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ON THE STABILITY OF RADICAL FUNCTIONAL EQUATION IN MODULAR SPACE

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Abstract. In this work, we prove the generalised Hyer Ulam stability of the following functional equation

$$\phi(x)+\phi(y)+\phi(z)=q\phi\left(\sqrt[s]{\frac{x^s+y^s+z^s}{q}}\right), \qquad |q|\leq 1 \tag{0.1}$$

and s is an odd integer such that $s\geq 3$, in modular space, using the direct method, and the fixed point theorem.

1. Introduction and preliminaries

The modulars and modular spaces as generalisation of normed space were by Nakano [10]. Since the 1950s, a number of eminent mathematicians [1, 6, 9, 11, 15] have focused extensively on it. Two applications of modular and modular spaces are found in [8, 7, 11, 15] interpolation theory and Orlicz spaces.

We begin by introducing the definition and some properties of modular and modular spaces.

Definition 1.1. Let Y be an arbitrary vector space. A functional $\rho : Y \to [0,\infty)$ is called a modular if for arbitrary $u, v \in Y$:

(1) $\rho(u)=0$ if and only if $u=0$.
(2) $\rho(\alpha u)=\rho(u)$ for every scalar α with $|\alpha|=1$.
(3) $\rho(\alpha u+\beta v)\leq \rho(u)+\rho(v)$ if and only if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$.
If (3) is replaced by:
(4) $\rho(\alpha u+\beta v)\leq \alpha\rho(u)+\beta\rho(v)$ if and only if $\alpha+\beta=1$ and $\alpha,\beta\geq 0$, then we say that ρ is a convex modular.
(5) It is said that the modular ρ has the Fatou property if and only if $\rho(x)\leq \liminf_{n\to\infty}\rho(x_n)$ whenever $x=\rho-\lim_{n\to\infty}x_n$.

A modular ρ defines a corresponding modular space, i.e., the vector space Y_ρ given by:

$$Y_\rho=\{u\in Y:\rho(\lambda u)\to 0 \text{ as } \lambda\to 0\}.$$

A function modular is said to satisfy the Δ_k-condition if there exist $\tau_k>0$ such that $\rho(ku)\leq \tau_k\rho(u)$ for all $u\in Y_\rho$.

Definition 1.2. Let $\{u_n\}$ and u be in Y_ρ. Then:

(1) The sequence $\{u_n\}$, with $u_n\in Y_\rho$, is ρ-convergent to u and write: $u_n \overset{\rho}{\to} u$ if $\rho(u_n-u)\to 0$ as $n\to\infty$.
(2) The sequence $\{u_n\}$, with $u_n\in Y_\rho$, is called ρ-Cauchy if $\rho(u_n-u_m)\to 0$ as $n,m\to\infty$.
(3) Y_ρ is called ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.

Proposition 1.3. *In modular space,*

- *If* $u_n \overset{\rho}{\to} u$ *and* a *is a constant vector, then* $u_n+a \overset{\rho}{\to} u+a$.

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- *If $u_n \xrightarrow{\rho} u$ and $v_n \xrightarrow{\rho} v$ then $\alpha u_n + \beta v_n \xrightarrow{\rho} \alpha u + \beta v$, where $\alpha + \beta \leq 1$ and $\alpha, \beta \geq 0$.*

Remark 1.4. Note that $\rho(u)$ is an increasing function, for all $u \in X$. Suppose $0 < a < b$, then property (4) of Definition 1.1 with $v = 0$ shows that $\rho(au) = \rho\left(\frac{a}{b}bu\right) \leq \rho(bu)$ for all $u \in Y$. Morever, if ρ is a convexe modular on Y and $|\alpha| \leq 1$, then $\rho(\alpha u) \leq \alpha\rho(u)$.

In general, if $\lambda_i \geq 0$, $i = 1, \ldots, n$ and $\sum \lambda_i, = 1$ then
$\rho(\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n) \leq \lambda_1 \rho(u_1) + \lambda_2 \rho(u_2) + \cdots + \lambda_n \rho(u_n)$.

If $\{u_n\}$ is ρ-convergent to u, then $\{cu_n\}$ is ρ-convergent to cu, where $|c| \leq 1$. But the ρ-convergent of a sequence $\{u_n\}$ to u does not imply that $\{\alpha u_n\}$ is ρ-convergent to αu_n for scalars α with $|\alpha| > 1$.

If ρ is a convex modular satisfying Δ_k condition with $0 < \tau_k < k$, then
$\rho(u) \leq \tau_k \rho(\frac{1}{k}u) \leq \frac{\tau_k}{k}\rho(u)$ for all u.

