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A SURVEY ON VOLUME-PRESERVING RIGIDITY

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This work is dedicated to those studying under difficult conditions and to those lacking the means to study.

ABSTRACT. This paper employs C^0-arguments to study the action of the identity component, topologized with the C^∞ Whitney topology, $Diff_{id}^{\Omega,\infty}(M)$ in the group of volume-preserving diffeomorphisms on the space $\mathcal{Z}^1(M)$ of all closed $1-$forms on a compact connected oriented manifold (M, Ω). When M is closed, we recover that $Diff_{id}^{\Omega,\infty}(M)$ is C^0-closed in the group $Diff^\infty(M)$ of all smooth diffeomorphisms of M. This implies that in two dimensions, the identity component in the group of symplectomorphisms is C^0-closed. We discuss several applications in the context of C^0 symplectic geometry for Lefschetz closed symplectic manifolds. This includes an attempt to solve the C^0 flux conjecture.

1. Introduction

Given a compact Riemannian manifold M, and a homeomorphism h of M, there are some interesting questions about the dynamical/geometrical properties of h which can be addressed as follows:

- **Uniform Approximation by Diffeomorphisms:** Can h be uniformly approximated by a sequence of diffeomorphisms? An answer to this question is given in Lemma 2 of [27].
- **Preservation of a Property (P):** If h has a property (P), can one uniformly approximate h by a sequence of diffeomorphisms having the same property (P)? A similar question was studied in [27], [28], and [32].
- **Inheritance of a Property** (Q)**:** If h can be uniformly approximated by a sequence of diffeomorphisms having a property (Q), under which conditions does h inherit the property (Q)? Answers to this question include the rigidity results found in [12] and [17].
- **Algebraic and Topological Properties of $\mathfrak{G}$:** More generally, if $\mathfrak{G}$ is a sub-group of the group of all homeomorphisms of M arising from a certain continuous deformation of a sub-group G of $Diff^\infty(M)$, are there any algebraic or topological properties of G inherited by $\mathfrak{G}$? This is akin to problems solved in [7], [29], also see Theorem A, Theorem B, and Theorem C found in [36].

The goal of this paper is to address some open cases of the aforementioned questions. Specifically, we gather the basic notions and results that extend the findings on Hamiltonian dynamical systems to topological Hamiltonian dynamical systems. The statement to be proven asserts that $Diff_{id}^{\Omega,\infty}(M)$ the identity component w.r.t the C^∞ Whitney topology of $Diff^{\Omega,\infty}(M)$ is C^0-closed in $Diff^\infty(M)$. This fact is likely known, but the proof proposed in this paper is quite elementary, relying on constructions based on metric geometry and global analysis.

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Theorem 1.1. *Let (M,Ω) be an oriented closed manifold. Let $\{\phi_i\} \subseteq Diff_{id}^{\Omega,\infty}(M)$ be a sequence such that $\{\phi_i\} \xrightarrow{C^0} \psi$. If $\psi \in Diff^{\infty}(M)$, then $\psi \in Diff_{id}^{\Omega,\infty}(M)$.*

Theorem 1.1 implies that in two dimensions, the identity component in the group of symplectomorphisms is C^0-closed within the group of symplectomorphisms. This result supports the C^0 flux conjecture in two dimensions. Generally, it is not known if the identity component in the group of symplectomorphisms is C^0-closed within the group of symplectomorphisms (see page 1 of [7]).

To simplify the proof of the main result, we shall need the following continuum of auxiliary results.

Lemma 1.2. *Let (M,g) be a smooth oriented compact Riemannian manifold, and let $\phi \in Diff^{\infty}(M)$. The pull-back map $\phi^* : \Omega^1(M) \to \Omega^1(M), \alpha \mapsto \phi^*\alpha$, is continuous with respect to the norm $\|.\|_{L^2}$, where $\Omega^1(M)$ denote the space of all $1-$forms on M.*

Lemma 1.2 is saying that the pull-back operation by a smooth diffeomorphism doesn't "blow up" the size of $1-$forms as measured by the $\|.\|_{L^2}$ norm. It preserves the "closeness" of $1-$forms. The uniformity coming from the compactness and smoothness is crucial in its proof.

Lemma 1.3. *Let (M,g) be a smooth oriented compact Riemannian manifold, and let $\alpha \in \mathcal{Z}^1(M)$ be fixed. Then, the pull-back map $\mathfrak{D}_\alpha : Diff^{\infty}(M) \to \mathcal{Z}^1(M), \phi \mapsto \phi^*\alpha$, is continuous with respect to the C^0-metric on $Diff^{\infty}(M)$, and the norm $\|.\|_{L^2}$ (resp. the norm $|.|_0$) on $\mathcal{Z}^1(M)$, where $\mathcal{Z}^1(M)$ denote the space of all closed $1-$forms on M.*

Lemma 1.3 states that small changes in the diffeomorphism ϕ will result in small changes in the pull-back $1-$form $\phi^*(\alpha)$ when measured in the L^2-sense (resp. C^0-sense).

2. Preliminaries

Let M be an n-dimensional manifold, and let $\Omega^p(M)$ be the space of all p-forms on M. A diffeomorphism $\phi : M \to M$ preserves a p-form α if $\phi^*(\alpha) = \alpha$. For any $\alpha \in \Omega^p(M)$, we shall denote by $Diff^{\alpha,\infty}(M)$ the group of all diffeomorphisms from M to M that preserve α. We equip the group $Diff^{\infty}(M)$ of all smooth diffeomorphisms of M with the C^∞ Whitney topology, and let $Diff_{id}^{\infty}(M)$ denote the identity component in $Diff^{\infty}(M)$. We refer the readers to [18] and [25] for further studies and illustrations of the groups of diffeomorphisms of smooth manifolds.

2.1. **Isotopies.** A smooth isotopy of M is defined as a smooth map $\Phi : [0,1] \times M \to M$ such that $\Phi(0,.) = id_M$ and for each $t \in [0,1]$, $\Phi(t,.)$ is an element of $Diff^{\infty}(M)$. Intuitively, Φ represents a 1-parameter family of diffeomorphisms of M. For each $\alpha \in \Omega^p(M)$, the space $Iso(M,\alpha)$ consists of all smooth isotopies $\Phi : [0,1] \times M \to M$ that preserve α, and $Diff_{id}^{\alpha,\infty}(M)$ denotes the identity component in $Diff^{\alpha,\infty}(M)$ topologized with the C^∞ Whitney topology [18]. If $Diff^{\alpha,\infty}(M)$ is locally connected by smooth arcs, then $Diff_{id}^{\alpha,\infty}(M)$ coincides with set of all time-one maps of all smooth isotopies in $Diff^{\alpha,\infty}(M)$. For any $\psi \in Diff_{id}^{\alpha,\infty}(M)$, the set $Iso(\psi)_\alpha$ is defined as $\{\Phi \in Iso(M,\alpha) \mid \Phi(1,.) = \psi\}$. When M is an orientable n-manifold with an orientation form Ω, an isotopy $\Phi = (\phi_t)$ of (M,Ω) is Ω-preserving if, for each t, the $(n-1)$-form $\imath(\dot{\phi_t})\Omega$ is closed. Throughout this paper, for each isotopy Φ and each point $p \in M$, $\mathcal{O}_p^\Phi$ denotes the orbit of p under the action of Φ, while $-\mathcal{O}_p^\Phi$ represents the orbit of p under the action of $\Phi^{-1} = (\phi_t^{-1})$.

2.2. **The C^0-topology.** Let d_g be the induced distance by the Riemannian metric g on M. Consider on $Homeo(M)$ the distance

$$d_0(f,h) := \max\left\{\sup_{x\in M} d_g(f(x),h(x)), \sup_{x\in M} d_g(f^{-1}(x),h^{-1}(x))\right\},$$

for all $f,h\in Homeo(M)$. On $\mathcal{P}(Homeo(M),id_M)$, the space of isotopies in $Homeo(M)$, we define the distance

$$\bar{d}(\lambda,\mu) := \max_{t\in[0,1]} d_0(\lambda(t),\mu(t)),$$

for all $\lambda,\mu\in\mathcal{P}(Homeo(M),id_M)$. The metric topology induced by the distance d_0 (resp. the distance $\bar{d}$) on $Homeo(M)$ (resp. $\mathcal{P}(Homeo(M),id_M)$) is the C^0-topology.

