

Claim. $\|T\|_{\mathcal{L}(^2\ \ell^2_{(\infty,\theta)}:\ell^2_{(\infty,\theta)})} = \|S_T\|_{\mathcal{L}(^2\ \ell^2_1:\ell^2_1)}$.

It follows form Lemma 2.1 that

$$\begin{aligned}
&\|T\|_{\mathcal{L}(^2\ \ell^2_{(\infty,\theta)}:\ell^2_{(\infty,\theta)})}\\
&\quad= \sup\{\|T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))W_1 + T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))W_2\|_{(\infty,\theta)} :\\
&\qquad\quad \|\phi(x_k,y_k)\|_{(\infty,\theta)} = 1,\ k=1,2\}\\
&\quad= \sup\{\|\phi(T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)),\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2)))\|_{(\infty,\theta)} :\\
&\qquad\quad \|\phi(x_k,y_k)\|_{(\infty,\theta)} = 1,\ k=1,2\}\\
&\quad= \sup\{\|(T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)),\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2)))\|_1 :\\
&\qquad\quad \|(x_k,y_k)\|_1 = 1,\ k=1,2\}\\
&\quad= \sup\{S_T((x_1,y_1),\ (x_2,y_2))\|_1 : \|(x_k,y_k)\|_1 = 1,\ k=1,2\}\\
&\quad= \|S_T\|_{\mathcal{L}(^2\ \ell^2_1:\ell^2_1)}.
\end{aligned}$$

Claim. $v(T) = v(S_T)$.

By Lemmas 2.1 and 2.2,

$$\begin{aligned}
v(T) &= \sup\{|f_{(t,s)}(T(\phi(x_1,y_1),\ \phi(x_2,y_2)))| : [f_{(t,s)},\ \phi(x_1,y_1),\ \phi(x_2,y_2)] \in \Pi(\ell^2_{(\infty,\theta)})\}\\
&= \sup\{|f_{(t,s)}(T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))W_1 + T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))W_2)| :\\
&\qquad [f_{(t,s)},\ \phi(x_1,y_1),\ \phi(x_2,y_2)] \in \Pi(\ell^2_{(\infty,\theta)})\}\\
&= \sup\{|t\ T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) + s\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| :\\
&\qquad [f_{(t,s)},\ \phi(x_1,y_1),\ \phi(x_2,y_2)] \in \Pi(\ell^2_{(\infty,\theta)})\}\\
&= \sup\{|t\ T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) + s\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| :\\
&\qquad [te_1^* + se_2^*,\ (x_1,y_1),\ (x_2,y_2)] \in \Pi(\ell^2_1)\}\\
&= \sup\{|(te_1^* + se_2^*)(S_T((x_1,y_1),\ (x_2,y_2)))| : [te_1^* + se_2^*,\ (x_1,y_1),\ (x_2,y_2)] \in \Pi(\ell^2_1)\}\\
&= v(S_T).
\end{aligned}$$

□

Let

$$\begin{aligned}
A_+ &= \Big\{(X,Y) \in S_{\ell^2_1} \times S_{\ell^2_1} : S_1(X,Y)S_2(X,Y) > 0\Big\},\\
A_- &= \Big\{(X,Y) \in S_{\ell^2_1} \times S_{\ell^2_1} : S_1(X,Y)S_2(X,Y) < 0\Big\},\\
B_1 &= \Big\{(X,Y) \in S_{\ell^2_1} \times S_{\ell^2_1} : S_1(X,Y) = 0\Big\},\\
B_2 &= \Big\{(X,Y) \in S_{\ell^2_1} \times S_{\ell^2_1} : S_2(X,Y) = 0\Big\}.
\end{aligned}$$

Note that

$$S_{\ell^2_1} \times S_{\ell^2_1} = A_+ \cup A_- \cup B_1 \cup B_2.$$

Let

$$\begin{aligned} W_+ = \Big\{ &\pm [f_{(1,1)}, \phi(\tilde{X}), \phi(\tilde{Y})] \in \Pi(\ell^2_{(\infty,\theta)}) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y, \\ &(X,Y) \in A_+ \cap \mathrm{Norm}(S_T)\Big\}, \\ W_- = \Big\{ &\pm [f_{(1,-1)}, \phi(\tilde{X}), \phi(\tilde{Y})] \in \Pi(\ell^2_{(\infty,\theta)}) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y, \\ &(X,Y) \in A_- \cap \mathrm{Norm}(S_T)\Big\}, \\ W_1 = \Big\{ &\pm [f_{(t,s)}, \phi(\tilde{X}), \phi(\tilde{Y})] \in \Pi(\ell^2_{(\infty,\theta)}) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y, \\ &(X,Y) \in B_1 \cap \mathrm{Norm}(S_T)\Big\}, \\ W_2 = \Big\{ &\pm [f_{(t,s)}, \phi(\tilde{X}), \phi(\tilde{Y})] \in \Pi(\ell^2_{(\infty,\theta)}) : \tilde{X} = X \text{ or } -X, \tilde{Y} = Y \text{ or } -Y, \\ &(X,Y) \in B_2 \cap \mathrm{Norm}(S_T)\Big\}. \end{aligned}$$

Note that W_+, W_-, W_1, W_2 are mutually disjoint.

