

VECTOR FOURIER ANALYSIS ON COMPACT GROUPS AND ASSIAMOUA SPACES

YAOGAN MENSAH

In memory of Professor VSK Assiamoua.

ABSTRACT. This paper shows how a family of function spaces, coined as Assiamoua spaces, plays a fundamental role in the Fourier analysis of vector-valued functions on compact groups. These spaces make it possible to transcribe the classical results of Fourier analysis in the framework of analysis of vector-valued functions and vector measures. The construction of Sobolev spaces of vector-valued functions on compact groups rests heavily on the members of the aforementioned family.

1. INTRODUCTION

The Fourier analysis has undergone many generalizations in various directions since the original idea of J. Fourier [6]. The extension of this analysis to complex-valued functions on a compact group has become a well documented classical theory. While Fourier used trigonometric series, today known as Fourier series, in the modeling of the heat phenomenon, his theory has been extended to abelian groups, compact groups, locally compact groups, then to other more complex topological and algebraic structures such as locally compact hypergroups. Each extension is accompanied by the delicate construction of a family of function spaces in which the Fourier transforms of the initial functions reside. For instance, in the case of complex-valued functions on the one-dimensional torus or on $\mathbb{R}^n$, Lebesgue spaces and their discrete analogues made it possible to report the results in terms of Riemann-Lebesgue lemma, Fourier inversion formula, Plancherel theorem, etc. For more details, we refer the books [4, 8].

In the noncommutative setting, for instance for compact non-necessarily abelian groups or for more general locally compact groups, the Fourier analysis of complex functions appeals classes of bounded operators on the representation spaces. The case of compact groups is described in the present paper. For more details on noncommutative Fourier analysis, we refer the books [5, 9].

The situation becomes more complicated when we consider vector-valued functions. The study of the Fourier analysis of vector-valued functions and vector measures was initiated by Assiamoua in [2] and developed by Assiamoua and Olubummo in [3]. A family of functions spaces, coined as Assiamoua spaces in the present paper, was introduced in order to conduct effectively the study. These spaces are quite useful and they recently made it possible to obtain an extension of the Fourier-Stieltjes transform from compact groups to locally compact groups through an induced representation [1].

The aim of this paper is to highlight these function spaces by emphasizing their roles and the potential of what they can still bring to the development of abstract harmonic analysis.

The rest of the paper is organized as follows. In Section 2, we recall basic definitions and facts in group representation theory. Section 3 provides a summary of the Fourier analysis of compact groups. Section 4 is devoted to the Fourier analysis of vector-valued

functions on compact groups and the highlighting of Assiamoua spaces. In Section 5, a tensor product approach is used to describe the Assiamoua spaces in a way equivalent to the previous construction. In Section 6, some topological properties of the Assiamoua spaces are provided. In Section 7, we describe a Sobolev space whose construction is based on an Assiamoua space. Finally, Section 8 collects some ideas and questions for further research.

2. BASICS OF REPRESENTATION THEORY

In this section, we recall some basic notions in group representation theory in order to make the paper more understandable.

Let G be a topological compact. A unitary representation σ of G on a complex Hilbert space H_σ is a continuous group homomorphism $\sigma : G \longrightarrow \mathcal{U}(H_\sigma)$ where $\mathcal{U}(H_\sigma)$ denotes the group of unitary operators on H_σ . The continuity is to be understood as the mapping

$$G \longrightarrow H_\sigma, x \longmapsto \sigma(x)\xi$$

is continuous for every $\xi \in H_\sigma$. The Hilbert space H_σ is called the representation space of σ and the dimension of H_σ is called the dimension of the representation σ and it is usually denoted by d_σ .

A subspace M of H_σ is said to be invariant by σ if $\forall x \in G, \forall \xi \in M, \sigma(x)\xi \in M$. A representation σ of G on H_σ is said to be irreducible if there is no proper closed subspace M of H_σ which is invariant by σ .

It is well known that the dimension of any unitary irreducible representation of a compact group is of finite dimension [9].

