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ESSENTIAL DESCENT SPECTRUM EQUALITY

MBARK ABKARI, HAMID BOUA, AND ABDELAZIZ TAJMOUATI

Abstract. A bounded operator T in a Banach space X is said to satisfy the essential descent spectrum equality, if the descent spectrum of T coincides with the essential descent spectrum of T. In this note, we give some conditions under which the equality $\sigma_{desc}(T) = \sigma^e_{desc}(T)$ holds for T.

1. Introduction

Throughout this paper, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. Let $T \in \mathcal{B}(X)$, we denote the adjoint of T, the range of T, the kernel of T, the resolvent set of T, the spectrum of T and the surjective spectrum of T by T^*, $R(T)$, $N(T)$, $\rho(T)$, $\sigma(T)$ and $\sigma_{su}(T)$ respectively.

An operator $T \in \mathcal{B}(X)$ is called semi-regular if $R(T)$ is closed and $N(T^n) \subseteq R(T)$ for every positive integer n. The operator $T \in \mathcal{B}(X)$ is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$ (for brevity, T has the SVEP at λ_0), if for every neighborhood $\mathcal{U}$ of λ_0, the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ is the constant function $f \equiv 0$. For an arbitrary operator $T \in \mathcal{B}(X)$, let $\mathcal{S}(T) = \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$. Note that $\mathcal{S}(T)$ is an open of the complex plane and is contained in the interior of the point spectrum $\sigma_p(T)$. The operator T is said to have the SVEP if $\mathcal{S}(T)$ is empty.

For $T \in \mathcal{B}(X)$, the local resolvent set $\rho_T(x)$ of T at the point $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $\mathcal{U}_\lambda$ of λ and an analytic function $f : \mathcal{U}_\lambda \to X$ such that $(T - \mu)f(\mu) = x$ for all $\mu \in \mathcal{U}_\lambda$. The local spectrum $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$. The local analytic solution occurring in the definition of the local resolvent set are unique for all $x \in X$ if and only if T has the SVEP.

A bounded linear operator $T \in \mathcal{B}(X)$ on a complex Banach space X is said to be decomposable if every open cover $\mathbb{C} = U \cup V$ of the complex plane $\mathbb{C}$ by two open sets U and V generates a splitting of the spectrum $\sigma(T)$ and the latter generates a decomposition of X, in the sense that there exists Y and Z which are closed T-invariant subspaces of X such that $\sigma(T_{|Y}) \subseteq U$, $\sigma(T_{|Z}) \subseteq V$, and $X = Y + Z$.

A bounded linear operator T on a complex Banach space X has the decomposition property (δ) if $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover U, V of $\mathbb{C}$, where $\mathcal{X}_T(F)$ is the vector space of all elements $x \in X$ for which there exists an analytic function $f : \mathbb{C}\backslash F \to X$ such that $(T - \mu)f(\mu) = x$, for all $\mu \in \mathbb{C}\backslash F$.

We define and denotes the descent of a bounded linear operator T by $d(T) = \min\{q : R(T^q) = R(T^{q+1})\}$, if no such q exists, we write $d(T) = \infty$, see [1], [8] and [9]. Also, we denote and define the descent spectrum of T by $\sigma_{desc}(T) = \{\lambda \in \mathbb{C} : d(\lambda - T) = \infty\}$.

Now we consider for $T \in \mathcal{B}(X)$ a decreasing sequence $c_n(T) := \dim(R(T^n)/R(T^{n+1}))$, $n \in \mathbb{N}$ (see [6]). Following M. Mbekhta and M. Müller [10], we say that T has a finite essential descent if $d_e(T) := \inf\{n \in \mathbb{N} : c_n(T) < \infty\}$ is finite, while we convey that the infimum over the empty set is equal to infinity. The latter class of operators contains every operator T with finite descent. In general $\sigma^e_{desc}(T) \subseteq \sigma_{desc}(T)$, where $\sigma^e_{desc}(T)$

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denotes the essential descent spectrum of T, also, the last inequality can be strict. Indeed, we take T the unilateral right shift operator T, according to [5], $\sigma_{desc}(T)$ contains strictly the closed unit disk, while $\sigma^e_{desc}(T)$ is contained in the unit circle.