We are in a position to classify $\mathrm{Nradius}(T)$ for every $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ in connection with $\mathrm{Norm}(S_T)$.

Theorem 2.5. *Let $0 \le \theta \le \frac{\pi}{2}$. Let $T \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})$ be such that $\|T\| = 1$ and*

$$T(\phi(x_1,y_1),\ \phi(x_2,y_2)) = T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))W_1 + T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))W_2$$

for some $T_j \in \mathcal{L}(^2\ \ell^2_{(\infty,\theta)})$. Then

$$\mathrm{Nradius}(T) = W_+ \cup W_- \cup W_1 \cup W_2.$$

Proof. ($\subseteq$). Let $X := [f_{(t,s)},\ \phi(x_1,y_1),\ \phi(x_2,y_2)] \in \mathrm{Nradius}(T)$. Without loss of generality we may assume that $t \ge 0$. Since $\|f_{(t,s)}\| = \|(t,s)\|_\infty = 1$, $t = 1$ or $|s| = 1$.

Case 1. $t = 1$. It follows that

$$\begin{aligned} (*)\ 1 = v(T) &= |f_{(1,s)}(T(\phi(x_1,y_1),\ \phi(x_2,y_2)))| \\ &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) + s\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &\le |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))| + |s|\ |T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &\le |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))| + |T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &= \|T(\phi(x_1,y_1),\ \phi(x_2,y_2))\| \le \|T\| = 1, \end{aligned}$$

which shows that $(\phi(x_1,y_1),\ \phi(x_2,y_2)) \in \mathrm{Norm}(T)$. Thus, $((x_1,y_1),\ (x_2,y_2)) \in \mathrm{Norm}(S_T)$.

Suppose that $((x_1,y_1),\ (x_2,y_2)) \in A_+$. By $(*)$,

$$\begin{aligned} 1 = v(T) &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) + T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) + s\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &= |S_1((x_1,y_1),\ (x_2,y_2)) + s\ S_2((x_1,y_1),\ (x_2,y_2))| \\ &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))| + |s||T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))|, \end{aligned}$$

which shows that $s = 1$. Hence, $X := [f_{(1,1)},\ \phi(x_1,y_1),\ \phi(x_2,y_2)] \in W_+$.

Suppose that $((x_1,y_1),\ (x_2,y_2)) \in A_-$. By $(*)$,

$$\begin{aligned} 1 = v(T) &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) - T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2)) + s\ T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))| \\ &= |S_1((x_1,y_1),\ (x_2,y_2)) + s\ S_2((x_1,y_1),\ (x_2,y_2))| \\ &= |T_1(\phi(x_1,y_1),\ \phi(x_2,y_2))| + |s||T_2(\phi(x_1,y_1),\ \phi(x_2,y_2))|, \end{aligned}$$

which shows that $s = -1$. Hence, $X = [f_{(1,-1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_-$.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in B_1$. By $(*)$,

$$1 = v(T) = |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| = |s||T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,$$

which shows that $|s| = 1$ and $S_1((x_1, y_1),\ (x_2, y_2)) = 0$. Hence,

$$X = [f_{(1,s)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_1.$$

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in B_2$. By $(*)$,

$$1 = v(T) = |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))|,$$

and $S_2((x_1, y_1),\ (x_2, y_2)) = 0$. which shows that $X = [f_{(1,s)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_2$. Therefore, $\text{Nradius}(T) \subseteq W_+ \cup W_- \cup W_1 \cup W_2$.

Case 2. $|s| = 1$. It follows that

$$\begin{aligned}(**)\ 1 = v(T) &= |f_{(t,s)}(T(\phi(x_1, y_1),\ \phi(x_2, y_2)))| \\ &= |t\ T_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) + s\ T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| + |s|\ |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &\leq |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| + |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= \|T(\phi(x_1, y_1),\ \phi(x_2, y_2))\| \leq \|T\| = 1,\end{aligned}$$

which shows that $(\phi(x_1, y_1),\ \phi(x_2, y_2)) \in \text{Norm}(T)$. Thus, $((x_1, y_1),\ (x_2, y_2)) \in \text{Norm}(S_T)$.