Hence $\rho = 0$. Consequently, we must have $\tau_k \geq k$ if ρ is convex modular.

In many settings, the functional equations are essential for studying stability issues.

The work on stability concerns began with Ulam's initial doubts about the stability of group homomorphisms [14]. Hyers [4] used Banach spaces to tackle this stability issue by taking in to account Cauchy's functional equation.

Aoki [2] extented the work of Hyers by assuming an infinite Cauchy difference. The additive mapping work was given by Rassias [12], and Gavruta [3] presented similar results.

In this work, we study the stability of the following functional equation:

$$f(x) + f(y) + f(z) = qf\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}}\right); \qquad |q| \leq 1$$

in modular space, using the direct method and fixed point theorem.

Proposition 1.5. *Let $\phi : \mathbb{R} \to Y$ be a mapping satisfying (0.1), where Y is a vector space. Then ϕ confirm: $\phi(\sqrt[s]{x^s + y^s}) = \phi(x) + \phi(y), \forall x, y \in \mathbb{R}$.*

Proof. Letting $x = y = z = 0$ in (0.1), we get: $3\phi(0) = q\phi(0)$.
Hence $\phi(0) = 0$. Letting $z = 0$ and $y = -x$ in (0.1), we obtain:

$$\phi(x) + \phi(-x) = 0,$$

Hence

$$\phi(-x) = -\phi(x)$$

Letting $z = -\sqrt[s]{x^s + y^s}$, we get

$$\phi(x) + \phi(y) - \phi\left(\sqrt[s]{x^s + y^s}\right) = 0,$$

then

$$\phi\left(\sqrt[s]{x^s + y^s}\right) = \phi(x) + \phi(y).$$

□

2. Stability of radical function equation (0.1) in modular space satisfying Δ_2-condition

Theorem 2.1. *Let Y_ρ be a ρ-complete convex modular space satisfying Δ_2-condition withτ and $\phi : \mathbb{R} \longrightarrow Y_\rho$ an old mapping such that:*

$$\rho\left(\phi(x) + \phi(y) + \phi(z) - q\phi\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}}\right)\right) \leq \alpha(x, y, z) \tag{2.2}$$

for all $x, y, z \in \mathbb{R}$ and all integer q with $|q| \leq 1$, and $\alpha : \mathbb{R}^3 \to [0, \infty)$ is a mapping such that:

$$\lim_{n\to\infty} \tau^n \alpha\left(\frac{x}{2^{n/s}}, \frac{y}{2^{n/s}}, \frac{z}{2^{n/s}}\right) = 0 \text{ and } \sum_{j=1}^{\infty} \left(\frac{\tau^2}{2}\right)^j \alpha\left(\frac{x}{2^{j/s}}, \frac{x}{2^{j/s}}, \frac{-x}{2^{\frac{j-1}{s}}}\right) < \infty \tag{2.3}$$

for all $x, y, z \in \mathbb{R}$. Then there exists a unique radical mapping $A : \mathbb{R} \to Y_\rho$ satisfying (0.1) and

$$\rho(\phi(x) - A(x)) \leq \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{\tau^2}{2}\right)^j \alpha\left(\frac{x}{2^{j/s}}, \frac{x}{2^{j/s}}, \frac{-x}{2^{\frac{j-1}{s}}}\right), x \in \mathbb{R}. \tag{2.4}$$

Proof. Putting $y = x$ and $z = -2^{1/s}x$ in (2.2), we get:

$$\rho\left(2\phi(x) - \phi\left(2^{1/s}x\right)\right) \leq \alpha\left(x, x, -2^{1/s}x\right).$$

Then

$$\rho\left(2\phi\left(\frac{x}{2^{1/s}}\right) - \phi(x)\right) \leq \alpha\left(\frac{x}{2^{1/s}}, \frac{x}{2^{1/s}}, -x\right).$$

Now, we have:

$$\begin{aligned} \rho\left(\phi(x) - 2^n\phi\left(\frac{x}{2^{n/s}}\right)\right) &= \rho\left(\sum_{j=1}^{n} \frac{1}{2^j}\left(2^{2j-1}\phi\left(\frac{x}{2^{\frac{j-1}{s}}}\right) - 2^{2j}\phi\left(\frac{x}{2^{j/s}}\right)\right)\right) \\ &\leq \frac{1}{\tau} \sum_{j=1}^{n} \left(\frac{\tau^2}{2}\right)^j \alpha\left(\frac{x}{2^{j/s}}, \frac{x}{2^{j/s}}, -\frac{x}{2^{\frac{j-1}{s}}}\right). \end{aligned}$$

