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ON SOME THEOREMS OF THE SECOND HANKEL-CLIFFORD LIPSCHITZ CLASS

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Dedicated to Professor Ahmed El Hamma.

Abstract. In this paper, we prove the generalization of Titchmarsh's theorem for the second Hankel-Clifford transform for functions satisfying the $(m, \mu, 2)$-Hankel-Clifford Lipschitz condition in the space $\mathrm{L}^2((0, +\infty), x^\mu)$, where $\mu \geq 0$.

1. Introduction and preliminaries

This work extends Titchmarsh's theorem by considering functions that satisfy a Hankel-Clifford Lipschitz condition within the weighted Hilbert space $\mathrm{L}^2((0, +\infty), x^\mu)$, where $\mu \geq 0$. The proof utilizes the translation operator, likely to facilitate the analysis of the transform's properties, such as continuity, invertibility, and the smoothness of the resulting functions. The result broadens the applicability of Titchmarsh's classical theorem to a wider class of functions in this specialized function space, making it a powerful tool for problems in harmonic analysis, mathematical physics, and engineering.

We point out that similar results have been established in the context of Jacobi transform, Fourier transform, Bessel transform, the first Hankel-Clifford transform, generalized Dunkl transform, q-Bessel transform and q-Dunkl transform (See [1, 4, 5, 6, 7, 8, 9, 10, 11, 21]).

In this section, we collect some basic facts of the Second Hankel-Clifford transform. More about the Hankel-Clifford analysis can be found in [11, 12, 17, 18, 19, 22].

From [13], we have the definition of Bessel-Clifford function of the first kind of order $\mu \geq 0$, by

$$c_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu + k + 1)},$$

this function is a solution of the differential equation

$$xy'' + (\mu + 1)y' + y = 0,$$

and is closely related with the Bessel function of first kind $J_\mu(x)$ by

$$c_\mu(x) = x^{-\frac{\mu}{2}} J_\mu(2\sqrt{x}), \ x \in (0, +\infty). \tag{1.1}$$

Consider the Banach space $\mathrm{L}^p_\mu = \mathrm{L}^p_\mu((0, +\infty), x^\mu dx)$, $1 \leq p < \infty$ and $\mu \geq 0$, with the norm

$$\|f\|_{\mathrm{L}^p_\mu} = \left(\int_0^\infty |f(x)|^p x^\mu dx\right)^{1/p} < \infty.$$

For $\mu \geq -\frac{1}{2}$, we consider the second Hankel-Clifford transform $h_{2,\mu}$ (See [15, 16]), defined on L^1_μ by

$$h_{2,\mu}(f)(\lambda) = \int_0^{+\infty} c_\mu(\lambda x) f(x) x^\mu dx, \ \lambda \in (0, +\infty).$$

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Theorem 1.1. *(Inversion formula for $h_{2,\mu}$, see [15, 16]) If $f \in L^1_\mu$ such that $h_{2,\mu}(f) \in L^1_\mu$, then*

$$f(x) = \int_0^{+\infty} c_\mu(\lambda x) h_{2,\mu}(f)(\lambda)\lambda^\mu d\lambda, \ a.e$$

From Méndez et al (See [16]), the Parseval's formula for the second Hankel-Clifford transform is written as

$$\|h_{2,\mu}(f)\|_{L^2_\mu} = \|f\|_{L^2_\mu},$$

where $f \in L^1_\mu \cap L^2_\mu$.

The normalized spherical Bessel function of index μ is defined by

$$j_\mu(x) = \frac{2^\mu \Gamma(\mu+1) J_\mu(x)}{x^\mu}. \tag{1.2}$$

From [2], we have the following lemma

Lemma 1.2. *Let $\mu \geq -\frac{1}{2}$. The following assertions are true*

(1) $|j_\mu(x)| \leq 1$,
(2) $1 - j_\mu(x) = O(x^2);\ 0 \leq x \leq 1$,
(3) $\sqrt{x} J_\mu(x) = O(1)$,

Lemma 1.3. *The following inequality is true*

$$|1 - j_\mu(x)| \geq c,$$

with $|x| \geq 1$, where $c > 0$ is certain constant.

Proof. (Analog of Lemma 2.9 in [3]) □

Therefore, by virtue of (1.1) and (1.2), we can write

$$c_\mu(x) = \frac{1}{\Gamma(\mu+1)} j_\mu(2\sqrt{x}). \tag{1.3}$$

Next, we define the translation operator. From [22, 14], we have

$$D_\mu(x,y,z) = \frac{\Delta^{2\mu+1}}{2^{2\mu}(xyz)^\mu \Gamma(\mu+\frac{1}{2})\sqrt{\pi}},$$

where $\Delta = \Delta(x,y,z)$ be the area of triangle with sides x, y, z. If Δ exists and zero otherwise. We note that $D_\mu(x,y,z) \geq 0$ and it is symmetric in x, y, z.