Subcase 1. $s = 1$. Suppose that $((x_1, y_1),\ (x_2, y_2)) \in A_+$. By $(**)$,

$$\begin{aligned}1 = v(T) &= |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) + T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t\ T_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) + T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| + |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,\end{aligned}$$

which shows that $t = 1$. Hence, $X = [f_{(1,1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_+$.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in A_-$. By $(**)$,

$$\begin{aligned}1 = v(T) &= |tT_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) - T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t\ T_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) + T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| + |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,\end{aligned}$$

which shows that $t = -1$. Hence, $X = [f_{(-1,1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_-$.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in B_1$. By $(**)$,

$$1 = v(T) = |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,$$

which shows that $X = [f_{(t,1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_1$.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in B_2$. By $(**)$,

$$1 = v(T) = |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| = |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))|,$$

which shows that $X = [f_{(t,1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_2$.

Therefore, $\text{Nradius}(T) \subseteq W_+ \cup W_- \cup W_1 \cup W_2$.

Subcase 2. $s = -1$. We claim that $((x_1, y_1),\ (x_2, y_2)) \notin A_+$. Indeed, by $(**)$,

$$\begin{aligned}1 = v(T) &= |t\ T_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) - T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| + |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,\end{aligned}$$

which shows that $t = -1$. This is a contradiction.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in A_-$. By $(**)$,

$$\begin{aligned}1 = v(T) &= |t\ T_1(\phi(x_1, y_1),\ \phi(x_2, y_2)) - T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))| \\ &= |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| + |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,\end{aligned}$$

which shows that $t = 1$. Hence, $X = [f_{(1,-1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_-$.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in B_1$. By $(**)$,

$$1 = v(T) = |T_2(\phi(x_1, y_1),\ \phi(x_2, y_2))|,$$

which shows that $X = [f_{(t,-1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_1$.

Suppose that $((x_1, y_1),\ (x_2, y_2)) \in B_2$. By $(**)$,

$$1 = v(T) = |t|\ |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))| = |T_1(\phi(x_1, y_1),\ \phi(x_2, y_2))|,$$

which shows that $X = [f_{(t,-1)},\ \phi(x_1, y_1),\ \phi(x_2, y_2)] \in W_2$.

Therefore, $\mathrm{Nradius}(T) \subseteq W_+ \cup W_- \cup W_1 \cup W_2$.

$(\supseteq)$. We claim that $W_+ \cup W_- \cup W_1 \cup W_2 \subseteq \mathrm{Nradius}(T)$.

Suppose that $[f_{(1,1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_+$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned}
1 = v(T) &\geq |f_{(1,1)}(T(\phi(\tilde{X}),\ \phi(\tilde{Y})))| \\
&= |T_1(\phi(\tilde{X}),\ \phi(\tilde{Y})) + T_2(\phi(\tilde{X}),\ \phi(\tilde{Y}))| \\
&= |T_1(\phi(X), -\phi(Y)) + T_2(\phi(X), -\phi(Y))| \\
&= |T_1(\phi(X), \phi(Y)) + T_2(\phi(X), \phi(Y))| \\
&= |S_1(X, Y) + S_2(X, Y)| \\
&= |S_1(X, Y)| + |S_2(X, Y)| \\
&= |T_1(\phi(X), \phi(Y))| + |T_2(\phi(X), \phi(Y))| \\
&= \|T(\phi(X), \phi(Y))\| = \|T\| = 1,
\end{aligned}$$

which shows that $[f_{(1,1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \mathrm{Nradius}(T)$. Hence, $W_+ \subseteq \mathrm{Nradius}(T)$.

Suppose that $[f_{(1,-1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_-$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned}
1 = v(T) &\geq |f_{(1,-1)}(T(\phi(\tilde{X}),\ \phi(\tilde{Y})))| \\
&= |T_1(\phi(\tilde{X}),\ \phi(\tilde{Y})) - T_2(\phi(\tilde{X}),\ \phi(\tilde{Y}))| \\
&= |T_1(\phi(X), -\phi(Y)) - T_2(\phi(X), -\phi(Y))| \\
&= |T_1(\phi(X), \phi(Y)) - T_2(\phi(X), \phi(Y))| \\
&= |S_1(X, Y) - S_2(X, Y)| \\
&= |S_1(X, Y)| + |S_2(X, Y)| \\
&= |T_1(\phi(X), \phi(Y))| + |T_2(\phi(X), \phi(Y))| \\
&= \|T(\phi(X), \phi(Y))\| = \|T\| = 1,
\end{aligned}$$

which shows that $[f_{(1,-1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \mathrm{Nradius}(T)$. Hence, $W_- \subseteq \mathrm{Nradius}(T)$.