Let m, n be positives integers such that $n > m$, we have:

$$\begin{aligned} \rho\left(2^n\phi\left(\frac{x}{2^{n/s}}\right) - 2^m\phi\left(\frac{x}{2^{m/s}}\right)\right) &\leq \tau^m \rho\left(2^{n-m}\phi\left(\frac{x}{2^{\frac{n-m}{s}} \cdot 2^{\frac{m}{s}}}\right) - \phi\left(\frac{x}{2^{\frac{m}{s}}}\right)\right) \\ &\leq \tau^{m-1} \sum_{j=1}^{n-m} \left(\frac{\tau^2}{2}\right)^j \alpha\left(\frac{x}{2^{\frac{m+j}{s}}}, \frac{x}{2^{\frac{m+j}{s}}}, -\frac{x}{2^{\frac{m+j-1}{s}}}\right) \\ &= \frac{1}{\tau}\left(\frac{2}{\tau}\right)^m \sum_{j=m+1}^{n} \left(\frac{\tau^2}{2}\right)^j \alpha\left(\frac{x}{2^{j/s}}, \frac{x}{2^{j/s}}, -\frac{x}{2^{\frac{j-1}{s}}}\right) \\ &\to 0 \text{ as } m \to \infty \end{aligned}$$

for all $x \in \mathbb{R}$.

Then $\left\{2^n\phi\left(\frac{x}{2^{n/s}}\right)\right\}$ is ρ-Cauchy sequence in Y_ρ which is ρ-complete. So, we can define a mapping $A : \mathbb{R} \to Y_\rho$ as: $A(x) = \rho - \lim_{n\to\infty} 2^n\phi\left(\frac{x}{2^{n/s}}\right)$, i.e. $\lim_{n\to\infty} \rho\left(2^n\phi\left(\frac{x}{2^{n/s}}\right) - A(x)\right) = 0$. Now, we prove the estimation (2.4), we have:

$$\begin{aligned} \rho(\phi(x) - A(x)) &\leq \frac{1}{2}\rho\left(2\phi(x) - 2^{n+1}\phi\left(\frac{x}{2^{n/s}}\right)\right) + \frac{1}{2}\rho\left(2^{n+1}\phi\left(\frac{x}{2^{n/s}}\right) - 2A(x)\right) \\ &\leq \frac{\tau}{2}\rho\left(\phi(x) - 2^n\phi\left(\frac{x}{2^{n/s}}\right)\right) + \frac{\tau}{2}\rho\left(2^n\phi\left(\frac{x}{2^{n/s}}\right) - A(x)\right) \\ &\leq \frac{1}{2}\sum_{j=1}^{n}\left(\frac{\tau^2}{2}\right)^j \alpha\left(\frac{x}{2^{j/s}}, \frac{x}{2^{j/s}}, \frac{-x}{2^{\frac{j-1}{s}}}\right) + \frac{\tau}{2}\rho\left(2^n\phi\left(\frac{x}{2^{n/s}}\right) - A(x)\right). \end{aligned}$$

for all $x \in \mathbb{R}$. Taking $n \to \infty$ we obtain the estimation (2.4).

We have:

$$\rho\left(2^n\phi\left(\frac{x}{2^{n/s}}\right)+2^n\phi\left(\frac{y}{2^{n/s}}\right)+2^n\phi\left(\frac{3}{2^{n/s}}\right)-2^nq\phi\left(\frac{1}{2^n}\sqrt[5]{\frac{x^s+y^s+z^s}{2}}\right)\right)\leq \tau^n\alpha\left(\frac{x}{2^{n/s}},\frac{y}{2^{n/s}},-\frac{z}{2^{n/s}}\right)$$

for all $x,y,z\in\mathbb{R}$ and all positive integer n. And we have:

