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MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

THE RESOLVENT OF THE GENERALIZED SUB-LAPLACIAN

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Abstract. We compute the resolvent operator of the generalized Sub-Laplacian.

1. Introduction

For $\alpha \geq 0$, let $\mathbb{K} = [0, \infty[\times\mathbb{R}$ equipped with the measure

$$d\mu_\alpha(x,t) = \frac{1}{\pi\Gamma(\alpha+1)} x^{2\alpha+1}\ \mathrm{d}x\ \mathrm{d}t.$$

Let us consider the generalized sublaplacian.

$$\mathcal{L} = -\left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}\right).$$

It is well-known that $\mathcal{L}$ is positive and symmetric in $L^2(\mathbb{K})$ and when see [3] $\alpha = n-1, \mathcal{L}$ is the radial part of the sublaplacian on the Heisenberg group $\mathbb{H}^n$. The Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group. In the literature there are a lot of works dealing with the generalized sublaplacian e.g [[2],[3]], however it's resolvent is not known.

The aim of this paper is to compute the resolvent operator of the generalized sub-laplacian.

2. Notations and preliminaries

By $< .,. >$ we denote the usual inner product in $L^2([0,\infty[, x^{2\alpha+1}dx)$ defined by

$$< f; g >= \int_0^\infty f(r)g(r)r^{2\alpha+1}dr$$

and by $< .,. >_\alpha$ we denote the usual inner product in $L^2(\mathbb{K}, d\mu_\alpha(x,t))$ defined by

$$< f; g >_\alpha= \int_0^\infty f(x,t)g(x,t)d\mu_\alpha(x,t).$$

Let $L_n^\alpha(x) = \sum_{k=0}^n \binom{n+\alpha}{n-k}\big(-1\big)^k \frac{x^k}{k!}, \alpha > -1$ be Laguerre polynomials of type n, which can be defined in terms of the generating by see [[5], p.101]

$$\sum_{j=0}^\infty L_j^\alpha(x)r^j = \frac{1}{(1-r)^{\alpha+1}} \exp\left(-\frac{xr}{1-r}\right), |r| < 1. \tag{2.1}$$

For $\xi \neq 0$, we set

$$\psi_j(x) = \left(\frac{2|\xi|^{\alpha+1}j!}{\Gamma(j+\alpha+1)}\right)^{\frac{1}{2}}\varphi_j(x).$$

where $\varphi_j(x) = e^{-\frac{|\xi|x^2}{2}}L_j^\alpha(|\xi|x^2)$.

We have the following known result

For any $\xi \neq 0$, the system

$$\{\psi_j(x) : j \in \mathbb{N}\}$$

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forms an orthonormal basis of the space $L^2\big([0,\infty),x^{2\alpha+1}\,\mathrm{d}x\big)$.

Special functions.

We start by the confluent hypergeometric [[1], p.204] is defined as

$${}_1F_1(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)}\frac{z^k}{k!}.$$

For $a\in\mathbb{C}$, a not a non-positive integer, ${}_1F_1(a,b,z)$ has the following asymptotic behavior [[1], p. 209]

$${}_1F_1(a,b,z)=\Gamma(b)\Big(\frac{(-z)^{-a}}{\Gamma(b-a)}+\frac{e^z z^{a-b}}{\Gamma(a)}\Big)\Big(1+O\Big(\frac{1}{|z|}\Big)\Big)\; as\, |z|\to\infty, -\pi<\arg(z)\leq\pi. \tag{2.2}$$

where by Γ we denote the usual Euler Gamma function.

In particular, when $\mathrm{Re}\,(z)>0$, we have

$${}_1F_1(a,b,z)\underset{|z|\to+\infty}{\sim}\frac{\Gamma(b)}{\Gamma(a)}e^z z^{a-b}. \tag{2.3}$$

Now we define the Tricomi ψ-function [[1], p.264], as a linear combination of two ${}_1F_1$-sums

$$\psi\,(a,c,x)=\frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,{}_1F_1(a,c,x)+\frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\,{}_1F_1(a-c+1,2-c,x).$$

By

$$u^{\xi}(x)=\int_{-\infty}^{\infty}u(x,t)e^{-\mathrm{i}\xi t}\,\mathrm{d}t$$

we note the Fourier transform of a suitable function $u(x,t)$ in the variable t.