2.3. **The geometry of displacements of closed $1-$forms.**

The supremum norm. Let us recall the definition of the supremum norm of a 1-form α: for each $x\in M$, we know that α induces a linear map $\alpha_x : T_xM\to\mathbb{R}$, whose norm is given by

$$\|\alpha_x\|^g := \sup\left\{|\alpha_x(X)|\ ;\ X\in T_xM, \|X\|_g = 1\right\}, \tag{2.1}$$

where $\|\cdot\|_g$ is the norm induced on each tangent space T_xM (at the point x) by the Riemannian metric g. The supremum norm of α, say $|\alpha|_0$, is defined as:

$$|\alpha|_0 := \sup_{x\in M}\|\alpha_x\|^g. \tag{2.2}$$

Given a Riemannian metric g on an oriented manifold M, let $dVol_g$ denote the volume element induced by the Riemannian metric, and by $\flat$ the isomorphism induced by g from the space of vector fields onto the space of all 1-forms so that $\flat(X) := \iota_X g$. The inverse mapping of $\flat$ will be denoted $\sharp$. Thus, one can equip the space of 1-forms with a metric tensor $\tilde{g}$ defined by:

$$\tilde{g}(\alpha_1,\alpha_2) := g(\sharp\alpha_1,\sharp\alpha_2),$$

for all differential 1-forms α_1 and α_2. The L^2-Hodge norm of α is defined as:

$$\|\alpha\|^2_{L^2} := \int_M g(\sharp\alpha,\sharp\alpha)dVol_g. \tag{2.3}$$

The L^2-Hodge norm of a de Rham cohomology class $[\alpha]$ of a closed 1-form α is defined as the norm $\|\mathcal{H}_\alpha\|_{L^2}$, where $\mathcal{H}_\alpha$ is the harmonic representative of $[\alpha]$ [38]. The space of all harmonic 1-forms of a compact Riemannian manifold is a finite dimensional real vector space: there exist positive constants L_0 and L_1 such that

$$L_1\|.\|_{L^2} \leq |.|_0 \leq L_0\|.\|_{L^2}. \tag{2.4}$$

Proof of Lemma 1.2. Let $\alpha\in\Omega^1(M)$, and $\phi\in Diff^\infty(M)$. Set $X_\alpha := \sharp\alpha$, and $Y_\phi := \sharp(\phi^*\alpha)$. The relation $\iota_{X_\alpha}g = \alpha$ implies $\phi^*(\iota_{X_\alpha}g) = \phi^*\alpha = \iota_{Y_\phi}g$, i.e., $g(Y_\phi,Y_\phi) = \phi^*(\iota_{X_\alpha}g)(Y_\phi)$. This yields,

$$g(Y_\phi,Y_\phi)^2 = (g(X_\alpha,\phi_*Y_\phi)\circ\phi)^2 \leq g(X_\alpha,X_\alpha)\circ\phi g(\phi_*Y_\phi,\phi_*Y_\phi)\circ\phi, \tag{2.5}$$

i.e., $g(Y_\phi,Y_\phi) \leq (g(X_\alpha,X_\alpha)\circ\phi)^{1/2}(\phi^*g(Y_\phi,Y_\phi))^{1/2}$. Integrating the latter inequality over M gives

$$\begin{aligned}\textstyle\int_M g(Y_\phi,Y_\phi)dVol_g &\leq \left(\textstyle\int_M g(X_\alpha,X_\alpha)\circ\phi dVol_g\right)^{1/2}\left(\textstyle\int_M \phi^*g(Y_\phi,Y_\phi)dVol_g,\right)^{1/2}\\ &= \left(\textstyle\int_M g(X_\alpha,X_\alpha)(\phi^{-1})^*dVol_g\right)^{1/2}\left(\textstyle\int_M \phi^*g(Y_\phi,Y_\phi)dVol_g\right)^{1/2}.\end{aligned}$$

There exist two smooth functions H_ϕ and K_ϕ such that $(\phi^{-1})^*dVol_g = e^{H_\phi}dVol_g$, and $\phi^*g \leq e^{K_\phi}g$. The above inequalities of integrals implies:

$$\int_M g(Y_\phi, Y_\phi)dVol_g \leq \left(\int_M e^{H_\phi}g(X_\alpha, X_\alpha)dVol_g\right)^{1/2}\left(\int_M e^{K_\phi}g(Y_\phi, Y_\phi)dVol_g\right)^{1/2}.$$

$$\|\phi^*\alpha\|^2_{L^2} \leq \left(\sup_{z\in M} e^{K_\phi(z)}\right)^{1/2}\left(\sup_{z\in M} e^{H_\phi(z)}\right)^{1/2}\|\alpha\|_{L^2}\|\phi^*\alpha\|_{L^2}, \tag{2.6}$$

i.e., $\|\phi^*\alpha\|_{L^2} \leq C_\phi\|\alpha\|_{L^2}$, with $C_\phi := \left(\sup_{z\in M} e^{K_\phi(z)}\right)^{1/2}\left(\sup_{z\in M} e^{H_\phi(z)}\right)^{1/2}$.

Proof of Lemma 1.3. Let $\{\phi_i\} \subseteq Diff^\infty(M)$ be a sequence such that $\phi_i \xrightarrow{C^0} \psi \in Diff^\infty(M)$, and let $x \in M$. Consider a chart $(U, x_1 \dots, x_n)$ whose domain contains x, and for i sufficiently large, we write : $\phi_i^*\alpha_{|U} = \sum_{k=1}^n h^i_k dx_k$, and $\psi^*\alpha_{|U} = \sum_{k=1}^n l_k dx_k$. Denote by g^{ab} the $(a,b)-$entry of the inverse of the matrix $(g_{ab})_{1\leq a\leq n, 1\leq b\leq n}$ with $g_{ab} := g(\frac{\partial}{\partial x_a}, \frac{\partial}{\partial x_b})$. W.l.o.g we may assume that $d_{C^0}(\phi_i, \psi) \leq r(g)/2$ for i sufficiently large where $r(g)$ is the injectivity radius of the Riemannian metric g. As in the proof of Theorem 3.1-[36], we derive that for any smooth embedded curve χ in M we have

$$\lim_{i\to\infty}\int_\chi \phi_i^*\alpha = \int_\chi \psi^*\alpha. \tag{2.7}$$

Fix a bump function f with value 1 on a neighborhood $W \subset U$ of x, for each k, let χ^t_k be the flow of the vector field $f\frac{\partial}{\partial x_k}$, and consider the smooth curve $t \mapsto \chi^t_k(x)$. Let s_1 be the first time that $t \mapsto \chi^t_k(x)$ escapes U, and s_2 be the first time that $t \mapsto (\chi^t_k)^{-1}(x)$ escapes U, and consider the curve $C_{s_0} : t \mapsto \chi^{ts_0}_k(x)$, where $s_0 = \min\{s_1, s_2\}$. Formula (2.7) applied to the curve C_{s_0} implies that

$$\lim_{i\to\infty}\int_0^v (fh^i_k)\circ\chi^u_k(x)du = \int_0^v (fl_k)\circ\chi^u_k(x)du, \tag{2.8}$$

for all k, and all $v \in [0, s_0[$. That is, as in the proof of Lemma 4.3-[35], relation (2.8) implies that $\lim_{i\to\infty} h^i_k = l_k$, for all k. On account of the compactness of M, we fix a finite open cover $\{\mathcal{V}_j\}$ of M. W.l.o.g we may assume that each $\mathcal{V}_j$ is a domain of a certain chart, and let $\{\tau_j\}$ be a partition of unity subordinated to the open cover $\mathcal{V}_j$. Let donote by $\{g^j_{ab}\}$ the components of the metric g restricted to $\mathcal{V}_j$, and let $\{g^{ab}_j\}$ also denote the components of the inverse metric restricted to $\mathcal{V}_j$. Compute,

$$\begin{aligned}
\|\phi_i^*\alpha - \psi^*\alpha\|^2_{L^2} &= \int_M g(\sharp\left(\phi_i^*\alpha - \psi^*\alpha\right), \sharp\left(\phi_i^*\alpha - \psi^*\alpha\right))dVol_g,\\
&= \sum_j \int_{support(\tau_j)} g(\tau_j\sharp\left(\phi_i^*\alpha - \psi^*\alpha\right), \tau_j\sharp\left(\phi_i^*\alpha - \psi^*\alpha\right))dVol_g,\\
&= \sum_j \int_{support(\tau_j)} \left(\tau_j\sum_{k,b}(h^i_{k,j} - l_{k,j})g^{kb}_j\right)^2 g^j_{bb}dVol_g,\\
&\leqslant \sum_j\left(\sup_{support(\tau_j)}\left(\tau_j\sum_k(h^i_{k,j} - l_{k,j})\right)^2\right)\left(\int_{support(\tau_j)}\sum_k g^{kb}_j\left(\sum_b g^{kb}_j g^j_{bb}\right)dVol_g\right),
\end{aligned}$$

$$\leqslant \sum_j \left(\sup_{support(\tau_j)} \left(\tau_j \sum_k (h^i_{k,j} - l_{k,j}) \right)^2 \right) \left(\int_{support(\tau_j)} \sum_k g_j^{kb} \delta_{b,k} dVol_g \right),$$

$$\leqslant \sum_j \left(\sup_{support(\tau_j)} \left(\tau_j \sum_k (h^i_{k,j} - l_{k,j}) \right)^2 \right) \left(\int_{support(\tau_j)} \sum_k g_j^{kk} dVol_g \right),$$

and since $\lim_{i\to\infty} h^i_{k,j} = l_{k,j}$, for each k and each j, we derive that $\sum_j \left(\sup_{support(\tau_j)} \left(\tau_j \sum_k (h^i_{k,j} - l_{k,j}) \right)^2 \right)$ tends to zero as i goes at infinity. This together with the boundedness of each of the integrals $\int_{support(\tau_j)} \left(\sum_k g_j^{kk}\right) dVol_g$, implies that $\|\phi_i^*\alpha - \psi^*\alpha\|_{L^2} \to 0, i \to \infty$. For the norm $|.|_0$, we compute

$$\begin{aligned} |\phi_i^*\alpha - \psi^*\alpha|_0 &= |\sum_j \tau_j \left(\phi_i^*\alpha - \psi^*\alpha\right)|_0, \\ &\leq \sum_j \sup_{support(\tau_j)} |\tau_j \left(\phi_i^*\alpha - \psi^*\alpha\right)|_0, \\ &= \sum_j \sup_{support(\tau_j)} |\tau_j \sum_k (h^i_{k,j} - l_{k,j}) dx^j_k|_0, \\ &\leq \left(\max_j \sum_k |dx^j_k|_0 \right) \sum_j \left(\sup_{support(\tau_j)} \left(\tau_j \sum_k (h^i_{k,j} - l_{k,j}) \right) \right). \end{aligned} \quad (2.9)$$