Suppose that $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_1$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned}
1 = v(T) &\geq |f_{(t,s)}(T(\phi(\tilde{X}),\ \phi(\tilde{Y})))| \\
&= |t\ T_1(\phi(\tilde{X}),\ \phi(\tilde{Y})) + s\ T_2(\phi(\tilde{X}),\ \phi(\tilde{Y}))| \\
&= |t\ T_1(\phi(X),\ \phi(-Y)) + s\ T_2(\phi(X),\ \phi(-Y))| \\
&= |t\ S_1(X,Y) + s\ S_2(X,Y)| \\
&= |s|\ |S_2(X,Y)| \leq |T_2(\phi(X),\ \phi(Y))| \\
&\leq \|T(X,Y)\| = \|T\| = 1,
\end{aligned}$$

which shows that $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \text{Nradius}(T)$. Hence, $W_1 \subseteq \text{Nradius}(T)$.

Suppose that $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_2$.

Without loss of generality we may assume that $\tilde{X} = X$ and $\tilde{Y} = -Y$ since the proofs for the other cases are similar. It follows that

$$\begin{aligned}
1 = v(T) &\geq |f_{(t,s)}(T(\phi(\tilde{X}),\ \phi(\tilde{Y})))| \\
&= |t\ T_1(\phi(\tilde{X}),\ \phi(\tilde{Y})) + s\ T_2(\phi(\tilde{X}),\ \phi(\tilde{Y}))| \\
&= |t\ T_1(\phi(X),\ \phi(-Y)) + s\ T_2(\phi(X),\ \phi(-Y))| \\
&= |t\ S_1(X,Y) + s\ S_2(X,Y)| \\
&= |t|\ |S_1(X,Y)| \leq |T_1(\phi(X),\ \phi(Y))| \\
&\leq \|T(X,Y)\| = \|T\| = 1,
\end{aligned}$$

which shows that $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \text{Nradius}(T)$. Hence, $W_2 \subseteq \text{Nradius}(T)$.

Therefore, $W_+ \cup W_- \cup W_1 \cup W_2 \subseteq \text{Nradius}(T)$. This completes the proof. □

Theorem 2.6. *Let* $0 \leq \theta \leq \frac{\pi}{2}$. *Then,* $NA(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)})) = NRA(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

Proof. ($\subseteq$). Let $T \in \text{NA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

Let $(X,Y) \in \text{Norm}(T)$. Note that $S_T \in \text{NA}(\mathcal{L}(^2\ \ell^2_1 : \ell^2_1))$ and $(\phi^{-1}(X), \phi^{-1}(Y)) \in \text{Norm}(S_T)$.

Case 1. $(\phi^{-1}(X), \phi^{-1}(Y)) \in A_+$.

Note that $(\phi^{-1}(X), \phi^{-1}(Y)) \in A_+ \cap \text{Norm}(S_T)$. Choose $\tilde{X} \in \{\phi^{-1}(X), -\phi^{-1}(X)\}$ and $\tilde{Y} \in \{\phi^{-1}(Y), -\phi^{-1}(Y)\}$ such that $[f_{(1,1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_+$. By Theorem 2.5, $[f_{(1,1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \text{Nradius}(T)$. Thus, $T \in \text{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

Case 2. $(\phi^{-1}(X), \phi^{-1}(Y)) \in A_-$.

Note that $(\phi^{-1}(X), \phi^{-1}(Y)) \in A_- \cap \text{Norm}(S_T)$. Choose $\tilde{X} \in \{\phi^{-1}(X), -\phi^{-1}(X)\}$ and $\tilde{Y} \in \{\phi^{-1}(Y), -\phi^{-1}(Y)\}$ such that $[f_{(1,-1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_-$. By Theorem 2.5, $[f_{(1,-1)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \text{Nradius}(T)$. Thus, $T \in \text{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

Case 3. $(\phi^{-1}(X), \phi^{-1}(Y)) \in B_1$.

Note that $(\phi^{-1}(X), \phi^{-1}(Y)) \in B_1 \cap \text{Norm}(S_T)$. Choose $\tilde{X} \in \{\phi^{-1}(X), -\phi^{-1}(X)\}$ and $\tilde{Y} \in \{\phi^{-1}(Y), -\phi^{-1}(Y)\}$ such that $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_1$. By Theorem 2.5, $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \text{Nradius}(T)$. Thus, $T \in \text{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

Case 4. $(\phi^{-1}(X), \phi^{-1}(Y)) \in B_2$.