3. The resolvent of the generalized Sub-Laplacian

We begin this section by computing the resolvent of the operator

$$\mathcal{L}_{\xi}=-\left(\frac{\partial^2}{\partial x^2}+\frac{2\alpha+1}{x}\frac{\partial}{\partial x}-\xi^2x^2\right)$$

on its maximal domaine $D_{\xi}=\{f\in L^2\big([0,\infty),x^{2\alpha+1}\,\mathrm{d}x\big),\mathcal{L}_{\xi}(f)\in L^2\big([0,\infty),x^{2\alpha+1}\,\mathrm{d}x\big)\}$. Note that by a simple calculation, we can prove that $\mathcal{L}_{\xi}$ is a self-adjoint and non-negative operator.

3.1. Eigenfunctions of the operator $\mathcal{L}_{\xi}$.

Theorem 3.1. *Let $\xi\in\mathbb{R}^*$. Then $f(x)=C.\,{}_1F_1\big(\frac{2(a+1)|\xi|-\mu}{4|\xi|},\alpha+1,|\xi|x^2\big)e^{-\frac{|\xi|}{2}x^2}$ is an eigenfunction of $\mathcal{L}_{\xi}$ with eigenvalue μ.*

Proof. Let $f(x)=g(-|\xi|x^2)e^{\frac{|\xi|x^2}{2}}$. So the equation $\mathcal{L}_{\xi}f(x)=\mu f(x)$ becomes after simplification to

$$sg''(s)+(\alpha+1-s)g'(s)-\frac{2(\alpha+1)+\mu}{4|\xi|}g(s)=0$$

which is the hypergeometric equation with parameters $\big(\alpha+1,\frac{2(\alpha+1)+\mu}{4|\xi|}\big)$ and the only solutions which are smooth at $s=0$ are $C\,{}_1F_1\big(\frac{2(a+1)|\xi|+\mu}{4|\xi|},\alpha+1,s\big)$.

Since ${}_1F_1(a,b,z)=e^z\,{}_1F_1(b-a,b,z)$, $f(x)=C.\,{}_1F_1\big(\frac{2(a+1)|\xi|-\mu}{4|\xi|},\alpha+1,|\xi|x^2\big)e^{-\frac{|\xi|}{2}x^2}$ is an eigenfunction of $\mathcal{L}_{\xi}$ with eigenvalue μ. ∎

3.2. **The spectrum of the operator $\mathcal{L}_\xi$.**

Corollary 3.2. *Let $\mu \in \mathbb{C}$ and $\xi \in \mathbb{R}^*$.*
If $f_\mu(x) = C.\,{}_1F_1\big(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi|x^2\big)e^{-\frac{|\xi|}{2}x^2} \in L^2(\mathbb{R}^+, x^{2\alpha+1})$, then $\mu = 2(\alpha+1+2j)|\xi|$ where $j \in \mathbb{N}$.

Proof. Since ${}_1F_1(a,b,z) \sim_\infty \frac{\Gamma(a)}{\Gamma(b)} z^{a-b}e^z$ if Re $(z) > 0$. Then

$$\int_0^\infty |f_\mu(x)|^2 x^{2\alpha+1} < \infty$$

if $\frac{2(a+1)|\xi|-\mu}{4|\xi|}$ is a pole of the function Γ.

Hence $\frac{2(a+1)|\xi|-\mu}{4|\xi|} = -j$ where $j \in \mathbb{N}$.

Therefore $\mu = 2(\alpha+1+2j)|\xi|$ where $j \in \mathbb{N}$. □

Corollary 3.3. *The spectrum of the operator $\mathcal{L}_\xi, \xi \neq 0$ is the set*

$$\sigma(\mathcal{L}_\xi) = \{2(\alpha+1+2j)|\xi|, j \in \mathbb{N}\}.$$

Proof. The result comes immediately since every $\psi_m(x)$ is an eigenvector of the self-adjoint operator $\mathcal{L}_\xi$ with eigenvalue $2(\alpha+1+2j)|\xi|$ and the system $\{\psi_m(x) : m \in \mathbb{N}\}$ forms an orthonormal basis of the space $L^2\big([0,\infty), x^{2\alpha+1}\,\mathrm{d}x\big)$. □

3.3. **The resolvent of the operator $\mathcal{L}_\xi$.** In order to compute the resolvent of the generalized sub-Laplacian $\mathcal{L}$, we need the following close formula

Theorem 3.4. *Assume β with Re $(\beta) > 0$. Then*

$$\sum_{n=0}^{\infty} \frac{n!}{(n+\beta)\Gamma(n+\alpha+1)} L_n^\alpha(x)L_n^\alpha(y)$$
$$= \frac{\Gamma(\beta)}{\Gamma(1+\alpha)}\, {}_1F_1(\beta, \alpha+1, \min(x,y))\Psi(\beta, \alpha+1, \max(x,y))$$

in the distributional sense.