That is,

$$|\phi_i^*\alpha - \psi^*\alpha|_0 \leq \left(\max_j \sum_k |dx^j_k|_0 \right) \sum_j \left(\sup_{support(\tau_j)} \left(\tau_j \sum_k (h^i_{k,j} - l_{k,j}) \right) \right) \to 0, i \to \infty.$$

2.4. **Displacement of closed $1-$forms along smooth curves in M.** Assume $\partial M = \emptyset$, for each fixed $p \in M$ and for all $z \in M$, let ξ_z denote a smooth curve from p to z. Let $\psi \in Diff^\infty_{id}(M)$, and $\alpha \in \mathcal{Z}^1(M)$, we have a smooth function

$$\nu_p^{\psi,\alpha} : M \to \mathbb{R}, z \mapsto \int_{\xi_z} \left(\psi^*\alpha - \alpha\right).$$

This function is well-defined since it does not depend on the choice of any curve from p to z. The real number $\int_{\xi_z} (\psi^*\alpha - \alpha)$ can be viewed, in a certain sense, as the displacement of α under the action of ψ along a curve ξ_z [34]. For each $p \in M$ and $\alpha \in \mathcal{Z}^1(M)$ (fixed), we assign to each $\psi \in Diff^\infty_{id}(M)$, the quantity $\Delta(\psi, \alpha)_p$ defined as:

$$\Delta(\psi,\alpha)_p := \begin{cases} 0, \quad if\ \alpha = 0 \\ \dfrac{1}{\|\alpha\|_{L^2}} \int_M \nu_p^{\psi,\alpha}\Omega \ \ , \ if\ \alpha \in \left(\mathcal{Z}^1(M)\backslash\{0\}\right). \end{cases} \quad (2.10)$$

A geometric meaning of the map Δ. Considering a continuous path in the group of all measure-preserving homeomorphisms of the 2-disk $\mathbb{D}^2$ with time-one map h, Fathi assigns to elements $x, y \in \mathbb{D}^2$, a real number $Ang_h(x,y)$ that "measures" the angle between the vector u from x to y, and the vector v from $h(x)$ to $h(y)$. This yields an invariant known as the angular variation invariant [14]. In the present context, let $\Phi = (\phi_t)$ be a smooth isotopy in $Diff^\infty_{id}(M)$. Pick $x, y \in M$. Rather than considering the angle between vectors as mentioned in the case of the 2-disk $\mathbb{D}^2$, here we instead consider smooth curves γ from x to y. To understand the dynamics of the gap variation created when ϕ_1 moves a curve γ, we need to compare the actions of the curves γ and $\phi_1 \circ \gamma$ on closed 1-forms:

$$\left(\int_\gamma \phi_1^*\alpha - \int_\gamma \alpha\right).$$

In other words, the map Δ seems to emanate from the obstruction that prevents a diffeomorphism isotopic to the identity map from always preserving closed 1-forms.

The map Δ has the following properties which can be found in [34].

Corollary 2.1. [34] *Let (M, Ω) be a closed connected orientable manifold with a volume form Ω. Fix $x \in M$, and $\alpha \in \left(\mathcal{Z}^1(M)\backslash\{0\}\right)$. For each $\psi \in Diff_{id}^{\Omega,\infty}(M)$, we have*

$$\Delta(\psi, \alpha)_x = \frac{1}{\|\alpha\|_{L^2}}\langle[\alpha], \widetilde{S}_\Omega(\Psi)\rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}}\int_{\mathcal{O}_x^\Psi}\alpha,$$

and the real number $\Delta(\psi, \alpha)_x$ does not depend on any choice of an isotopy Ψ from the identity to ψ, where $\mathcal{O}_x^\Psi$ is the orbit of x under Ψ, and $\widetilde{S}_\Omega$ is the flux homomorphism for the volume form Ω.

Lemma 2.2. [34] *For each $x \in M$, and for each $\alpha \in \mathcal{Z}^1(M)$ both fixed, the map $\Delta(., \alpha)_x : Diff_{id}^\infty(M) \to \mathbb{R}, \psi \mapsto \Delta(\psi, \alpha)_x$, is continuous with respect to the C^0-metric on $Diff_{id}^\infty(M)$.*

Let us provide a sketch of the construction of the flux homomorphism $\widetilde{S}_\Omega$: As a consequence of Moser's theorem, the group $Diff_{id}^{\Omega,\infty}(M)$ is locally connected by smooth arcs [4]-page 124. Thus, the identity component $Diff_{id}^{\Omega,\infty}(M)$ consists of elements which are isotopic to the identity through smooth paths in $Diff^{\Omega,\infty}(M)$. The universal covering $\widetilde{Diff_{id}^{\Omega,\infty}}(M)$ is represented by homotopy classes of volume-preserving isotopies relative to fixed endpoints [37]. We denote the equivalence class represented by an isotopy Φ as $\{\Phi\}$. There is an epimorphism $\widetilde{S}_\Omega$ defined on the space $\widetilde{Diff_{id}^{\Omega,\infty}}(M)$ by the following formula:

$$\widetilde{S}_\Omega : \widetilde{Diff_{id}^{\Omega,\infty}}(M) \to H^{n-1}(M, \mathbb{R}), \{\Phi\} \mapsto \left[\int_0^1 (i_{\dot{\phi}_t}\Omega)\, dt\right].$$

The subgroup $\Gamma_\Omega := \widetilde{S}_\Omega(\pi_1(Diff_{id}^{\Omega,\infty}(M)))$, called the volume flux group, is a discrete subgroup of $H^{n-1}(M, \mathbb{R})$ [34], [37]. The map $\widetilde{S}_\Omega$ also induces a surjective map $S_\Omega : Diff_{id}^{\Omega,\infty}(M) \to H^{n-1}(M, \mathbb{R})/\Gamma_\Omega$ called the flux homomorphism for volume-preserving diffeomorphisms, with $Ham_\Omega(M) := \ker S_\Omega$.

Proposition 2.3. *Let $\phi \in Ham_\Omega(M)$ (fixed), and $h \in Diff_{id}^{\Omega,\infty}(M)$. We have,*

$$|\Delta(\phi \circ h \circ \phi^{-1}, \alpha)_{\phi(x)}| = \left(\frac{\|\phi^*\alpha\|_{L^2}}{\|\alpha\|_{L^2}}\right)|\Delta(h, \phi^*\alpha)_x|,$$

for each $x \in M$, and all non-trivial $\alpha \in \mathcal{Z}^1(M)$.

Proof. Let $\phi \in Ham_\Omega(M)$ (fixed), and $h \in Diff_{id}^{\Omega,\infty}(M)$. Consider H to be a smooth isotopy in $Diff_{id}^{\Omega,\infty}(M)$ from the identity to h, hence $\phi \circ H \circ \phi^{-1}$ is a smooth isotopy in

$Diff_{id}^{\Omega,\infty}(M)$ from the identity to $\phi\circ h\circ\phi^{-1}$. Corollary 2.1 implies that

$$\begin{aligned} |\Delta(\phi\circ h\circ\phi^{-1},\alpha)_x| &= |\frac{1}{\|\alpha\|_{L^2}}\langle[\alpha],\widetilde{S}_\Omega(H)\rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}}\int_{\phi\left(\mathcal{O}^H_{\phi^{-1}(x)}\right)}\alpha|, && (2.11)\\ &= |\frac{1}{\|\alpha\|_{L^2}}\langle[\phi^*\alpha],\widetilde{S}_\Omega(H)\rangle - \frac{Vol_\Omega(M)}{\|\alpha\|_{L^2}}\int_{\mathcal{O}^H_{\phi^{-1}(x)}}\phi^*\alpha|, && (2.12)\\ &= \left(\frac{\|\phi^*\alpha\|_{L^2}}{\|\alpha\|_{L^2}}\right)|\Delta(h,\phi^*\alpha)_{\phi^{-1}(x)}|. && (2.13)\end{aligned}$$

□

2.5. **A norm on the group $Diff_{id}^{\Omega,\infty}(M)$.** For each $\psi\in Diff_{id}^{\Omega,\infty}(M)$, set

$$\|\psi\|^\infty := \sup_{\alpha\in\mathcal{B}(1)}\left(\sup_{z\in M}|\tilde{\Delta}(\psi,\alpha)_z|\right), \qquad (2.14)$$

where $\tilde{\Delta}(\psi,\alpha)_z := \|\alpha\|_{L^2}\Delta(\psi,\alpha)_z$ and $\mathcal{B}(1) := \{\alpha\in\mathcal{Z}^1(M) : \|\alpha\|_{L^2} = 1\}$.

We have the following facts.

Proposition 2.4. [34] *The rule $\|.\|^\infty$ has the following properties:*

1. *Positivity: $\|\psi\|^\infty\geq 0$, for all $\psi\in Diff_{id}^{\Omega,\infty}(M)$.*
2. *Triangle inequality: $\|\psi\circ\phi\|^\infty\leq\|\psi\|^\infty+\|\phi\|^\infty$, for all $\psi,\phi\in Diff_{id}^{\Omega,\infty}(M)$.*
3. *Duality: $\|\psi^{-1}\|^\infty=\|\psi\|^\infty$, for all $\psi\in Diff_{id}^{\Omega,\infty}(M)$.*
4. *If $\|\psi\|^\infty=0$, then $\psi=id_M$.*

The following key property of the norm $\|.\|^\infty$ shall be needed.