$$\begin{aligned}
&\rho\left(\frac{1}{5}A(x)+\frac{1}{5}A(y)+\frac{1}{5}A(z)-\frac{1}{5}qA\left(\sqrt[s]{\frac{x^s+y^s+z^s}{q}}\right)\right)\\
&\leq\frac{1}{5}\rho\left(A(x)-2^n\phi\left(\frac{x}{2^{n/s}}\right)\right)+\frac{1}{5}\rho\left(A(y)-2^n\phi\left(\frac{y}{2^{n/s}}\right)\right)\\
&+\frac{1}{5}\rho\left(A(z)-2^n\phi\left(\frac{z}{2^{n/s}}\right)\right)+\frac{1}{5}\rho\left(qA\left(\sqrt[s]{\frac{x^s+y^s+z^s}{q}}\right)-2^nq\phi\left(\frac{1}{2^{n/s}}\sqrt[5]{\frac{x^s+y^s+z^s}{q}}\right)\right)\\
&+\frac{1}{5}\rho\left(2^n\phi\left(\frac{x}{2^{n/s}}\right)+2^n\phi\left(\frac{y}{2^{n/s}}\right)+2^n\phi\left(\frac{z}{2^{n/s}}\right)-q\cdot 2^n\phi\left(\frac{1}{2^{n/s}}\sqrt[5]{\frac{x^s+y^s+z^s}{q}}\right)\right)\\
&\leq\frac{1}{5}\rho\left(A(x)-2^n\phi\left(\frac{x}{2^{n/s}}\right)\right)+\frac{1}{5}\rho\left(A(y)-2^n\phi\left(\frac{y}{2^{n/s}}\right)\right)+\frac{1}{5}\rho\left(A(z)-2^n\phi\left(\frac{z}{2^{n/s}}\right)\right)\\
&+\frac{|q|}{5}\rho\left(A\left(\sqrt[5]{\frac{x^s+y^s+z^s}{q}}\right)-2^n\phi\left(\frac{1}{2^{n/s}}\sqrt[5]{\frac{x^s+y^s+z^s}{q}}\right)\right)\\
&+\frac{1}{5}\tau^n\alpha\left(\frac{x}{2^{n/s}},\frac{y}{2^{n/s}},\frac{z}{2^{n/s}}\right)\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence:

$$A(x)+A(y)+A(z)=qA\left(\sqrt[s]{\frac{x^s+y^s+z^s}{q}}\right).$$

Now, we prove the uniqueness of A.
Let $A':\mathbb{R}\to Y_\rho$ be an other mapping satisfying the estimation (2.4).
We have:

$$\begin{aligned}
\rho\left(A(x)-A'(x)\right)&\leq\frac{1}{2}\rho\left(2^{n+1}A\left(\frac{x}{2^{n/s}}\right)-2^{n+1}\phi\left(\frac{x}{2^{n/s}}\right)\right)\\
&+\frac{1}{2}\rho\left(2^{n+1}\phi\left(\frac{x}{2^{n/s}}\right)-2^{n+1}A'\left(\frac{x}{2^{n/s}}\right)\right)\\
&\leq\frac{\tau^{n+1}}{2}\rho\left(A\left(\frac{x}{2^{n/s}}\right)-\phi\left(\frac{x}{2^{n/s}}\right)\right)+\frac{\tau^{n+1}}{2}\rho\left(\phi\left(\frac{x}{2^{n/s}}\right)-A'\left(\frac{x}{2^{n/s}}\right)\right)\\
&\leq\frac{\tau^{n+1}}{2}\sum_{j=1}^{\infty}\left(\frac{\tau^2}{2}\right)^j\alpha\left(\frac{x}{2^{\frac{n+j}{s}}},\frac{x}{2^{\frac{n+j}{s}}},-\frac{x}{2^{\frac{n+j-1}{s}}}\right)\\
&\leq\left(\frac{2}{\tau}\right)^{n-1}\sum_{i=n+1}^{\infty}\left(\frac{\tau^2}{2}\right)^j\alpha\left(\frac{x}{2^{j/s}},\frac{x}{2^{j/s}},-\frac{x}{2^{j-1/s}}\right)
\end{aligned}$$

$\to 0$ as $n\to\infty$ (becaure $\frac{2}{\tau}\leq 1$) forall $x\in\mathbb{R}$.
Hence $A=A'$. □

Corollary 2.2. *Let $\theta>0$ be a real numbers and $p>s\log_2\left(\frac{\tau^2}{2}\right)$. If $\phi:\mathbb{R}\to Y_\rho$ is a mapping such that:*

$$\rho\left(\phi(x)+\phi(y)+\phi(z)-q\phi\left(\sqrt[s]{\frac{x^s+y^s+z^s}{q}}\right)\right)\leq\theta\left(|z|^p+|y|^p+|z|^p\right)$$

for all $x, y, z \in \mathbb{R}$, then there exists a unique radical mapping $A : \mathbb{R} \to Y_\rho$ such that:

$$\rho(\phi(x) - A(x)) \leq \frac{\left(2 + 2^{p/s}\right)\tau^2}{2\left(2^{p/s+1} - \tau^2\right)}|x|^p.$$

3. Stability of (0.1) in modulon space withont Δ_2- condition

The following theorem, presents the stability results of (0.1) in modular space without using the Δ_2-condition.