Proof. Assume $x \leq y$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{n!}{(n+\beta)\Gamma(n+\alpha+1)} L_n^\alpha(x)L_n^\alpha(y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!L_n^\alpha(y)}{(n+\beta)\Gamma(n-k+1)\Gamma(k+\alpha+1)} \frac{(-x)^k}{k!} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(m+k+1)}{(m+k+\beta)\Gamma(m+1)\Gamma(k+\alpha+1)} L_{m+k}^\alpha(y)\frac{(-x)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma(k+\alpha+1)} \sum_{j=k}^{\infty} \frac{\Gamma(j+1)}{(j+\beta)\Gamma(j-k+1)} L_j^\alpha(y)\end{aligned} \tag{3.4}$$

Now we are going to find the close formula of this sum $\sum_{j=k}^{\infty} \frac{\Gamma(j+1)}{(j+\beta)\Gamma(j-k+1)} L_j^\alpha(y)$ in the distributional sense. Since the power series $\sum_{j=0}^{\infty} L_j^\alpha(x)r^j$ in r verify

$$\sum_{j=0}^{\infty} L_j^\alpha(x)r^j = \frac{1}{(1-r)^{\alpha+1}} \exp\left(-\frac{xr}{1-r}\right), |r| < 1. \tag{3.5}$$

So the series $\sum_{j=0}^{\infty} L_j^\alpha(x)r^j$ in r converges uniformly on each compact set of $]0,1[$, hence we have the convergence in the distributional sense.

By differentiating "k" times to both sides of 3.5 we get

$$\Big(\sum_{j=0}^{\infty} L_j^{(\alpha)}(y)r^j\Big)^{(k)} = \Big(\frac{1}{(1-r)^{\alpha+1}}\exp\Big(-\frac{yr}{1-r}\Big)\Big)^{(k)}.$$

Hence

$$\sum_{j=k}^{\infty} L_j^{(\alpha)}(y)\frac{\Gamma(j+1)}{\Gamma(j-k+1)}r^j = r^k\Big(\frac{1}{(1-r)^{\alpha+1}}\exp\Big(-\frac{yr}{1-r}\Big)\Big)^{(k)}.$$

Thus

$$\sum_{j=k}^{\infty} L_j^{(\alpha)}(y)\frac{\Gamma(j+1)}{\Gamma(j-k+1)}r^{j+\beta-1} = r^{k+\beta-1}\Big(\frac{1}{(1-r)^{\alpha+1}}\exp\Big(-\frac{yr}{1-r}\Big)\Big)^{(k)}.$$

So

$$\sum_{j=k}^{\infty} \frac{\Gamma(j+1)}{(j+\beta)\Gamma(j-k+1)}L_j^{(\alpha)}(y) = \int_0^1 r^{k+\beta-1}\Big(\frac{1}{(1-r)^{\alpha+1}}\exp\Big(-\frac{yr}{1-r}\Big)\Big)^{(k)}dr$$

Using $\int_a^b u(x)v^{(n)}(x)\mathrm{d}x = \Big[\sum_{k=0}^{n-1}(-1)^k u^{(k)}v^{(n-k-1)}\Big]_a^b + (-1)^n\int_a^b u^{(n)}(x)v(x)\mathrm{d}x$ with $(x^{k+\beta-1})^{(j)} = \frac{\Gamma(\beta+j)}{\Gamma(k+\beta-j)}x^{k+\beta-1-j}$ and $(x^{k+\beta-1})^{(k)} = \frac{\Gamma(\beta+k)}{\Gamma(\beta)}x^{\beta-1}$, we obtain by using [[4],p.1023] $\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}dt:\ \mathrm{Re}\ (\alpha)>0,\ \mathrm{Re}\ (z)>0$