Proposition 2.5. *Let $\phi\in Ham_\Omega(M)$ (fixed). Then, the norms $h\mapsto\|\phi\circ h\circ\phi^{-1}\|^\infty$ and $h\mapsto\|h\|^\infty$ are equivalent.*

Proof. Proposition 2.3 implies that

$$\|\phi\circ h\circ\phi^{-1}\|^\infty\leq \sup_{\alpha\in\mathcal{B}(1)}\|\phi^*\alpha\|_{L^2}\|h\|^\infty,$$

and applying Lemma 1.2 we obtain

$$\|\phi\circ h\circ\phi^{-1}\|^\infty\leq C_\phi\|h\|^\infty.$$

With the identity, $h=\phi^{-1}\left(\phi\circ h\circ\phi^{-1}\right)\phi$, we derive that

$$\|h\|^\infty=\|\phi^{-1}\left(\phi\circ h\circ\phi^{-1}\right)\phi\|^\infty\leq C_{\phi^{-1}}\|\phi\circ h\circ\phi^{-1}\|^\infty.$$

Thus,

$$\frac{1}{C_{\phi^{-1}}}\|h\|^\infty\leq\|\phi\circ h\circ\phi^{-1}\|^\infty\leq C_\phi\|h\|^\infty.$$

The constant C_ϕ is defined as in the proof of Lemma 1.2.

□

A norm $\|.\|$ on a group G induces a metric $d_{\|.\|}(A,B) := \|AB^{-1}\|$, for all $A,B\in G$. Let (M,Ω) and $(N,\overline{\Omega})$ be two closed connected oriented manifolds, $\|.\|_M^\infty$ (resp. $\|.\|_N^\infty$) be the norm on $Diff_{id}^{\Omega,\infty}(M)$ (resp. $Diff_{id}^{\overline{\Omega},\infty}(N)$), and consider
$\mathcal{B}_M(1) := \{\alpha\in\mathcal{Z}^1(M) : \|\alpha\|_{L^2,M} = 1\}$ (resp. $\mathcal{B}_N(1) := \{\alpha\in\mathcal{Z}^1(N) : \|\alpha\|_{L^2,N} = 1\}$).
Question (1): If $\phi : M\longrightarrow N$ is a smooth diffeomorphism such that $\phi^*(\overline{\Omega}) = \Omega$, then are the metric spaces $(Diff_{id}^{\Omega,\infty}(M), d_{\|.\|_M^\infty})$ and $(Diff_{id}^{\overline{\Omega},\infty}(N), d_{\|.\|_N^\infty})$ always isometric?
Question (2): Let (M,Ω) and $(N,\overline{\Omega})$ be two closed connected oriented manifolds.

Assume that $I : Diff_{id}^{\Omega,\infty}(M) \to Diff_{id}^{\overline{\Omega},\infty}(N)$ is a group isomorphism that preserves distance, then what can we say about the manifolds (M, Ω) and $(N, \overline{\Omega})$?

2.6. **The volume-preserving displacement energy.**

Definition 2.6. For each non-empty open subset $\mathcal{U} \subset M$, we define the displacement energy of $\mathcal{U}$ as:

$$E^{\Omega}(\mathcal{U}) := \inf\{\|\psi\|^{\infty} \colon \psi \in Diff_{id}^{\Omega,\infty}(M), \psi(\mathcal{U}) \cap \mathcal{U} = \emptyset\}, \tag{2.15}$$

if some elements of $Diff_{id}^{\Omega,\infty}(M)$ displace $\mathcal{U}$, and $E^{\Omega}(\mathcal{U}) := +\infty$ if no element of $Diff_{id}^{\Omega,\infty}(M)$ can displace $\mathcal{U}$.

Theorem 2.7. *For each non-empty open subset $\mathcal{U} \subset M$, we have $E^{\Omega}(\mathcal{U}) > 0$.*

Proposition 2.8. [34] *For any non-empty open subset $B \subset M$, there exists a pair $\phi, \psi \in Ham_{\Omega}(M)$ each of which is supported in B such that the commutator $[\phi, \psi] := \psi^{-1} \circ \phi^{-1} \circ \psi \circ \phi$ is different from the identity map.*

We shall say that a map ψ completely displaces a non-empty subset $\mathcal{U} \subset M$ if $\psi(\mathcal{U}) \cap \mathcal{U} = \emptyset$. Let us denote by $D_{\Omega}(\mathcal{U})$ the set of all $\phi \in Diff_{id}^{\Omega,\infty}(M)$ that completely displace $\mathcal{U}$.

Proposition 2.9. [34] *Consider a non-empty open subset B in M. Let ϕ and ψ be defined as in Proposition 2.8 and let $f \in D_{\Omega}(B)$. We have $\big[[f, \phi^{-1}], \psi\big] = [\phi, \psi]$.*

Proof of Theorem 2.7 : For each non-empty open subset $\mathcal{U} \subset M$, either $D_{\Omega}(\mathcal{U})$ is empty or non-empty. In the case $D_{\Omega}(\mathcal{U}) = \emptyset$, we have $E^{\Omega}(\mathcal{U}) = +\infty$: We are done. Assume that $D_{\Omega}(\mathcal{U})$ is non-empty. By the characterization of the infimum, for each positive integer k, there exists a volume-preserving diffeomorphism f_k such that

$$\|f_k\|^{\infty} < E^{\Omega}(\mathcal{U}) + \frac{1}{k}, \tag{2.16}$$

and $f_k \in D_{\Omega}(\mathcal{U})$. Fix the integer k large enough, and apply Proposition 2.8 with $B := \mathcal{U}$, to derive that there exists a pair $\phi, \psi \in Ham_{\Omega}(M)$ each of which is supported in $\mathcal{U}$ such that the commutator $[\phi, \psi] := \psi^{-1} \circ \phi^{-1} \circ \psi \circ \phi$ is different from the identity map, and since $f_k \in D_{\Omega}(\mathcal{U})$, we derive from Proposition 2.9 that for k large enough, we equally have $[[f_k, \phi^{-1}], \psi] = [\phi, \psi]$. Apply Lemma 1.2 and Proposition 2.5 twice in a suitable way to get

$$\begin{aligned}
\|[\phi, \psi]\|^{\infty} &= \|[\phi, \psi]^{-1}\|^{\infty}, \\
&= \|[[f_k, \phi^{-1}], \psi]^{-1}\|^{\infty}, \\
&= \|[f_k, \phi^{-1}] \circ \psi^{-1} \circ [f_k, \phi^{-1}]^{-1} \circ \psi\|^{\infty}, \\
&\leq (C_{\psi^{-1}} + 1)\|[f_k, \phi^{-1}]\|^{\infty}, \\
&= (C_{\psi^{-1}} + 1)(C_{\phi} + 1)\|f_k\|^{\infty}.
\end{aligned} \tag{2.17}$$

Combining (2.16) together with (2.17) implies

$$0 < \frac{\|[\phi, \psi]\|^{\infty}}{(C_{\psi^{-1}} + 1)(C_{\phi} + 1)} \leqslant \|f_k\|^{\infty} < E^{\Omega}(\mathcal{U}) + \frac{1}{k}, \tag{2.18}$$

for k large enough. □

Theorem 2.10. *Let $\{\phi_i\} \subset Diff_{id}^{\Omega,\infty}(M)$, $\phi \in Diff_{id}^{\Omega,\infty}(M)$, and $\psi : M \longrightarrow M$ be a map such that*

- *$\{\phi_i\}$ uniformly converges to ψ and,*
- *$\|\phi_i \circ \phi^{-1}\|^{\infty} \longrightarrow 0, i \longrightarrow \infty$.*

Then, $\psi = \phi$.

Proof. Assume that $\psi\circ\phi^{-1} \neq id_M$. That is, there exists a non-empty open subset $B \subset M$ such that $\psi \circ \phi^{-1} \in D_\Omega(B)$. Fix an integer k large enough, since $\phi_i \xrightarrow{C^0} \psi$, we equally have $\phi_i \circ \phi^{-1} \xrightarrow{C^0} \psi \circ \phi^{-1}$. Thus, we may assume $\phi_k \circ \phi^{-1} \in D_\Omega(B)$. The positivity of the volume-preserving displacement energy implies that $0 < E^\Omega(B) \leqslant \|\phi_k \circ \phi^{-1}\|^\infty$, for each k large enough. The latter inequalities contradict each other as k tends to infinity. □

2.7. **The proof of theorem 1.1.** We start this subsection with the following important remark.

Remark 2.11. Let ϕ be a diffeomorphism, X be a vector field and θ be any differential form, we have

$$(\phi^{-1})^*[i_X(\phi^*\theta)] = i_{\phi_*X}\theta. \tag{2.19}$$

On an oriented manifold (M,Ω), formula (2.19) tells us that, for $\theta = \Omega$, if ϕ preserves the volume form Ω, then $i_{\phi_*X}\Omega = (\phi^{-1})^*(i_X\Omega)$, for all vector field X. For instance, assume that $\phi \neq id_M$ is not volume-preserving. Then there exists a non-empty open subset $\mathfrak{U}$ in M which is completely displaced by ϕ. Consider β to be a differential form of degree $l := (\dim M - 2)$, f to be a smooth function with small support in an open subset σ strictly contained in $\mathfrak{U}$, and then set $\bar\beta := f\beta$. The vector field Y defined as $i_Y\Omega = d\bar\beta$ is a divergence-free vector field supported in σ. Let $z \in \mathfrak{U}$ so that $z \notin \sigma$, and consider vector fields $X^1,\ldots,X^{l+1}$ such that $\Omega(\phi_*Y, X^1,\ldots,X^{l+1})(\phi(z)) \neq 0$, i.e., $(i_{\phi_*Y}\Omega)\left(X^1,\ldots,X^{l+1}\right)|_{\phi(z)} \neq 0$. On the other hand, $(\phi^{-1})^*(i_Y\Omega)\left(X^1,\ldots,X^{l+1}\right)|_{\phi(z)} = \Omega_z\left(0, T_z\phi^{-1}X^1_z,\ldots,T_z\phi^{-1}X^{l+1}_z\right) = 0$, because $z \notin \sigma$. When ϕ is not volume-preserving, there exists a divergence-free vector field Y such that the form $i_Y\Omega$ is exact, and in this case $(\phi^{-1})^*(i_Y\Omega)$ is not equal to $i_{\phi_*Y}\Omega$.