Theorem 3.1. *Let $\phi : \mathbb{R} \to Y_\rho$ be a mapping satifies*

$$\rho\left(\phi(x) + \phi(y) + \phi(z) - q\phi\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}}\right)\right) \leq \alpha(x, y, z) \tag{3.5}$$

and: $\alpha : \mathbb{R}^3 \longrightarrow [0, \infty)$ is a mapping such that:

$$\lim_{n\to\infty} \frac{\alpha\left(2^{n/s}x, 2^{n/s}y, 2^{n/s}z\right)}{2^n} = 0; \qquad \sum_{j=0}^{\infty} \frac{\alpha\left(2^{j/s}x, 2^{j/s}x, -2^{(j+1)/s}x\right)}{2^j} < \infty \tag{3.6}$$

for all $x, y, z \in \mathbb{R}$. Then there exists a unique radical functional mapping $\rho : \mathbb{R} \to Y_\rho$ such that :

$$\rho(\phi(x) - q\phi(0) - A(x)) \leq \frac{1}{2}\sum_{j=0}^{\infty} \frac{1}{2^j}\alpha\left(2^{j/s}x, 2^{j/s}, 2^{\frac{j+1}{s}}\right) \tag{3.7}$$

for all $x \in \mathbb{R}$.

Proof. Taking $y = x$ and $z = -2^{1/s}x$ in (3.5) we get

$$\rho\left(2\phi(x) - \phi\left(2^{1/s}x\right) - q\phi(0)\right) \leq \alpha\left(x, x, -2^{1/s}x\right)$$

Hence:

$$\rho\left(\hat{\phi}\left(2^{1/s}x\right) - 2\hat{\phi}(x)\right) \leq \alpha\left(x, x, -2^{1/s}x\right)$$

where $\hat{\phi}(x) = \phi(x) - q\phi(0)$.

Then we obtain from the convexity of ρ and $\sum_{i=0}^{n-1} \frac{1}{2^{j+1}} \leq 1$

$$\begin{aligned}
\rho\left(\hat{\phi}(x) - \frac{\hat{\phi}\left(2^{n/s}x\right)}{2^n}\right) &= \rho\left(\sum_{j=0}^{n-1} \frac{1}{2^{j+1}}\left(2\hat{\phi}\left(2^{j/s}x\right) - \hat{\phi}\left(2^{(j+1)/s}x\right)\right)\right) \\
&\leq \sum_{j=0}^{n-1} \frac{1}{2^{j+1}}\rho\left(2\hat{\phi}\left(2^{j/s}x\right) - \hat{\phi}\left(2^{\frac{j+1}{s}}x\right)\right) \\
&\leq \frac{1}{2}\sum_{j=0}^{n-1} \frac{1}{2^j}\alpha\left(2^{j/s}x, 2^{j/s}x, -2^{\frac{j+1}{s}}x\right)
\end{aligned}$$

for all $x \in \mathbb{R}$ and all positive integers $n > 1$.

Then,

$$\begin{aligned}
\rho\left(\frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} - \frac{\hat{\phi}\left(2^{\frac{m}{s}}x\right)}{2^m}\right) &\leq \frac{1}{2^m}\rho\left(\frac{\hat{\phi}\left(2^{\frac{n-m}{s}} \cdot 2^{\frac{m}{s}}x\right)}{2^{n-m}} - \hat{\phi}\left(2^{\frac{m}{s}}x\right)\right) \\
&\leq \frac{1}{2^m}\sum_{j=0}^{n-m-1} \frac{1}{2^{j+1}}\alpha\left(2^{\frac{m+j}{s}}x, 2^{\frac{m+j}{s}}x, -2^{\frac{m+j+1}{s}}x\right) \\
&= \frac{1}{2}\sum_{j=m}^{n-1} \frac{1}{2^j}\alpha\left(2^{j/s}x, 2^{j/s}x, -2^{\frac{j+1}{s}}x\right)
\end{aligned}$$

for all $x \in \mathbb{R}$ and all $m, n \in \mathbb{N}$ with $n > m$.

Then $\left\{ \dfrac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} \right\}$ is a ρ-Cauchy sequence in Y_ρ wish is ρ-complete. Then it's ρ-convergent to a mapping $A : \mathbb{R} \to Y_\rho$. Thus we can write:

$$A(x) = \rho - \lim_{n\to\infty} \frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} \text{ i.e. } \lim_{n\to\infty} \rho \left(\frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} - A(x) \right) = 0$$

for all $x \in \mathbb{R}$. By apply the Fatou property, we have:

$$\begin{aligned} \rho(\hat{\phi}(x) - A(x)) &\leq \liminf_{n\to\infty} \rho \left(\hat{\phi}(x) - \frac{\hat{\phi}\left(2^{n/s}x\right)}{2^n} \right) \\ &\leq \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{2^j} \alpha \left(2^{j/s}x, 2^{j/s}x, -2^{\frac{j+1}{s}}x \right) \end{aligned}$$

for all $x \in \mathbb{R}$.