$$\begin{aligned}\sum_{j=k}^{\infty} \frac{\Gamma(j+1)}{(j+\beta)\Gamma(j-k+1)}L_j^{(\alpha)}(y) =& \frac{\Gamma(\beta+k)}{\Gamma(\beta)}\int_0^1 r^{\beta-1}\frac{1}{(1-r)^{\alpha+1}}\exp\Big(-\frac{yr}{1-r}\Big)dr\\ =& \frac{\Gamma(\beta+k)}{\Gamma(\beta)}\int_0^\infty e^{-ys}s^{\beta-1}(1+s)^{\alpha-\beta-1}ds \quad (s=\frac{r}{1-r})\\ =&(-1)^k\Gamma(\beta+k)\Psi(\beta,\alpha+1,y).\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma(k+\alpha+1)}\sum_{j=k}^{\infty}\frac{\Gamma(j+1)}{(j+\beta)\Gamma(j-k+1)}L_j^{\alpha}(y) =& \Gamma(\beta)\Psi(\beta,\alpha+1,y)\sum_{k=0}^{\infty}\frac{\Gamma(\beta+k)(-x)^k}{k!\Gamma(k+\alpha+1)}x^k\\ =&\frac{\Gamma(\beta)}{\Gamma(1+\alpha)}\,{}_1F_1(\beta,\alpha+1,x)\Psi(\beta,\alpha+1,y).\end{aligned}$$

□

For $f\in D_\xi$ we have the following expansion

$$f(x) = \sum_{j=0}^{\infty}\langle f,\psi_j\rangle\psi_j(x)$$

Therefore since $\mathcal{L}_\xi(f)\in L^2\big([0,\infty),x^{2\alpha+1}\ \mathrm{d}x\big)$ (understood in the distributional sense) we get

$$\mathcal{L}_\xi(f)(x) = \sum_{j=0}^{\infty} 2(\alpha+1+2j)|\xi|\langle f,\psi_j\rangle\psi_j(x) \tag{3.6}$$

$$= \int_0^\infty \sum_{j=0}^{\infty} 2(\alpha+1+2j)|\xi|\psi_j(x)\psi_j(r)f(r)r^{2\alpha+1}dr \tag{3.7}$$

Now we are in position to compute the resolvent of the operator $\mathcal{L}_\xi$, we have

Theorem 3.5. *Let* $\mu \notin \{2(\alpha+1+2j)|\xi|, j \in \mathbb{N}\}$. *Then*

$$\begin{aligned}(\mathcal{L}_\xi - \mu)^{-1} f(x) =& \frac{|\xi|^\alpha}{\Gamma(\alpha+1)} \Gamma\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}\right) e^{-\frac{|\xi|x^2}{2}} \\ &. \int_0^\infty {}_1F_1\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi| \min(r^2, x^2)\right) \\ &. \Psi\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi| \max(r^2, x^2)\right) f(r) e^{-\frac{|\xi|r^2}{2}} r^{2\alpha+1} dr\end{aligned}$$

where $f \in L^2\big([0,\infty), x^{2\alpha+1}\ \mathrm{d}x\big)$.

Proof. According to the spectral decomposition 3.6 we have for $f \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ and $\mathrm{Re}\ (\mu) > 0$.

$$\begin{aligned}(\mathcal{L}_\xi - \mu)^{-1} f(x) =& \sum_{j=0}^\infty \frac{1}{2(\alpha+1+2j)|\xi|-\mu} \langle f, \psi_j \rangle \psi_j(x) \\ =& \int_0^\infty \sum_{j=0}^\infty \frac{1}{2(\alpha+1+2j)|\xi|-\mu} \frac{2|\xi|^{\alpha+1} j!}{\Gamma(j+\alpha+1)} f(r) \varphi_j(r) r^{2\alpha+1} \varphi_j(x) dr \\ =& \int_0^\infty \sum_{j=0}^\infty \frac{1}{2(\alpha+1+2j)|\xi|-\mu} \frac{2|\xi|^{\alpha+1} j!}{\Gamma(j+\alpha+1)} f(r) \varphi_j(r) r^{2\alpha+1} \varphi_j(x) dr \\ =& \int_0^\infty \sum_{j=0}^\infty \frac{1}{2(j+\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|})} \frac{|\xi|^\alpha j!}{\Gamma(j+\alpha+1)} f(r) \psi_j(r) \psi_j(x) r^{2\alpha+1} dr \\ (\textit{by Theorem } 3.4) =& \frac{|\xi|^\alpha}{\Gamma(\alpha+1)} \Gamma\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}\right) e^{-\frac{|\xi|x^2}{2}} \\ &. \int_0^\infty {}_1F_1\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi| \min(r^2, x^2)\right) \\ &. \Psi\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi| \max(r^2, x^2)\right) f(r) r^{2\alpha+1} e^{-\frac{|\xi|r^2}{2}} dr\end{aligned}$$