We shall also need the following result.

Theorem 2.12. [27, 32] *Let M be a closed connected smooth manifold of dimension n. If $n \leq 3$, then any homeomorphism can be uniformly approximated by diffeomorphisms. If $n \geq 5$, then a homeomorphism h of M can be uniformly approximated by a diffeomorphism ϕ if and only if h is isotopic to a diffeomorphism.*

Remark 2.13. **(The set $\mathfrak{AHom}^\Omega_\delta(\Psi)$).**

(a) The isotopies concerned in Theorem 2.12 are continuous isotopies: a continuous map $\Phi : [0,1] \to Homeo(M), t \mapsto \phi_t$.

(b) When M is orientable, if the homeomorphism h in Theorem 2.12 is volume-preserving, then it can be uniformly approximated by a volume-preserving diffeomorphism [27], [32]. It follows that any continuous path $\Psi := \{\psi^t\}$ in the identity component of the group of all volume-preserving homeomorphisms (w.r.t the C^0 metric) can be uniformly approximated by a continuous path $\Theta := \{\theta_t\}$ in $Diff^{\Omega,\infty}_{id}(M)$. Let $r(g)$ denote the injectivity radius of the Riemannian metric g on M. For each $\delta \in]0, r(g)[$ we may choose Θ such that $d_{C^0}(\Theta,\Psi) < \delta$, and if the time-one map of Ψ is a volume-preserving diffeomorphism, then we may assume that it coincides with the time-one map of Θ.

(c) The continuous path $\Theta := \{\theta_t\}$ in $Diff^{\Omega,\infty}_{id}(M)$ is homotopic relatively to fixed endpoints to a smooth path $\Phi := \{\phi^t\}$ in $Diff^{\Omega,\infty}_{id}(M)$ [4], [18].

(d) When Ψ is a continuous path in the identity component of the group of all volume-preserving homeomorphisms (w.r.t the C^0 metric), we shall let $\mathfrak{AHom}^\Omega_\delta(\Psi)$ be the set consisting of all smooth volume-preserving isotopies homotopic relatively to fixed endpoints to an approximate Θ (equally volume-preserving) of Ψ such that $d_{C^0}(\Theta,\Psi) < \delta$.

Proof of Theorem 1.1: We shall proceed step by step.

• **Step 0: Uniform convergence and divergence-free vector fields.**

Let $\{\phi_i\} \subseteq Diff^{\Omega,\infty}_{id}(M)$ such that $\{\phi_i\} \xrightarrow{C^0} \psi \in Diff^{\infty}(M)$. Assume that ψ is not volume-preserving. Remark 2.11 suggests that there is a divergence-free vector field Y such that

$$i_{\psi_* Y}\Omega \neq (\psi^{-1})^* (i_Y\Omega) = d\left((\psi^{-1})^*\beta\right), \tag{2.20}$$

with $i_Y\Omega = d\beta$. Let $\Theta := \{\varphi^t\}$ be the flow of Y, and $\Xi = \{E_t\}$ be the flow of a vector field Z defined as $i_Z\Omega = d\left((\psi^{-1})^*\beta\right)$. Formula (2.20) also implies that $\Lambda := \psi \circ \Theta \circ \psi^{-1} \neq \Xi$. It is clear that the sequence of volume-preserving isotopies $\Lambda_i := \phi_i \circ \Theta \circ \phi_i^{-1}$ converges uniformly to Λ. Let s_0 be a time such that

$$\psi \circ \varphi^{s_0} \circ \psi^{-1} \neq E_{s^0}. \tag{2.21}$$

For $t = s^0$, the sequence volume-preserving diffeomorphisms $\phi_i \circ \varphi^{s_0} \circ \phi_i^{-1}$ converges uniformly to $\psi \circ \varphi^{s_0} \circ \psi^{-1}$.

• **Step 1: A path in $Diff^{\Omega,\infty}_{id}(M)$ with time-one map $\left(\phi_i \circ \varphi^{s_0} \circ \phi_i^{-1}\right)^{-1} \circ E_{s^0}$.**

Let $u : [0,1] \to [0,1]$ be a smooth function such that its restriction to an interval $[0,b]$ vanishes while its restriction to $[(1-b),1]$ is the constant function 1 with $0 < b \leq \dfrac{1}{8}$. Consider the smooth functions $\lambda(s) := u(2s)$ for all $0 \leq s \leq \frac{1}{2}$, and $\tau(s) := u(2s-1)$ for all $\frac{1}{2} \leq s \leq 1$. Then, let $\Upsilon_{s^0} : t \mapsto E_{ts^0}$, and $\varPhi_i^{s^0} : t \mapsto \phi_i \circ \varphi^{ts^0} \circ \phi_i^{-1}$, and define

$$((\varPhi_i^{s^0})^{-1} *_r \Upsilon_{s^0})(a) = \begin{cases} (\varPhi_i^{s_0}(\lambda(a)))^{-1}, & \text{if } 0 \leq a \leq \frac{1}{2}, \\ (\varPhi_i^{s^0}(1))^{-1} \circ \Upsilon_{s^0}(\tau(a)), & \text{if } \frac{1}{2} \leq a \leq 1, \end{cases}$$

for all i. We have $((\varPhi_i^{s^0})^{-1} *_r \Upsilon_{s^0})(0) = id_M$, and the time-one map of the path $(\varPhi_i^{s^0})^{-1} *_r \Upsilon_{s^0}$ is $\left(\phi_i \circ \varphi^{s^0} \circ \phi_i^{-1}\right)^{-1} \circ E_{s^0}$. Since $E_{s^0} \in Ham_\Omega(M)$, Proposition 2.5 implies that the norm $\| \left(\phi_i \circ \varphi^{s^0} \circ \phi_i^{-1}\right)^{-1} \circ E_{s^0}\|^\infty$ is equivalent to the norm $\|E_{s^0} \circ \left(\left(\phi_i \circ \varphi^{s^0} \circ \phi_i^{-1}\right)^{-1} \circ E_{s^0}\right) \circ E_{s^0}^{-1}\|^\infty$, and so we shall estimate the norm $\|E_{s^0} \circ \left(\phi_i \circ \varphi^{s^0} \circ \phi_i^{-1}\right)^{-1}\|^\infty$ in the next step.

• **Step 2: Estimating the norm $\|E_{s^0} \circ \left(\phi_i \circ \varphi^{s_0} \circ \phi_i^{-1}\right)^{-1}\|^\infty$.**

If necessary, smooth the path $\Upsilon_{s^0} *_r (\varPhi_i^{s^0})^{-1}$ at the point $a = 1/2$ [18]. The path $\Upsilon_{s^0} *_r (\varPhi_i^{s^0})^{-1}$ has a trivial flux, so from the definition of the norm $\|.\|^\infty$, there exist $z \in M$ and for each positive integer k, there exists $\alpha_k \in \mathcal{B}(1)$ such that

$$\|E_{s^0} \circ \left(\phi_i \circ \varphi^{s^0} \circ \phi_i^{-1}\right)^{-1}\|^\infty \leq \left|\int_{\mathcal{O}_z^{\Upsilon_{s^0} *_r (\varPhi_i^{s^0})^{-1}}} \alpha_k\right| + \frac{1}{k}.$$

We have,

$$\left|\int_{\mathcal{O}_z^{\Upsilon_{s^0} *_r (\varPhi_i^{s^0})^{-1}}} \alpha_k\right|$$
$$= \left|\int_0^1 (\alpha_k(Z) \circ E_{us^0}(z) - (E_{s^0}^*(\alpha_k))\,(\psi_i \circ (\varphi^{s^0})^{-1})_*(Y)) \circ G_{us^0,i}(z))du\right|,$$

for each i , and each positive integer k where $G_{us^0,i} := \left(\phi_i \circ \varphi^{us^0} \circ \phi_i^{-1}\right)^{-1}$. On the other hand, from the equality $\alpha \wedge \Omega = 0$, we derive that

$$\alpha_k(Z) \circ E_{us^0}(z)\Omega_z = ((\psi^{-1})^*\alpha_k \wedge d\beta)_{|z}.$$