The translation operator T_h of function f is defined by (see [19]).

$$T_h(f)(x) = \int_0^{+\infty} f(z) D_\mu(h,x,z) z^\mu dz, \ 0 < x, h < \infty.$$

Proposition 1.4. [19] *Let $f \in L^1_\mu \cap L^2_\mu$. Then*

$$h_{2,\mu}(T_h f(.)))(\lambda) = c_\mu(h\lambda) h_{2,\mu}(f(.))(\lambda), \ \lambda \in (0, +\infty). \tag{1.4}$$

From [19], we have the following differential operator

$$B_\mu = xD^2 + (1+\mu)D,$$

where $D = \frac{d}{dx}$ and $D^2 = \frac{d^2}{dx^2}$, and we have the following relation

$$h_{2,\mu}(B_\mu f)(\lambda) = -\lambda h_{2,\mu}(f)(\lambda), \ f \in L^1_\mu. \tag{1.5}$$

The finite differences of the first and higher orders are defined as follow

$$\Delta_h f(x) = \Gamma(\mu+1) T_h f(x) - f(x) = (\Gamma(\mu+1)T_h - I) f(x).$$

$$\Delta_h^m f(x) = (\Gamma(\mu+1)T_h - I)^m f(x) = \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} (\Gamma(\mu+1)T_h)^i f(x), \tag{1.6}$$

where $\mathrm{T}_h^0 f(x)=f(x)$, $\mathrm{T}_h^i f(x)=\mathrm{T}_h(\mathrm{T}_h^{i-1}f(x))$, $(i=1,2,...,m; m=1,2,...)$ and I is a unit operator in L^2_μ.

Let $\mathrm{W}^m_{2,\mu}$ be the Sobolev space of $f\in \mathrm{L}^2_\mu$ such that $\mathrm{B}^i_\mu f\in \mathrm{L}^2_\mu;\ \ i=1,2,...,m$ (the action of differential operator B^i_μ is understood in the sense of distributions), i.e.,

$$\mathrm{W}^m_{2,\mu}=\{f\in \mathrm{L}^2_\mu \ /\ \mathrm{B}^i_\mu f\in \mathrm{L}^2_\mu;\ i=1,2,...,m\}.$$

2. Main results

In this section, we present the main results of this paper. We first need to define the $(m,\mu,2)$-Hankel-Clifford Lipschitz class.

Definition 2.1. Let $\alpha\in(0,m)$. A function $f\in \mathrm{W}^m_{2,\mu}$ is said to be in the $(m,\mu,2)$-Hankel-Clifford Lipschitz class, denoted by $Lip_m(\alpha,2,\mu)$, if

$$\|\Delta^m_h \mathrm{B}^r_\mu f(x)\|_{\mathrm{L}^2_\mu}=O(h^\alpha)\ as\ h\to 0,$$

where $r=0,1,2,...,m$ and $m=1,2,.....$

Lemma 2.2. *Let $f\in \mathrm{W}^m_{2,\mu}$. Then*

$$\|\Delta^m_h \mathrm{B}^r_\mu f(x)\|^2_{\mathrm{L}^2_\mu}=\int_0^{+\infty}\lambda^{2r}|1-j_\mu(2\sqrt{\lambda h}|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda.$$

Proof. By formula (1.4), we have

$$h_{2,\mu}(\mathrm{B}^r_\mu f)(\lambda)=(-1)^r\lambda^r h_{2,\mu}(f)(\lambda). \tag{2.7}$$

From formula (1.5) and Proposition 1.4, we obtain

$$h_{2,\mu}((\Gamma(\mu+1)\mathrm{T}_h)^i\mathrm{B}^r_\mu f)(\lambda)=(-1)^r\lambda^r j^i_\mu(2\sqrt{\lambda h})h_{2,\mu}(f)(\lambda). \tag{2.8}$$

By (2.8) and (1.6) follows that the second Hankel-Clifford transform of $\Delta^m_h\mathrm{B}^r_\mu f(x)$ is $(-1)^r\lambda^r(j_\mu(2\sqrt{\lambda h})-1)^m h_{2,\mu}(f)(\lambda)$, by Parseval's identity we have the result. □

Theorem 2.3. *Let $f\in \mathrm{W}^m_{2,\mu}$. Then the following are equivalents*

(1) $f\in Lip_m(\alpha,2,\mu)$,
(2) $\int_t^{+\infty}\lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda=O(t^{-2\alpha})\ as\ s\to+\infty$,

Proof. 1) $\Longrightarrow$ 2) Assume that $f\in Lip_m(\alpha,2,\mu)$. Then we have

$$\|\Delta^m_h \mathrm{B}^r_\mu f(x)\|^2_{\mathrm{L}^2_\mu}=\int_0^{+\infty}\lambda^{2r}|1-j_\mu(2\sqrt{\lambda h}|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda.$$