In order to prove that A satisfies (0.1), we note that

$$\begin{aligned}
&\rho \left(\frac{1}{5}A(x) + \frac{1}{5}A(y) + \frac{1}{5}A(z) - \frac{1}{5}qA\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) \right) \\
&\leq \frac{1}{5}\rho\left(A(x) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} \right) + \frac{1}{5}\rho\left(A(y) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}y\right)}{2^n} \right) \\
&+ \frac{1}{5}\rho\left(A(z) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}z\right)}{2^n} \right) + \frac{1}{5}\rho\left(qA\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) - \frac{q}{2^n}\hat{\phi}\left(2^{n/s}\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) \right) \\
&\frac{1}{5}\rho\left(\frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} + \frac{\hat{\phi}\left(2^{\frac{n}{s}}y\right)}{2^n} + \frac{\hat{\phi}\left(2^{\frac{n}{s}}z\right)}{2^n} - \frac{q}{2^n}\hat{\phi}\left(2^{n/s}\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) \right) \\
&\leq \frac{1}{5}\rho\left(A(x) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n} \right) + \frac{1}{5}\rho\left(A(y) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}y\right)}{2^n} \right) \\
&+ \frac{1}{5}\rho\left(A(z) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}z\right)}{2^n} \right) + \frac{1}{5}\rho\left(qA\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) - \frac{q}{2^n}\hat{\phi}\left(2^{n/s}\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) \right) \\
&\frac{1}{5}\rho\left(\frac{\phi\left(2^{\frac{n}{s}}x\right)}{2^n} + \frac{\phi\left(2^{\frac{n}{s}}y\right)}{2^n} + \frac{\phi\left(2^{\frac{n}{s}}z\right)}{2^n} - \frac{q}{2^n}\phi\left(2^{n/s}\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) - (q^2 - 3q)\frac{\phi(0)}{2^n} \right) \\
&\leq \frac{1}{5}\rho\left(A(x) - \frac{\hat{\phi}\left(2^{\frac{n}{5}}x\right)}{2^n} \right) + \frac{1}{5}\rho\left(A(y) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}y\right)}{2^n} \right) \\
&+ \frac{1}{5}\rho\left(A(z) - \frac{\hat{\phi}\left(2^{\frac{n}{s}}z\right)}{2^n} \right) + \frac{1}{5}\rho\left(qA\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) - \frac{q}{2^n}\hat{\phi}\left(2^{n/s}\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right) \right) \\
&+ \frac{1}{5}\frac{\alpha(2^{n/s}x, 2^{n/s}y, 2^{n/s}z)}{2^n} + \frac{2^n - 1}{5.2^n}\rho\left(\frac{q^2 - 3q}{2^n - 1}\phi(0) \right) \to 0 \text{ as } n \to \infty \text{ (because } \phi(0) \in Y_p
\end{aligned}$$

and $\lim_{n\to\infty} \frac{q^2-3q}{2^n} = 0$).

Then

$$A(x) + A(y) + A(z) = qA\left(\sqrt[s]{\frac{x^s + y^s + z^s}{q}} \right)$$

Now, we prove that A is unique Let A' be an other radical mapping satisfying the estimation (3.7).

We have

$$\begin{aligned}\rho\left(\frac{1}{2}A(x)-\frac{1}{2}A'(x)\right) &\leq \frac{1}{2}\rho\left(\frac{A\left(2^{\frac{n}{s}}x\right)}{2^n}-\frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n}\right)+\frac{1}{2}\rho\left(\frac{\hat{\phi}\left(2^{\frac{n}{s}}x\right)}{2^n}-\frac{A'\left(2^{\frac{n}{s}}x\right)}{2^n}\right)\\ &\leq \frac{1}{2^{n+1}}\rho\left(A\left(2^{\frac{n}{s}}x\right)-\hat{\phi}\left(2^{\frac{n}{s}}x\right)\right)+\frac{1}{2^{n+1}}\rho\left(\hat{\phi}\left(2^{\frac{n}{s}}x\right)-A'\left(2^{\frac{n}{s}}x\right)\right)\\ &\leq \frac{1}{2^n}\sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\alpha\left(2^{\frac{n+j}{s}}x,2^{\frac{n+j}{s}}x,-2^{\frac{n+j+1}{s}}x\right)\\ &\leq \sum_{j=0}^{\infty}\frac{1}{2^{j+1}}\alpha\left(2^{\frac{j}{s}}x,2^{\frac{j}{s}}x,-2^{\frac{j+1}{s}}x\right)\to 0 \text{ as } n\to\infty\end{aligned}$$

Then $A=A'$, this complete the proof $\square$

4. Stability of (0.1) in modular space using the point five theorem

The following theorem of fixed point in modular space will play in important role in establishing our stability remits of (0.1).