The equality $E^*_{s^0}(\alpha_k)\wedge\Omega = 0$ also implies that

$$\left(E^*_{s^0}(\alpha_k)\right)(\psi_i\circ(\varphi^{s^0})^{-1})_*(Y))\circ\left(\phi_i\circ\varphi^{us^0}\circ\phi_i^{-1}\right)^{-1}(z)\Omega_z = ((\phi_i^{-1})^*\alpha_k\wedge d\beta)_{|z},$$

for each i, and each positive integer k. Fix vector fields $X^1,\dots,X^{\dim M}$ such that $C^z_\Omega := \Omega(X^1,\dots,X^{\dim M})(z) > 0$, (w.l.o.g we may choose X^j so that $\sup_x \|X^j_x\|_g = 1$), and compute

$$\begin{aligned}
&|\int_0^1 \alpha_k(Z)\circ E_{us^0}(z)du - \int_0^1 \left(E^*_{s^0}(\alpha_k)\right)(\phi_i\circ(\varphi^{s_0})^{-1})_*(Y))\circ\left(\phi_i\circ\varphi^{us_0}\circ\phi_i^{-1}\right)^{-1}(z)du| \\
&= \frac{|(\int_0^1 \alpha_k(Z)\circ E_{us^0}(z)du - \int_0^1 \left(E^*_{s^0}(\alpha_k)\right)(\phi_i\circ(\varphi^{s_0})^{-1})_*(Y))\circ\left(\phi_i\circ\varphi^{us_0}\circ\phi_i^{-1}\right)^{-1}(z)du)C^z_\Omega|}{C^z_\Omega} \\
&= \frac{|(\left((\phi_i^{-1})^*\alpha_k - (\psi^{-1})^*\alpha_k\right)\wedge d\beta)(X^1,\dots,X^{\dim M})(z)|}{C^z_\Omega} \\
&\leq \frac{|\left((\phi_i^{-1})^*\alpha_k - (\psi^{-1})^*\alpha_k\right)\wedge d\beta|_0}{C^z_\Omega},
\end{aligned}$$

for each positive integer k.
Since $|\left((\phi_i^{-1})^*\alpha_k - (\psi^{-1})^*\alpha_k\right)\wedge d\beta|_0 \leq \dim(M)|d\beta|_0|(\phi_i^{-1})^*\alpha_k - (\psi^{-1})^*\alpha_k|_0$, for each positive integer k, it follows from Lemma 1.3 that
$|\left((\phi_i^{-1})^*\alpha_k - (\psi^{-1})^*\alpha_k\right)\wedge d\beta|_0 \to 0, i\to\infty$, each positive integer k. Combining the above estimates together yields

$$\left|\int_0^1 \alpha_k(Z)\circ E_{us^0}(z)du - \int_0^1 \left(E^*_{s^0}(\alpha_k)\right)(\phi_i\circ(\varphi^{s_0})^{-1})_*(Y))\circ\left(\phi_i\circ\varphi^{us_0}\circ\phi_i^{-1}\right)^{-1}(z)du\right| \underset{i\to\infty}{\to} 0, \tag{2.22}$$

for each positive integer k. This implies that $\|E_{s^0}\circ\left(\phi_i\circ\varphi^{s_0}\circ\phi_i^{-1}\right)^{-1}\|^\infty \to 0, i\to\infty$.

- **Step 3: Determining the volume-preserving nature of ψ.**

Summarising **Step 0** and **Step 2** yields that $\phi_i\circ\varphi^{s^0}\circ\phi_i^{-1} \xrightarrow{C^0} \psi\circ\varphi^{s^0}\circ\psi^{-1}$, and $\|E_{s^0}\circ\left(\phi_i\circ\varphi^{s^0}\circ\phi_i^{-1}\right)^{-1}\|^\infty \to 0,$ as $i\to\infty$. This implies through Theorem 2.10 that

$$E_{s^0} = \psi\circ\varphi^{s^0}\circ\psi^{-1}. \tag{2.23}$$

Therefore, (2.21) and (2.23) contradicts each other: Thus, $\psi \in Diff^{\Omega,\infty}(M)$. Let us show the existence of a smooth volume-preserving isotopy from the identity map to ψ. Fix i sufficiently large, consider a volume-preserving isotopy $\Phi_i = \{\phi_i^t\}$ from the identity map to ϕ_i, and set $\Upsilon^i_t := \psi\circ\phi_i^t\circ\psi^{-1}$. Since $\psi\circ\phi_j^1\circ\psi^{-1}$ uniformly converges to ψ inside the group $Homeo(M,\Omega)$ of volume-preserving homeomorphisms, then from the locally path-connectedness of $Homeo(M,\Omega)$, we can find a volume-preserving isotopy $\mathit{\Phi}_i$ in $Homeo(M,\Omega)$ from $\psi\circ\phi_i^1\circ\psi^{-1}$ to ψ for i sufficiently large. Apply Theorem 2.12 to approximate $\mathit{\Phi}_i$ by a continuous isotopy $\bar{\mathit{\Phi}}_i$ in $Diff^{\Omega,\infty}(M)$ (relatively to fixed endpoints). Juxtaposing the paths Υ^i and $\bar{\mathit{\Phi}}_i$ in a suitable way yields a continuous isotopy in $Diff^{\Omega,\infty}(M)$ from the identity map to ψ. The latter isotopy is homotopic relatively for fixed endpoints to a smooth volume-preserving isotopy from the identity map to ψ. Therefore, $\psi \in Diff^{\Omega,\infty}_{id}(M)$.□

3. Application to C^0 symplectic geometry

Among the most celebrated discoveries in symplectic geometry/topology of the last three decades are the Hofer norm on the group $Ham(M,\omega)$ of Hamiltonian diffeomorphisms of a symplectic manifold (M,ω) (1990) [19], [20] and the Eliashberg-Gromov rigidity theorem (1986, 1987) [13], [17]. The latter asserts that the group $Symp(M,\omega)$ of symplectomorphisms of (M,ω) forms a closed subset of the group $Diff^\infty(M)$ of all diffeomorphisms of M equipped with the C^0-topology. This implies that there is symplectic geometry underlying the C^0-topology, which is now called C^0-symplectic topology. Let $Sympeo(M,\omega)$ be the closure of $Symp(M,\omega)$ in the group $Homeo(M)$ of all homeomorphisms of M equipped with the C^0-topology. The set $Sympeo(M,\omega)$ forms a group called the group of symplectic homeomorphisms of (M,ω).

3.1. Symplectic homeomorphisms.

Using a blend of the C^0-topology and the Hofer norm, Oh and Müller initiated the study of topological Hamiltonian dynamical systems by introducing the notion of Hamiltonian homeomorphisms [29].

Definition 3.1. (Oh-Müller, [29]) A homeomorphism h is called a Hamiltonian homeomorphism if there exists $\lambda \in \mathcal{P}(Homeo(M), id)$ with $\lambda(1) = h$, and there exists a Cauchy sequence Φ_i of smooth Hamiltonian isotopies in the Hofer metric such that Φ_i converges uniformly to λ.

The set $Hameo(M,\omega)$ consisting of all Hamiltonian homeomorphisms of any closed connected symplectic manifold (M,ω) forms a group [29].
Banyaga generalized the Hofer metric into a metric on the group of all smooth symplectic isotopies of a closed connected symplectic manifold (M,ω), and used it to defined the group of all strong symplectic homeomorphisms [1].

Definition 3.2. (Banyaga, [5]) A homeomorphism h is called a strong symplectic homeomorphism if there exists a sequence $\Psi_i := \{\psi_i^t\}$, of smooth symplectic isotopies which is Cauchy in both the Hofer-like metric and the C^0-metric, such that $\psi_i^1 \xrightarrow{C^0} h$.

Let $SSympeo(M,\omega)$ denote the group of all strong symplectic homeomorphisms. This group coincides with $Hameo(M,\omega)$ when M is simply connected. In [5], the author claimed that there was a topological flux geometry underlying the group $SSympeo(M,\omega)$: An affirmative answer to this claim for closed connected symplectic manifolds of Lefschetz type is given in [36]. The group $SSympeo(M,\omega)$ contains $Hameo(M,\omega)$ as a normal subgroup [5].

We have the following open conjectures which can be found in [5].

Conjecture (A): The commutator subgroup $[SSympeo(M,\omega), SSympeo(M,\omega)]$ coincides with $Hameo(M,\omega)$.

Let $Sympeo_0(M,\omega)$ be the identity component in $Sympeo(M,\omega)$ with respect to the C^0-topology.

Conjecture (B): The inclusion $SSympeo(M,\omega) \subseteq Sympeo_0(M,\omega)$ is strict.

It has been shown in [11, 10] that conjecture (B) holds for compact symplectic surfaces of genus 0. Recently, this result has been generalized to all closed symplectic surfaces in [23].

Question (A) (A weak C^0 flux conjecture, [7]): If $\psi \in Symp(M,\omega) \cap Hameo(M,\omega)$, then is $\psi \in Ham(M,\omega)$?

The shadow of smooth symplectic diffeomorphisms in $Hameo(M,\omega)$ still need illumination. In [7], Question (A) was called a weak version of the C^0 flux conjecture. Here is an attempt to an affirmative answer. It uses similar technique as in the proof of Theorem 1.1. Let $\rho \in Symp(M,\omega) \cap Hameo(M,\omega)$ be nontrivial, and assume that $\rho \notin Ham(M,\omega)$. There exists a non-empty open subset $\mathfrak{U}$ in M which is completely displaced by ρ. Let H be a smooth function with small support in an open subset σ strictly contained in $\mathfrak{U}$. The vector field Y such that $i_Y\omega = dH$ is Hamiltonian, and it is supported in σ. The symplectic form ω being nondegenerate, pick $z \in \mathfrak{U}$ such that $z \notin \sigma$, and a vector field X such that $\omega(\rho_*Y, X)(\rho(z)) \neq 0$. Compute,

$$(\rho^{-1})^*\left(i_Y\omega\right)(X)\,|_{\rho(z)} = \omega_z\left(Y_z, T_z\rho^{-1}X_z\right) = \omega_z\left(0, T_z\rho^{-1}X_z\right) = 0.$$

So, there exists a Hamiltonian vector field Y such that $i_Y\omega = dH$, and $(\rho^{-1})^*\left(i_Y\omega\right)$ is not equal to $i_{\rho_*Y}\omega$. Since $\rho \in Hameo(M,\omega)$, there exists a Cauchy sequence $\Psi_i = (\psi_t^i)$ of Hamiltonian isotopies in the Hofer length such that ψ_1^i converges C^0 to ρ. Let $\Theta := \{\varphi^t\}$ be the flow of Y, and $\Xi = \{E_t\}$ be the flow of a vector field Z defined as $i_Z\omega = d\left(H \circ \rho^{-1}\right)$: we have $\Lambda := \rho \circ \Theta \circ \rho^{-1} \neq \Xi$. The sequence of Hamiltonian isotopies $\Lambda_i := \psi_1^i \circ \Theta \circ (\psi_1^i)^{-1}$ uniformly converges to Λ, and a straightforward calculation implies that $\Lambda_i := \psi_1^i \circ \Theta \circ (\psi_1^i)^{-1}$ converges to Ξ in the Hofer topology. The conclusion follows exactly as in the proof of Theorem 1.1. Instead of appealing to Theorem 2.10, we appeal to Theorem 3.1 found in [33] or the Hamiltonian version of Theorem 3.1$-$[33] found by Oh-Müller.