We have by Cauchy-Schwarz inequality

$$\begin{aligned}\|(\mathcal{L}_\xi - \mu)^{-1} f\|_2^2 =& \sum_{j=0}^\infty \Big| \frac{1}{2(\alpha+1+2j)|\xi|-\mu} \langle f, \psi_j \rangle \Big|^2 \\ \leq& \Big(\sum_{j=0}^\infty \Big| \frac{1}{2(\alpha+1+2j)|\xi|-\mu} \Big|^2 \Big) \|f\|^2 \quad (\|\psi_j\| = 1) \\ =& C \|f\|^2\end{aligned}$$

but $\overline{\mathcal{C}_0^\infty(\mathbb{R}^+)}^{\|\|_2} = L^2([0,\infty[, x^{2\alpha+1}dx)$, then $(\mathcal{L}_\xi - \mu)^{-1}$ considered initially on $\mathcal{C}_0^\infty(\mathbb{R}^+)$, extends uniquely to a bounded operator on $L^2([0,\infty[, x^{2\alpha+1}dx)$.

Finally note that $(\mathcal{L}_\xi - \mu)^{-1}$ can be extended meromorphically for all $\mu \notin \{2(\alpha+1+2j)|\xi|, j \in \mathbb{N}\}$. □

4. The resolvent of the operator $\mathcal{L}$

In this section we prove our main result, we have

Theorem 4.1. *For $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ and $f \in L^2\big([0,\infty), d\mu_\alpha(x,t)\big)$. We have*

$$\begin{aligned}&\big(\mathcal{L}-\mu\big)^{-1} f(x,t)\\ &= \frac{1}{2\pi\Gamma(\alpha+1)} \int_{\mathbb{R}} \Bigg(\int_0^\infty {}_1F_1\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi|\min(r^2,x^2)\right)\\ &\cdot\Psi\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi|\max(r^2,x^2)\right) f^\xi(r) r^{2\alpha+1} e^{-\frac{|\xi|r^2}{2}} dr\Bigg)\\ &\cdot\Gamma\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}\right) e^{-\frac{|\xi|x^2}{2}} |\xi|^\alpha e^{i\xi.t} d\xi\end{aligned}$$

Proof. For $\mu \in \mathbb{C} \setminus \mathbb{R}_+$ and $f \in L^2\big([0,\infty), d\mu_\alpha(x,t)\big)$. Put

$$\big(\mathcal{L}-\mu\big) f(x,t) = g(x,t).$$

by Taking the Fourier transform to both sides we get

$$\big(\mathcal{L}_\xi-\mu\big) f^\xi(x) = g^\xi(x)$$

Hence

$$f^\xi(x) = \big(\mathcal{L}_\xi-\mu\big)^{-1} g^\xi(x).$$

Now by taking the inverse Fourier transform, we have

$$\begin{aligned}f(x,t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \big(\mathcal{L}_\xi-\mu\big)^{-1} g^\xi(x) e^{i\xi.t} d\xi\\ &= \frac{1}{2\pi\Gamma(\alpha+1)} \int_{\mathbb{R}} \Bigg(\int_0^\infty {}_1F_1\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi|\min(r^2,x^2)\right)\\ &\quad .\Psi\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}, \alpha+1, |\xi|\max(r^2,x^2)\right) g^\xi(r) r^{2\alpha+1} e^{-\frac{|\xi|r^2}{2}} dr\Bigg)\\ &\quad .\Gamma\left(\frac{2(\alpha+1)|\xi|-\mu}{4|\xi|}\right) e^{-\frac{|\xi|x^2}{2}} |\xi|^\alpha e^{i\xi.t} d\xi\end{aligned}$$

This proves the theorem. □

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