Theorem 4.1 ([5]). *Let Y_ρ be a modular space and C be a p-complete nonempty subset of Y_ρ. A mapping $T:C\to Y_\rho$ is called quasi-contration if and only if there exits $K<1$ such that:*

$$\rho(T(x)-T(y))\leq K\max\{\rho(x-y),\rho(x-T(x)),\rho(y-T(y)),\rho(x-T(y)),\rho(y-T(x))\}.$$

Let $x\in C$ such that: $\delta_p(x)=\sup\{\rho\left(T^n(x)-T^m(x)\right);m,n\in\mathbb{N}\}$.

Then $\{T^n(x)\}$ is ρ-convergent to a point $w\in C$. Moreover, if $\rho(\omega-T(\omega))<\infty$ and $\rho(x-T(\omega))<\infty$, Then the ρ-limit of $T(x)^n$ is a fixed point of T and if $\bar{\omega}$ is any fixed point of T in C such that $\rho(\omega-\bar{\omega})<\infty$, then $\omega=\bar{\omega}$.

Theorem 4.2. *Let Y_ρ be a ρ-complete modular space satisfying Δ_2-condition and $\phi:\mathbb{R}\to Y_\rho$ be a mapping such that:*

$$\rho\left(\phi(x)+\phi(y)+\phi(z)-q\phi\left(\sqrt[s]{\frac{x^s+y^s+z^s}{q}}\right)\right)\leq\alpha(x,y,z) \tag{4.8}$$

for all $x,y,z\in\mathbb{R}$, where $\alpha:\mathbb{R}^3\to[0,\infty)$ is a mapping such that:

$$\alpha\left(2^{1/s}x,2^{1/s}x,-2^{2/s}x\right)\leq 2L\alpha(x,x,-2^{1/s}x)\qquad(0<L<1)$$

and

$$\lim_{n\to\infty}\frac{\alpha\left(2^{n/s}x,2^{n/s}y,2^{n/s}z\right)}{2^n}=0.. \tag{4.9}$$

Then there exists a unique radical mapping $\hat{\phi}:\mathbb{R}\to Y_\rho$ such that $\rho(\phi(x)-\hat{\phi}(x))\leq \dfrac{1}{2(1-L)}\alpha(x,x,-2^{1/s}x)$ for all $x\in\mathbb{R}$.

Proof. We define the set: $N=\{g:\mathbb{R}\to Y_p\}$ and we introduce the mapping $\hat{\rho}$ on N as :

$$\hat{\rho}(g)=\inf\{\lambda>0:\rho(g(x))\leq\lambda\alpha(x,x,-2^{1/s}x)\}$$

by same methode as in the proof of Theorem 2.1 in [13], we deduce that $\hat{\rho}$ is convex modular on N, and N is $\hat{\rho}$-complete.

Now, we introduce the mapping $A:N_{\hat{\rho}}\to N_{\hat{\rho}}$ as:

$$\Lambda(g)(x)=\frac{1}{2}g\left(2^{1/s}x\right);\qquad g\in N \text{ and } x\in\mathbb{R}.$$

Let $f,g\in N_{\hat{\rho}}$ and c be positive real number with: $\hat{\rho}(f-g)\leq c$, we have: $\rho(f(x)-g(x))\leq$

$c\alpha(x, x, -2^{1/s}x)$ for all $x \in \mathbb{R}$.
Then

$$\begin{aligned}\rho\left(\frac{f\left(2^{1/s}x\right)}{2} - \frac{g\left(2^{1/s}x\right)}{2}\right) &\leq \frac{1}{2}\rho\left(f\left(2^{1/s}x\right) - g\left(2^{1/s}x\right)\right) \\ &\leq \frac{1}{2}c\alpha\left(\left(2^{1/s}x\right), \left(2^{1/s}x\right), \left(-2^{2/s}x\right)\right) \\ &\leq Lc\alpha(x, x, -2^{1/s}x)\end{aligned}$$

for all $x \in \mathbb{R}$.
Hence

$$\hat{\rho}(\Lambda f - \Lambda g) \leq L\hat{\rho}(f - g)$$

for all $f, g \in N_{\hat{\rho}}$.
So A is $\hat{\rho}$-strict contraction.
Letting $y = x$ and $z = -2^{1/s}x$ in (4.8), we get:

$$\rho\left(2\phi(x) - \phi\left(2^{1/s}x\right)\right) \leq \alpha\left(x, x, -2^{1/s}x\right)$$