In [4] Olfa Bel Hadj Fredj has proved that, if $T \in \mathcal{B}(X)$ with a spectrum $\sigma(T)$ at most countable, then $\sigma_{desc}(T) = \sigma^e_{desc}(T)$. It is easy to construct an operator T satisfying the essential descent spectrum equality such that $\sigma(T)$ is uncountable. For example, let H be the Hilbert space $\ell^2(\mathbb{N})$ provided by the canonical basis $\{e_1, e_2, ...\}$ and let $T \in \mathcal{B}(H)$ be defined as $T(x_1, x_2, ...) = (\frac{x_1}{2}, 0, x_2, x_3, ...)$, $(x_n)_n \in \ell^2(\mathbb{N})$. From [2], we have $\sigma_{su}(T) = \Gamma \cup \{\frac{1}{2}\}$, where Γ denotes the unit circle. Then $\sigma_{su}(T)$ is of empty interior. According to [4], $\sigma_{desc}(T)\backslash\sigma^e_{desc}(T)$ is open. Since $\sigma_{desc}(T)\backslash\sigma^e_{desc}(T) \subseteq \sigma_{su}(T)$, we have $\sigma_{desc}(T) = \sigma^e_{desc}(T)$. Motivated by the previous example, we wil study the following question:

Question 1. *Let $T \in \mathcal{B}(X)$. If $\sigma(T)$ is uncountable, under which conditions on T does $\sigma_{desc}(T) = \sigma^e_{desc}(T)$?*

2. Main results

We start by the following results:

Theorem 2.1. [4] Let $T \in \mathcal{B}(X)$ be an operator for which $d_e(T)$ is finite. Then there exists $\delta > 0$ such that for $0 < |\lambda| < \delta$ and $p := p(T)$, we have the following assertions:

(1) $T - \lambda$ is semi regular;
(2) $\dim N(T-\lambda)^n = n \dim N(T^{p+1})/N(T^p)$ for all $n \in \mathbb{N}$;
(3) $\text{codim} R(T-\lambda)^n = n \dim R(T^p)/R(T^{p+1})$ for all $n \in \mathbb{N}$.

Corollary 2.2. [4] Let $T \in \mathcal{B}(X)$ be an operator of finite descent d. Then there exists $\delta > 0$ such that the following assertions hold for $0 < |\lambda| < \delta$:

(1) $T - \lambda$ is onto;
(2) $\dim N(T-\lambda) = \dim N(T^{d+1})/N(T^d)$.

We have the following theorem.

Theorem 2.3. *Let $T \in \mathcal{B}(X)$, then*

$$\sigma_{desc}(T) = \sigma^e_{desc}(T) \cup \overline{\mathcal{S}(T^*)}.$$

Proof. If $\lambda \notin \sigma_{desc}(T)$, then $\lambda \notin \sigma^e_{desc}(T)$. From [1, Theorem 3.8], T^* has the SVEP at λ, which establish $\sigma^e_{desc}(T) \cup \mathcal{S}(T^*) \subseteq \sigma_{desc}(T)$, then it follows that $\sigma^e_{desc}(T) \cup \overline{\mathcal{S}(T^*)} = \overline{\sigma^e_{desc}(T) \cup \mathcal{S}(T^*)} \subseteq \overline{\sigma_{desc}(T)} = \sigma_{desc}(T)$. For the other inclusion, let λ be a complex number such that $T - \lambda$ has finite essential descent and $\lambda \notin \overline{\mathcal{S}(T^*)}$. According to theorem 2.2, there is $\delta > 0$, such that for $0 < |\lambda - \mu| < \delta$ and $p \in \mathbb{N}$, the operator $T - \mu$ is semi-regular and codim $R(T-\mu) = \dim R(T-\lambda)^p/R(T-\lambda)^{p+1}$. Let $D^*(\lambda, \delta) = \{\mu : 0 < |\lambda - \mu| < \delta\}$. But since $\lambda \notin \overline{\mathcal{S}(T^*)}$, $D^*(\lambda,\delta)\backslash\mathcal{S}(T^*)$ is non-empty, it follows that there exists $\mu_0 \in D^*(\lambda, \delta)$ such that $T - \mu_0$ is semi-regular and T^* has the SVEP at μ_0. From [1, Theorem 3.17], $T - \mu_0$ is of finite descent. Also by Corollary 2.3, there exists $\mu_1 \in D^*(\lambda, \delta)$ such that $T - \mu_1$ is surjective. Therefore $\text{codim} R(T-\mu) = \dim R(T-\lambda)^p/R(T-\lambda)^{p+1} = 0$. It follows that $R(T-\lambda)^p = R(T-\lambda)^{p+1}$, which yields $\lambda \notin \sigma_{desc}(T)$. □

Corollary 2.4. *Let $T \in \mathcal{B}(X)$. If T^* has the SVEP, then $\sigma_{desc}(T) = \sigma^e_{desc}(T)$.*

