Methods of Functional Analysis and Topology
Vol. 30 (2024), no. 3-4, pp. 105–107
doi.org/10.31392/MFAT-npu26_3-4.2024.02

CHARACTERIZATION OF COMPACT SETS IN THE COMPLEX PLANE WITH SPECIFIC BOUNDARY CONDITIONS AND AN APPLICATION TO THE SPECTRUM OF OPERATORS VERIFYING ISOMETRIC CONDITIONS

MOHAMED AMINE AOUICHAOUI

Abstract. This paper delves into the investigation of compact sets within the complex plane under a boundary constraint. Specifically, we focus on scenarios where a compact set A is enclosed by a curve ∂A that lies within the boundary of the augmented unit disk $\partial\mathbb{D}$, including the origin. The main goal is to establish a theorem that characterizes the possible configurations of such sets. By interweaving the principles of topology and operator theory, this study not only enhances our comprehension of compact sets under specialized boundary conditions but also underscores a practical implication in the realm of operator theory. This connection is particularly evident in the examination of the spectrum of operators that meet specific isometric conditions.

1. Introduction

In this paper, we adopt the notation $\mathbb{D}$ to depaper the open unit disc within the complex plane. We further utilize $\overline{\mathbb{D}}$ and $\partial\mathbb{D}$ to respectively represent the closure and the boundary of $\mathbb{D}$. In the context of a complex Hilbert space $\mathcal{H}$, equipped with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$, we use $\mathcal{B}(\mathcal{H})$ to depaper the algebra encompassing all bounded linear operators defined upon $\mathcal{H}$. Moreover, we use $\sigma(T)$ and $\sigma_{ap}(T)$ to depaper the spectrum and the approximate point spectrum of T, respectively.

In the realm of mathematical analysis, the interplay between compact sets and their boundary constraints has long captured the attention of researchers due to its profound implications in diverse mathematical contexts. The theorem proposed in this paper takes a significant stride towards unraveling the intricate relationships that emerge when studying compact sets within the complex plane under distinctive boundary conditions. By examining the constraints that characterize a compact set A whose boundary is included in $\partial\mathbb{D}\cup\{0\}$, the theorem establishes an elegant dichotomy: either A is entirely contained within $\partial\mathbb{D}\cup\{0\}$, or it extends to coincide with the entire unit disc.

Theorem 1.1. *Consider A as a compact set in $\mathbb{C}$ such that its boundary ∂A satisfies $\partial A\subseteq\partial\mathbb{D}\cup\{0\}$. Then, either $A\subseteq\partial\mathbb{D}\cup\{0\}$, or $A=\overline{\mathbb{D}}$.*

In the pursuit of elucidating the theorem's significance and applicability, this research paper will not only present one, but two elegant and illuminating proofs. To firmly establish the theorem's practical relevance, a compelling application within the realm of operator theory will be showcased. This application will artfully demonstrate how the theorem's principles find resonance in real-world operator scenarios, specifically those characterized by isometric conditions.

In passing, it is worth noting that several results concerning the spectrum of such operators (see, for example, [7, Corollary 3.1] and [2, Proposition 4.1]) are derived by combining a result established regarding the approximate point spectrum and results

2020 *Mathematics Subject Classification.* 30E25, 54C10, 54D30, 47A10.

Keywords. Compact sets, complex plane, boundary constraints, unit disk, spectrum of isometric operators.

from these two papers [5, 6]. I have previously employed this line of reasoning in my paper [3] while transitioning from Theorem 4.3 to Corollary 4.5, and similarly in another work [4] in the context of Theorem 4.6. Sincerely, this deduction might not jump to the eye. Therefore, I am inclined to compose this paper and present a simpler alternative method to arrive at these consequences. Thus, the reasoning provided deserves to be shared with the mathematical community.

2. Proof of Theorem 1.1 and an application

Proof of Theorem 1.1. First, we show that $A \subseteq \overline{\mathbb{D}}$. The mapping

$$A \longrightarrow \mathbb{R}$$
$$z \longmapsto |z|$$

is continuous; it attains its maximum at a point w. This point w is not an interior point of A, as otherwise, A would contain a certain ball $B(w, \varepsilon)$. However, in this case, the point $u = w + \varepsilon \frac{w}{|w|} \in A$ would have a norm greater than that of w, which contradicts the maximality of w. Consequently, w lies on the boundary of A, and therefore, $A \subseteq \overline{\mathbb{D}}$.

Now, we will demonstrate that either A is the closed unit disk or is contained within the unit circle union $\{0\}$. The aim is to provide two ways of doing that.

(1) First proof: There are two cases to distinguish:
   (a) If A lacks interior points, then A is contained within the unit circle union $\{0\}$.
   (b) Suppose A possesses an interior point z. We will prove that A is the closed unit disk. Otherwise, we would find $b \notin A$ with $|b| \leq 1$. Since $b \notin A$, there exists a ball $B(b, \delta) \subseteq A^c$. If necessary, we can adjust the choice of b or z so that 0 does not lie on the segment $[z, b]$. Thus, the boundary of A contains a portion within the unit disk other than $\{0\}$, which contradicts the initial assumption. This latter deduction requires improvement for greater reader conviction: We consider the segment $z + t(b - z); t \in [0, 1]$, and we depaper $t_0 := \sup\{t \mid z + t(b - z) \in A\}$. t_0 must necessarily lie in the interval $(0, 1)$. The point $z + t_0(b - z)$ belongs to the boundary of A and furthermore, $0 < |z + t_0(b - z)| < \max(|z|, |b|) \leq 1$, which contradicts the assumption regarding the boundary of A.

(2) Second proof: Set

$$U := \{z \in \mathbb{C} \,:\, 1 > |z| > 0\}.$$

   One can see that the set $A \cap U$ is both a closed and an open subset of U. Since U is connected, there are two possible scenarios: either A contains U, in which case A is the closed unit disk, or A is contained within the complement of U. In the latter case, A is contained within the unit circle union $\{0\}$. This concludes the proof.

□

The forthcoming result is a direct application of Theorem 1.1.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ such that $\sigma_{ap}(T) \subseteq \partial\mathbb{D} \cup \{0\}$. Then $\sigma(T) \subseteq \partial\mathbb{D} \cup \{0\}$ or $\sigma(T) = \overline{\mathbb{D}}$.*

Proof. Given that $\sigma(T)$ is always compact (according to [1, Theorem 6.10]) and $\partial\sigma(T) \subseteq \sigma_{ap}(T)$ (as per [1, Theorem 6.18]), the result can be directly inferred from Theorem 1.1. □

Conclusion

In conclusion, Theorem 2.1 represents a significant advancement in understanding the spectrum of operators. By employing techniques from topology, this work provides an intuitive and accessible approach that contrasts with the intricate arguments found in the cited works [5, 6]. Notably, the proposed method demonstrates how information about the spectrum of an operator (such as one satisfying certain isometric conditions) can be derived from hypotheses concerning its approximate point spectrum.

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Mohamed Amine Aouichaoui: amine.aouichaoui@fsm.rnu.tn, amine.aouichaoui@fsm.u-monastir.tn
Department of Mathematics, University of Monastir, Faculty of Sciences of Monastir, Monastir, 5019, Tunisia