Then

$$\rho\left(\frac{1}{2^2}\phi\left(2^{2/s}x\right) - \phi(x)\right) \leq 1/2\rho\left(1/2\phi\left(2^{2/s}x\right) - \phi\left(2^{1/s}x\right)\right) + 1/2\rho\left(\phi\left(2^{1/s}\right) - 2\phi(x)\right)$$
$$\leq \frac{1}{2^2}\alpha\left(2^{1/s}x, 2^{1/s}x, -2^{2/s}x\right) + \frac{1}{2}\alpha\left(x, x, -2^{1/s}x\right)$$

by mathematical induction we have:

$$\begin{aligned}\rho\left(\frac{1}{2^n}\phi\left(2^{n/s}x\right) - \phi(x)\right) &\leq \sum_{i=1}^{n}\frac{1}{2^i}\alpha\left(2^{\frac{i-1}{s}}x, 2^{\frac{i-1}{s}}x, -2^{\frac{i}{s}}x\right) \\ &= \sum_{i=0}^{n-1}\frac{1}{2^{i+1}}\alpha\left(2^{\frac{i}{s}}, 2^{\frac{i}{s}}x, -2^{\frac{i+1}{s}}x\right) \\ &\leq \sum_{i=0}^{n-1}\frac{1}{2^{i+1}}(2L)^i\alpha\left(x, x, -2^{1/s}x\right) \\ &\leq \frac{1}{2(1-L)}\alpha\left(x, x, -2^{1/s}x\right)\end{aligned}$$

Then

$$\begin{aligned}\rho\left(\frac{\phi\left(2^{n/s}x\right)}{2^n} - \frac{\phi\left(2^{\frac{m}{s}}x\right)}{2^m}\right) &\leq \frac{1}{2}\rho\left(2\frac{\phi\left(2^{\frac{n}{s}}x\right)}{2^n} - 2\phi(x)\right) + \frac{1}{2}\rho\left(2\frac{\phi\left(2^{\frac{m}{s}}\right)}{2^m} - 2\phi(x)\right) \\ &\leq \frac{\tau}{2}\rho\left(\frac{\phi\left(2^{\frac{n}{s}}x\right)}{2^n} - \phi(x)\right) + \frac{\tau}{2}\rho\left(\frac{\phi\left(2^{\frac{m}{s}}x\right)}{2^m} - \phi(x)\right) \\ &\leq \frac{\tau}{2(1-L)}\alpha\left(x, x, -2^{1/s}x\right)\end{aligned}$$

Hence $\hat{\rho}\left(\Lambda^n\phi - \Lambda^m\phi\right) \leq \dfrac{\tau}{2(1-L)}$.
Thus $\delta_{\hat{\rho}}(\phi) < 0$ and $\{\Lambda^n\phi\}$ is $\hat{\rho}$-convergent to $\hat{\phi} \in N_{\hat{\rho}}$.
By $\hat{\rho}$-contraction of Λ we have:

$$\hat{\rho}\left(\Lambda^n\phi - \Lambda\hat{\phi}\right) \leq L\hat{\rho}\left(\Lambda^{n-1}\phi - \hat{\phi}\right)$$

Letting $n \to \infty$ and apply $\hat{\rho}$-atom property:

$$\begin{aligned}\hat{\rho}(\Lambda\hat{\phi} - \hat{\phi}) &\leq \liminf_{n\to\infty} \hat{\rho}\left(\Lambda\hat{\phi} - \Lambda^n \phi\right) \\ &\leq L \liminf_{n\to\infty} \hat{\rho}\left(\hat{\phi} - \Lambda^{n-1}\phi\right) = 0.\end{aligned}$$

Then, we conclude that $\hat{\phi}$ is a fixed puint of Λ. Then, we conclude that ϕ is a fxed poin of A. By same methode as in the proof of Theorem 2.1, we deduce that $\hat{\phi}$ satifies (0.1). Now, we must prove that $\hat{\phi}$ is unique.
Let $\bar{\phi}$ be another fixed point of Λ.
We have

$$\begin{aligned}\hat{\rho}(\bar{\phi} - \hat{\phi}) &= \hat{\rho}(\Lambda\bar{\phi} - \Lambda\hat{\phi}) \\ &\leq L\hat{\rho}(\bar{\phi} - \hat{\phi}).\end{aligned}$$

Hence $\bar{\phi} = \hat{\phi}$. □

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