Example 2.5. We consider the forward bilateral shift T on $\ell^2(\mathbb{Z})$ defined by $Te_n = e_{n+1}$ for all $n \in \mathbb{Z}$. It is well known that

$$\sigma(T^*) = \sigma(T) = \{\lambda \ : \ |\lambda| = 1\}.$$

As the interior of $\sigma(T^*)$ is empty. Therefore, taking into account that $S(T^*)$ is open and contained in the interior of $\sigma(T^*)$, we conclude that $S(T^*) = \emptyset$. Hence, T^* admits the SVEP, and by the last corollary, we conclude that

$$\sigma_{desc}(T) = \sigma^e_{desc}(T).$$

Example 2.6. Let $T \in \mathcal{B}(X)$ and T is a compact operator. It is well known that $\sigma(T)$ is countable so, T^* is compact and $\sigma(T^*)$ is countable. Therefore the interior of $\sigma(T^*)$ is empty. It follows that T^* has the SVEP, and by corollary 2.4, we conclude that $\sigma_{desc}(T) = \sigma^e_{desc}(T)$.

Remark 2.7. If $T \in \mathcal{B}(X)$ and T is a quasinilpotent operator, then we have: $\sigma_{desc}(T) = \sigma^e_{desc}(T)$.

Now, in order to give a corollary about multipliers, we recall that a mapping $T : \mathcal{A} \to \mathcal{A}$ on a commutative complex Banach algebra $\mathcal{A}$ is said to be a multiplier if

$$u(Tv) = (Tu)v \text{ for all } u, v \in \mathcal{A}.$$

Any element $a \in \mathcal{A}$ provides an example, since, if $L_a : \mathcal{A} \to \mathcal{A}$ denotes the mapping given by $L_a(u) := au$ for all $u \in \mathcal{A}$, then the multiplication operator L_a is clearly a multiplier on $\mathcal{A}$. The set of all multipliers of $\mathcal{A}$ is denoted by $M(\mathcal{A})$. We recall that an algebra $\mathcal{A}$ is said to be semi-prime if $\{0\}$ is the only two-sided ideal J for which $J^2 = 0$.

Corollary 2.8. *Let $T \in M(\mathcal{A})$ be a multiplier on a semi-prime, regular and commutative Banach algebra $\mathcal{A}$ then*

$$\sigma_{desc}(T) = \sigma^e_{desc}(T).$$

Proof. From [1, Corollary 6.52], then T^* has the SVEP. Therefore by corollary 2.4, we have $\sigma_{desc}(T) = \sigma^e_{desc}(T)$. □

A bounded operator $T \in \mathcal{B}(X)$ is said to be supercyclic if for some $x \in X$, the homogeneous orbit $\mathbb{C}.O(x, T) = \{\lambda T^n(x) : n \in \mathbb{N}\}$ is dense in X.

Corollary 2.9. *Let $T \in \mathcal{B}(X)$ be supercyclic operator. Then*

$$\sigma_{desc}(T) = \sigma^e_{desc}(T).$$

Proof. If $T \in \mathcal{B}(X)$ is supercyclic, according to [3, Proposition 1.26], either $\sigma_p(T^*) = \emptyset$ or $\sigma_p(T^*) = \{\lambda\}$ for some $\lambda \neq 0$, hence $\mathrm{int}(\sigma_p(T^*)) = \emptyset$, so $S(T^*) = \emptyset$, from corollary 2.4, we have $\sigma_{desc}(T) = \sigma^e_{desc}(T)$. □

Corollary 2.10. *Let $T \in \mathcal{B}(X)$. If T satisfies one of the following properties:*

(1) *T is decomposable,*
(2) *T has the property (δ),*

then

$$\sigma_{desc}(T) = \sigma^e_{desc}(T).$$

Proof. If T is decomposable or T has the property (δ). We know from [7, Theorem 1.2.7] and [7, Theorem 2.5.5] that T^* has the SVEP, by corollary 2.4, we have $\sigma_{desc}(T) = \sigma^e_{desc}(T)$. □

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Mbark Abkari: mbark.abkari@gmail.com
Faculty of Sciences Dhar Al Mahraz, Laboratory of Mathematical Sciences and Applications, Fez, Morocco

Hamid Boua: hamid12boua@yahoo.com
Multidisciplinary Faculty, Laboratory Ibn Al Banna, Nador, Morocco

Abdelaziz Tajmouati: abdelaziz.tajmouati@usmba.ac.ma
Sidi Mohamed Ben Abdellah University, Fez, Morocco