If $\lambda\in[\frac{1}{4h},\frac{2}{4h}]$, then $2\sqrt{\lambda h}\geq 1$ and Lemma 1.3, we obtain

$$1\leq\frac{1}{c^{2m}}|1-j_\mu(2\sqrt{h\lambda})|^{2m}.$$

Then

$$\begin{aligned}\int_{\frac{1}{4h}}^{\frac{2}{4h}}\lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda\\ &\leq \frac{1}{c^{2m}}\int_{\frac{1}{4h}}^{\frac{2}{4h}}\lambda^{2r}|1-j_\mu(2\sqrt{h\lambda})|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda\\ &\leq \frac{1}{c^{2m}}\int_0^{+\infty}\lambda^{2r}|1-j_\mu(2\sqrt{h\lambda})|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda\\ &= O(h^{2\alpha}).\end{aligned}$$

Furthemore, we have

$$\int_t^{2t} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda = O(t^{-2\alpha}) \ as \ t \to +\infty.$$

and there exists a positive constant C such that

$$\int_t^{2t} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda \leq Ct^{-2\alpha}.$$

So that

$$\begin{aligned}
\int_t^{+\infty} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda &= \left(\int_t^{2t} + \int_{2t}^{4t} + \int_{4t}^{8t} + ...\right) |h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda \\
&\leq Ct^{-2\alpha} + C(2t)^{-2\alpha} + C(4t)^{-2\alpha} + ... \\
&\leq C\left(t^{-2\alpha} + (2t)^{-2\alpha} + (4t)^{-2\alpha} + ...\right) \\
&\leq Ct^{-2\alpha}\left(1 + 2^{-2\alpha} + (2^{-2\alpha})^2 + (2^{-2\alpha})^3 + ...\right) \\
&\leq CK_\alpha t^{-2\alpha},
\end{aligned}$$

where $K_\alpha = (1-2^{-2\alpha})^{-1}$ since $2^{-2\alpha} < 1$.

This proves that

$$\int_t^{+\infty} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda = O(t^{-2\alpha}) \ as \ t \to +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_t^{+\infty} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda = O(t^{-2\alpha}) \ as \ t \to +\infty.$$

We have to show that

$$\int_0^{+\infty} \lambda^{2r}|1-j_\mu(2\sqrt{h\lambda})|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda = O(h^{2\alpha}) \ as \ h \to 0.$$

We write

$$\int_0^{+\infty} \lambda^{2r}|1-j_\mu(2\sqrt{h\lambda})|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_0^{\frac{1}{4h}} \lambda^{2r}|1-j_\mu(2\sqrt{h\lambda})|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda,$$

and

$$\mathrm{I}_2 = \int_{\frac{1}{4h}}^{+\infty} \lambda^{2r}|1-j_\mu(2\sqrt{h\lambda})|^{2m}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda.$$

Estimate the summands I_1 and I_2 from above. It follows from (1) of Lemma 1.2 that

$$\begin{aligned}
\mathrm{I}_2 &\leq 2^{2m}\int_{\frac{1}{4h}}^{+\infty} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda \\
&= O(h^{2\alpha}) \ as \ h \to 0.
\end{aligned}$$

Set

$$\psi(x) = \int_x^{+\infty} \lambda^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda.$$

From (1) and (2) of Lemma 1.2 and integration by parts, we obtain

$$\mathrm{I}_1 \leq -C_1h^{2m}\int_0^{\frac{1}{4h}} x^{2m}\psi'(x)dx.$$

Then

$$\begin{aligned}
\mathrm{I}_1 &\leq -C_1 h^{2m}\int_0^{\frac{1}{4h}} x^{2m}\psi'(x)dx\\
&\leq -C_1 h^{2m}\left((\frac{1}{4h})^{2m}\psi(\frac{1}{4h}) - 2m\int_0^{\frac{1}{4h}} x^{2m-1}\psi(x)dx\right)\\
&\leq 2mC_2 h^{2m}\int_0^{\frac{1}{4h}} x^{2m-1}\psi(x)dx.
\end{aligned}$$

Since $\psi(x) = O(x^{-2\alpha})$, we have $x^{2m-1}\psi(x) = O(x^{2m-2\alpha-1})$. Then

$$\begin{aligned}
\mathrm{I}_1 &\leq 2mC_2 h^{2m}\int_0^{\frac{1}{4h}} x^{2m-1}\psi(x)dx\\
&\leq C_3 h^{2m}\int_0^{\frac{1}{4h}} x^{2m-2\alpha-1}dx \ (\textit{the integral exists since } \alpha < m)\\
&\leq C_3 h^{2m}\left(\frac{1}{4h}\right)^{2m-2\alpha}\\
&\leq C_4 h^{2\alpha},
\end{aligned}$$

where C_1, C_2, C_3 and C_4 are a positive constants and this ends the proof. □

Corollary 2.4. *Let $f \in Lip_m(\alpha, 2, \mu)$. Then*

$$\int_t^{+\infty} |h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda = O(t^{-2r-2\alpha}) \text{ as } s \to +\infty.$$

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