Question (B): Is $SSympeo(M,\omega)$ a normal subgroup of $Sympeo_0(M,\omega)$?

This question was originally posed by Banyaga [5] and has been addressed in [23].

Question (C): Does $[Sympeo_0(M,\omega), Sympeo_0(M,\omega)]$ coincide with $Hameo(M,\omega)$?

For compact surfaces, it has been established that $[Sympeo_0(M,\omega), Sympeo_0(M,\omega)]$ is simple (Theorem 1.11$-$[9]), while $Hameo(M,\omega)$ is not simple (Theorem 1.3$-$[9] and Theorem 4$-$[24]). Therefore, the answer to Question (C) is negative for compact surfaces.

3.2. **Basic notions on symplectic geometry.** Let M be a smooth manifold of dimension $2n$. A differential form ω of degree two is called a symplectic form if it is closed and non-degenerate. A symplectic manifold is an even-dimensional manifold that admits a symplectic form. In the rest of this paper, we assume that M is a closed symplectic manifold with symplectic form ω, equipped with a fixed Riemannian metric g.

3.2.1. *Symplectic vector fields.* A symplectic form ω, being non-degenerate, induces an isomorphism between vector fields and 1-forms. This isomorphism is given by: for each vector field Y on M, one assigns the 1-form $\iota(Y)\omega := \omega(Y,\cdot)$, where ι is the usual interior product. A vector field Y on M is symplectic if the 1-form $\iota(Y)\omega$ is closed. A symplectic vector field Y is said to be a Hamiltonian vector field if the 1-form $\iota(Y)\omega$ is exact. From the definition of symplectic vector fields, if the first de Rham cohomology group of M is trivial, then any symplectic vector field on M is Hamiltonian.

3.2.2. *Symplectic diffeomorphisms and symplectic isotopies.* A diffeomorphism $\phi : M \to M$ is called symplectic if $\phi^*(\omega) = \omega$. The group of all symplectic diffeomorphisms of (M,ω) is denoted by $Symp(M,\omega)$. An isotopy $\{\phi_t\}$ of a symplectic manifold (M,ω) is said to be symplectic if $\phi_t \in Symp(M,\omega)$ for each t. Equivalently, the vector field $\dot{\phi}_t := \frac{d\phi_t}{dt} \circ \phi_t^{-1}$, is symplectic for each t. A symplectic isotopy $\{\psi_t\}$ is called a Hamiltonian isotopy if, for each t, the vector field $\dot{\psi}_t := \frac{d\psi_t}{dt} \circ \psi_t^{-1}$, is Hamiltonian. This means there exists a smooth function $F : [0,1] \times M \to \mathbb{R}$, called the generating Hamiltonian, such that $\iota(\dot{\psi}_t)\omega = dF_t$, for each t. Let $Ham(M,\omega)$ (resp. $Symp_0(M,\omega)$) denote the group consisting of all time-one maps of all Hamiltonian (resp. symplectic) isotopies. Weinstein showed that $Symp(M,\omega)$ is locally connected by smooth arcs; in fact, it is locally contractible. Thus, $Symp_0(M,\omega)$ is the identity component in $Symp(M,\omega)$ equipped with the C^∞ Whitney topology. For more details, refer to [1], [18], and [26].

3.2.3. *Flux homomorphism.* Let $\Phi = (\phi_t)$ be a symplectic isotopy. According to Cartan's Magic formula,

$$0 = \phi_1^*\omega - \omega = d\left(\int_0^1 \phi_t^*(i_{\dot{\phi}_t}\omega)\,dt\right),$$

which implies that the 1-form $\Sigma(\Phi) := \int_0^1 \phi_t^*(i_{\dot{\phi}_t}\omega)\,dt$ is closed. Thus, $\Sigma(\Phi)$ defines a cohomology class that depends only on the homotopy class of the isotopy Φ with fixed extremities in $Symp_0(M,\omega)$ [4]. In [3], the author defined the flux epimorphism $\widetilde{\mathfrak{S}}$ from the universal cover $\widetilde{Symp_0(M,\omega)}$ onto $\mathbb{H}^1(M,\mathbb{R})$ by assigning to each homotopy class $\{\Phi = (\phi_t)\}$ the de Rham cohomology class of the form $\Sigma(\Phi)$, i.e.,
$\widetilde{\mathfrak{S}} : \widetilde{Symp_0(M,\omega)} \to \mathbb{H}^1(M,\mathbb{R})$, $\{\Phi = (\phi_t)\} \mapsto [\Sigma(\Phi)]$, where $[,]$ denotes the de Rham cohomology class. Consider the covering projection

$$\pi : \widetilde{Symp_0(M,\omega)} \to Symp_0(M,\omega), \quad \{\Phi = (\phi_t)\} \mapsto \phi_1,$$

and define $\Gamma_\omega := \widetilde{\mathfrak{S}}(\ker\pi)$. Ono has shown that Γ_ω is a discrete subgroup of $\mathbb{H}^1(M,\mathbb{R})$ [30].

The map $\widetilde{\mathfrak{S}}$ induces another epimorphism $\mathfrak{S}$ from the group $Symp_0(M,\omega)$ onto the quotient $\mathbb{H}^1(M,\mathbb{R})/\Gamma_\omega$ (see [3, 4], [8] for more details).

Theorem 3.3. *(*C^0-**approximation of a cohomology class by flux***). Let (M,ω) be a closed connected symplectic manifold, and $X^i := \{X_t^i\}$ be a sequence of smooth families of symplectic vector fields (i.e., for each i, the map $X^i : M \times [0,1] \to TM$ is smooth and for every t, X_t^i is a smooth section of TM) which uniformly converges to a continuous family of symplectic vector fields $X := \{X_t\}$ (i.e., the map $X : M \times [0,1] \to TM$ is continuous in the variable t and for every t, X_t is a smooth section of TM). The following hold:*

1. *The sequence $\Phi^i := \{\phi_t^i\}$ of symplectic isotopies generated by $X^i := \{X_t^i\}$ uniformly converges to a continuous family $\Phi := \{\phi_t\}$ of symplectic diffeomorphisms which depends on X (i.e., the map $[0,1] \ni t \mapsto \phi_t \in Symp(M,\omega)$ is continuous).*
2. *For each $\epsilon \in [0,1[$, there exists a symplectic isotopy $\Theta_\epsilon := \{\theta_\epsilon^t\}$ with $\theta_\epsilon^1 = \phi_1$ such that*

$$\int_0^1 [\iota(X_t)\omega]\,dt = \lim_{\epsilon\to 0} \widetilde{\mathfrak{S}}(\Theta_\epsilon).$$

Proof. The sequence of smooth 1-parameter family of vector fields X^i converges uniformly to a continuous 1-parameter family of smooth symplectic vector fields $\{X_t\}$: Thus, by a result due to R. Abraham and J. Robbin [2], the sequence of flows generated by the sequence of vector fields X^i uniformly converges to a continuous family $\Phi : t \mapsto \phi_t$ of smooth diffeomorphisms which depends on X. Eliashberg-Gromov's rigidity theorem ensures that Φ is a continuous path in $Symp_0(M,\omega)$ [13]. On account of the local contractibility of $Symp_0(M,\omega)$, the isotopy Φ is homotopic relatively to fixed endpoints to a smooth symplectic isotopy Θ in $Symp_0(M,\omega)$ (see [4]). For the second item, the homotopy between Φ and Θ induces a uniform approximation of Φ by a smooth symplectic isotopy relatively to fixed endpoints. Namely, for all ϵ, we can choose a smooth symplectic isotopy $\Theta_\epsilon := \{\theta_\epsilon^t\}$ with $\theta_\epsilon^0 = id_M$ and $\theta_\epsilon^1 = \phi_1$ such that $d_{C^0}(\Phi,\Theta_\epsilon) \leq \frac{c\epsilon}{2(C+1)(c+1)}$, where

c and C are positive constants found in Theorem 1.6−[7]. To construct Θ_ϵ for ϵ small, on account of the homotopy between Θ and Φ, imagine a path you describe when moving along Φ so that whenever you meet a point where Φ is not smooth, you go around the point smoothly, but remaining as close as possible to Φ in the C^0 metric. The sequence $\Phi^i := \{\phi_t^i\}$ of symplectic isotopies generated by $X^i := \{X_t^i\}$ being uniformly convergent to $\Phi := \{\phi_t\}$, we may assume that $d_{C^0}(\Phi^i, \Theta_\epsilon) \leq \frac{c\epsilon}{(C+1)(c+1)}$, for all i sufficiently large. Argue in a similar way as in the proof of Theorem 1.6−[7] to obtain

$$|\widetilde{\mathfrak{S}}(\Theta_\epsilon \circ (\Phi^i)^{-1})| \leq \epsilon, \tag{3.24}$$

for all i sufficiently large, and for $\epsilon \in [0, \frac{1}{i+1}[$, where $|.|$ is any norm on the first De Rham cohomology group of M with real coefficients. As i goes to infinity, (3.24) implies that $\int_0^1 [\iota(X_t)\omega]\, dt = \lim_{\epsilon \to 0} \widetilde{\mathfrak{S}}(\Theta_\epsilon)$. □

In fluid dynamics, the concepts of flux and vector fields are fundamental. Theorem 3.3 can be used to understand how fluid flow evolves over time, especially in cases where the flow needs to be approximated by smooth vector fields. This is particularly useful in simulations where small perturbations need to be controlled and approximations need to be accurate. In quantum mechanics, symplectic geometry is often used to describe the phase space of systems. It seems that the approximation properties in Theorem 3.3 can be applied to ensure that the quantum states and their evolutions are accurately represented and approximated.