Note that $(\phi^{-1}(X), \phi^{-1}(Y)) \in B_2 \cap \text{Norm}(S_T)$. Choose $\tilde{X} \in \{\phi^{-1}(X), -\phi^{-1}(X)\}$ and $\tilde{Y} \in \{\phi^{-1}(Y), -\phi^{-1}(Y)\}$ such that $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in W_2$. By Theorem 2.5, $[f_{(t,s)},\ \phi(\tilde{X}),\ \phi(\tilde{Y})] \in \text{Nradius}(T)$. Thus, $T \in \text{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

($\supseteq$). Let $T \in \mathrm{NRA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$.

Then $v(T) = |f_{(t,s)}(T(Z_1, Z_2))|$ for some $[f_{(t,s)}, Z_1, Z_2] \in \Pi(\ell^2_{(\infty,\theta)})$. It follows that by Theorem 2.4,

$$\|T\| = v(T) = |f_{(t,s)}(T(Z_1, Z_2))| \leq \|T(Z_1, Z_2)\|_{(\infty,\theta)} \leq \|T\|\ \|Z_1\|_{(\infty,\theta)}\ \|Z_2\|_{(\infty,\theta)} = \|T\|,$$

which implies that $(Z_1, Z_2) \in \mathrm{Norm}(T)$ and $T \in \mathrm{NA}(\mathcal{L}(^2\ \ell^2_{(\infty,\theta)} : \ell^2_{(\infty,\theta)}))$. $\square$

References

[1] R. M. Aron, C. Finet and E. Werner, *Some remarks on norm-attaining n-linear forms,* Function spaces (Edwardsville, IL, 1994), 19–28, Lecture Notes in Pure and Appl. Math., **172**, Dekker, New York, 1995. `doi:10.1016/j.jfa.2005.09.010`.

[2] E. Bishop and R. Phelps, *A proof that every Banach space is subreflexive,* Bull. Amer. Math. Soc. **67** (1961), 97–98, `doi:10.1090/s0002-9904-1961-10514-4`.

[3] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. (2) **54** (1996), 135–147. `doi:10.1112/jlms/54.1.135`.

[4] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London 1999, `doi:10.1007/978-14471-0869-6`.

[5] M. Jiménez Sevilla and R. Payá, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces,* Studia Math. **127** (1998), 99–112, `doi:10.4064/sm-127-2-99-112`.

[6] Y. S. Choi, D. Garcia, S. G. Kim and M. Maestre, *Norm or numerical radius attaining polynomial on* $\mathcal{C}(K)$, J. Math. Anal. Appl. **295**(2004), 80–96, `doi:10.1016/j.jmaa.2004.03.005`.

[7] S. G. Kim, *The norming set of a bilinear form on* ℓ^2_∞, Comment. Math. **60** (1-2) (2020), 37–63, `doi:10.14708/cm.v60i1-2.70715`.

[8] S. G. Kim, *The norming set of a polynomial in* $\mathcal{P}(^2\ell^2_\infty)$, Honam Math. J. **42** (3) (2020), 569–576, `doi:10.583/HMJ.2020.42.3.569`.

[9] S. G. Kim, *The norming set of a symmetric bilinear form on the plane with the supremum norm,* Mat. Stud. **55** (2) (2021), 171–180, `doi:10.30970/ms.55.2.171-180`.

[10] S. G. Kim, *The norming set of a symmetric 3-linear form on the plane with the* ℓ_1*-norm,* New Zealand J. Math. **51** (2021), 95–108, `doi:10.53730/177`.

[11] S. G. Kim, *Numerical radius points of* $\mathcal{L}(^m\ell^n_\infty : \ell^n_\infty)$*,* New Zealand J. Math. **53** (2022), 1–10, `doi:10.53733/179`.

[12] S. G. Kim, $NA(\mathcal{L}(^n l_1 : l_1)) = NRA(\mathcal{L}(^n\ell_1 : \ell_1))$, Acta Sci. Math. (Szeged) **88** (3-4) (2022), 769–775, `doi:10.1007/s44146-022-00048-5`.

Sung Guen Kim: `sgk317@knu.ac.kr`
Department of Mathematics, Kyungpook National University, Daegu, Republic of Korea

Chang Yeol Lee: `cy435@naver.com`
Department of Mathematics, Kyungpook National University, Daegu, Republic of Korea