Note that the flux group Γ_ω has a topological counterpart denoted $\mathcal{S}\Gamma_\omega$ that is defined in [36] : $\Gamma_\omega \subset \mathcal{S}\Gamma_\omega$. The following result generalizes Theorem D−[36]. Its proof is a consequence of Theorem 3.3 combined with the equality

$$Symp_0(M, \omega) \cap Hameo(M, \omega) = Ham(M, \omega).$$

Lemma 3.4. *Let (M, ω) be a closed connected symplectic manifold. Then, $\Gamma_\omega = \mathcal{S}\Gamma_\omega$.*

3.3. C^0 **flux conjecture.** The C^0 flux conjecture states that: $Ham(M, \omega)$ is C^0−closed in $Symp_0(M, \omega)$. In [21], the authors established certain conditions under which this conjecture holds. They identified a class of manifolds that satisfy these conditions, specifically closed Kähler manifolds of nonpositive curvature with fundamental groups that have no center. Subsequently, Buhovsky [7] generalized the results from [21], extending the framework to include symplectically aspherical symplectic manifolds whose fundamental groups also possess a trivial center. In this subsection, we show that the C^0 flux conjecture holds on any closed connected symplectic manifold of Lefschetz type.

Lemma 3.5. *Let (M, ω) be a closed connected symplectic manifold of Lefschetz type. The C^0 flux conjecture holds.*

We shall need the following facts.

(1) The Arnold conjecture: Fixed points of Hamiltonian diffeomorphisms: [15], [22], [16], and [31].
(2) Lemma 3.5 found in [34] (a factorization result).
(3) Let $H = \{h_t\}$ be a uniform limit of a sequence of smooth isotopies Φ_k. For each $p \in M$ the orbit $\mathcal{O}_p^H$ can be written as a collection $\bigsqcup_{i \in I} C_i(p)$ of smooth curves $C_i(p)$. So, for all closed 1-form α, we interpret the integral $\int_{\mathcal{O}_p^H} \alpha$ as $\sum_{i \in I} \int_{C_i(p)} \alpha$. It follows that if for each k, Φ_k is volume-preserving, then for each $\Phi = \{\phi_t\} \in \mathfrak{AHom}_\delta^\Omega(H)$, we have

$$\int_{\mathcal{O}_p^H} \alpha = \int_{\mathcal{O}_p^\Phi} \alpha + \int_{\chi_\delta^p} \alpha,$$

for a unique minimizing geodesic χ^p_δ with extremities $h_1(p)$ and $\phi_1(p)$.

(4) A consequence of Theorem A-[6], on the transitivity of certain automorphism groups states that on a manifold M of dimension at least 2 with volume element: if $C_1, \dots, C_k$ is an arbitrary collection of disjoint closed curves on M, each a differentiably embedded image of the circle, then there is a 1-parameter group of volume-preserving transformations on M with these curves as orbits (see [6]).

Proof of Lemma 3.5: Let $\{\psi_i\}$ be a sequence of Hamiltonian diffeomorphisms such that $\psi_i \xrightarrow{C^0} \phi \in \mathrm{Symp}_0(M,\omega)$, and Ω be the symplectic volume form on M. Let $\Phi := \{\phi_t\}$ be any symplectic isotopy from the identity to ϕ, and for each i, let $\Psi_i := \{\psi_i^t\}$ be a Hamiltonian isotopy with time-one map ψ_i. Consider the boundary flat smooth functions $\mu_1 : [0, \frac{1}{2}] \to [0,1]$ with $\mu_1 = 0$ near 0 and $\mu_1 = 1$ near $\frac{1}{2}$, and $\mu_2 : [\frac{1}{2}, 1] \to [0,1]$ with $\mu_2 = 0$ near $\frac{1}{2}$ and $\mu_2 = 1$ near 1. These functions are used to define a sequence of isotopies $F_i := \{f_i^t\}$ as follows: For each i, set

$$f_i^t := \begin{cases} (\psi_i^{\mu_1(t)})^{-1} & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \psi_i^{-1} \circ \phi_{\mu_2(t)} & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

and it is clear that $f_i^1 = \psi_i^{-1} \circ \phi$, for each i. Assume that $\mathrm{Vol}_\Omega(M) = 1$. Corollary 2.1, together with the convergence $\|\psi_i^{-1} \circ \phi\|^\infty \to 0$ as $i \to \infty$ (Lemma 2.2), implies that $\lim_{i\to\infty} \left| \langle [\alpha], \mathrm{Flux}_\Omega(F_i) \rangle - \int_{\mathcal{O}_x^{F_i}} \alpha \right| = 0$, for all $x \in M$, and all closed 1-forms α. For each i, let $\hat{x}_i$ denote a fixed point for ψ_i (see [15], [22]) and compute $d_g((\psi_i^{-1} \circ \phi)(\hat{x}_i), \hat{x}_i) \to 0$ as $i \to \infty$, where d_g is the distance induced by the Riemannian metric g on M. Furthermore, as a C^0-limit of a Hamiltonian diffeomorphism, ϕ has at least a fixed point z_0, and we equally have $d_g(z_0, \hat{x}_i) \to 0$ as $i \to \infty$. Thus, there exists a 1-cycle, denoted $\gamma(\Phi)$, in M such that $\lim_{i\to\infty} \int_{\mathcal{O}_{\hat{x}_i}^{F_i}} \beta = \int_{\gamma(\Phi)} \beta$, for all closed 1-forms β. As a consequence of Theorem A in [6], $\gamma(\Phi)$ is an orbit of a certain loop at the identity map in $\mathrm{Diff}^{\Omega,\infty}(M)$. Here is another way to justify the existence of such a loop Θ: Let $F := \{f^t\}$ be the C^0-limit of the sequence of smooth isotopies $F_i := \{f_i^t\}$. Since $\psi_i^{-1} \circ \phi \xrightarrow{C^0} \mathrm{id}_M$, then $F \in \pi_1(\mathrm{Homeo}(M,\Omega))$. Hence, choose Θ to be any loop in $\mathfrak{AHom}_\delta^\Omega(F)$ for some $\delta \in (0, r(g))$. For each $x \in M$, the orbits $\mathcal{O}_x^\Theta$ and $\mathcal{O}_x^F$, being homotopic relatively to fixed extremities, we have

$$\int_M \left(\int_{\mathcal{O}_{(.)}^\Theta} \alpha \right) \Omega = \int_M \left(\int_{\mathcal{O}_{(.)}^F} \alpha \right) \Omega, \tag{3.25}$$

for each closed 1-form α. Similarly, it follows from the proof of Theorem 4.1−[36] that

$$\lim_{i\to\infty} \int_M \left(\int_{\mathcal{O}_{(.)}^{F_i}} \alpha \right) \Omega = \int_M \left(\int_{\mathcal{O}_{(.)}^F} \alpha \right) \Omega, \tag{3.26}$$

for each closed 1-form α. Lemma 3.5−[34] also implies that

$$\lim_{i\to\infty} \int_M \left(\int_{\mathcal{O}_{(.)}^{F_i}} \alpha \right) \Omega = \langle [\alpha], \widetilde{S}_\Omega(\Phi) \rangle, \tag{3.27}$$

and

$$\int_M \left(\int_{\mathcal{O}_{(.)}^\Theta} \alpha \right) \Omega = \langle [\alpha], \mathrm{Flux}_\Omega(\Theta) \rangle = \int_{\mathcal{O}_x^\Theta} \alpha, \tag{3.28}$$

for each closed 1-form α, and all $x \in M$. Therefore, (3.25), (3.26), (3.27), and (3.28) together imply $\langle [\alpha], \widetilde{S}_\Omega(\Phi) \rangle = \int_{\mathcal{O}_{z_0}^\Theta} \alpha = \langle [\alpha], \mathrm{Flux}_\Omega(\Theta) \rangle$, for each closed 1-form α; hence, $\widetilde{S}_\Omega(\Phi) = \widetilde{S}_\Omega(\Theta)$. The Lefschetz assumption, the factorization result $\widetilde{S}_\Omega(.) = \frac{1}{n!}\omega^{n-1} \wedge \widetilde{\mathfrak{S}}(.)$

found in [3], and the equality $\widetilde{S}_{\Omega}(\Phi) = \widetilde{S}_{\Omega}(\Theta)$ together implies that, regarded as a symplectic path, the isotopy Φ has its flux in the flux group Γ_{ω}, which is equivalent to require that $\phi = \phi_1 \in \mathrm{Ham}(M, \omega)$. □

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