Two unitary representations $\sigma_i, i = 1, 2$ of G on $H_{\sigma_i}, i = 1, 2$ are said to be unitary equivalent if there exists a unitary operator $S : H_{\sigma_1} \longrightarrow H_{\sigma_2}$ such that $\forall x \in G, \sigma_2(x)S = S\sigma_1(x)$.

Throughout the paper, $\widehat{G}$ stands for the set of equivalent classes of unitary irreducible representations of G . It is called the unitary dual of G . If G is compact, then $\widehat{G}$ is discrete.

3. FOURIER ANALYSIS ON COMPACT GROUPS

In this section, fundamental concepts and results of the Fourier analysis of complex-valued functions on compact groups are recaped. More details can be find in many abstract harmonic analysis books. However, we refer the preliminary notes of the paper [10, Section 2]. Results in this section are generalized in Section 4.

Let G be a compact group (assumed Hausdorff once for all) equipped with its normalized Haar measure. Let us denote by $L^1(G)$ the space of complex-valued integrable functions on G and by $L^2(G)$ the space of complex-valued square integrable functions on G . The Fourier transform of $f \in L^1(G)$ is defined by

$$\widehat{f}(\sigma) = \int_G f(x)\sigma(x)^* dx, \sigma \in \widehat{G},$$

where $\sigma(x)^*$ stands for the Hilbert adjoint of the operator $\sigma(x)$. The object $\widehat{f}(\sigma)$ is a bounded linear operator on H_σ . Then, $\widehat{f} = \left(\widehat{f}(\sigma)\right)_{\sigma \in \widehat{G}}$ is a family of bounded linear operators. Let us denote by $B(H_\sigma)$ the space of bounded linear operators on H_σ endowed with the operator norm. To express results about the Fourier transform of complex-valued functions on compact groups, the following spaces are considered.

- (1) $\ell^p - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma) = \left\{ (T_\sigma) \in \prod_{\sigma \in \widehat{G}} B(H_\sigma) : \sum_{\sigma \in \widehat{G}} d_\sigma \|T_\sigma\|_{B(H_\sigma)}^p < \infty \right\}$, $1 \leq p < \infty$,
endowed with the norm

$$\|(T_\sigma)\|_{\ell^p - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)} = \left(\sum_{\sigma \in \widehat{G}} d_\sigma \|T_\sigma\|_{B(H_\sigma)}^p \right)^{\frac{1}{p}},$$

- (2) $\ell^\infty - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma) = \left\{ (T_\sigma) \in \prod_{\sigma \in \widehat{G}} B(H_\sigma) : \sup_{\sigma \in \widehat{G}} \|T_\sigma\|_{B(H_\sigma)} < \infty \right\}$, endowed with
the norm

$$\|(T_\sigma)\|_{\ell^\infty - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)} = \sup_{\sigma \in \widehat{G}} \|T_\sigma\|_{B(H_\sigma)},$$

- (3) $c_0 - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma) = \left\{ (T_\sigma) \in \ell^\infty - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma) : T_\sigma \rightarrow 0 \text{ as } \sigma \rightarrow \infty \right\}$.

The spaces thus constructed make it possible to efficiently transcribe the results of harmonic analysis on compact groups. Let us mention some well known results.

Theorem 3.1. *Let G be a compact group. If $f \in L^1(G)$, then $\widehat{f} \in \ell^\infty - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)$.*

Moreover,

$$\|\widehat{f}\|_{\ell^\infty - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)} \leq \|f\|_{L^1(G)}.$$

Theorem 3.2. *Let G be a compact group. If $f \in L^1(G)$, then $\widehat{f} \in c_0 - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)$.*

The above theorem is known as the Riemann-Lebesgue lemma.

Theorem 3.3. *Let G be a compact group. The mapping $f \rightarrow \widehat{f}$ is an isomorphism from $L^2(G)$ onto $\ell^2 - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)$. Moreover, if $f \in L^2(G)$, then*

$$\|\widehat{f}\|_{\ell^2 - \bigoplus_{\sigma \in \widehat{G}} B(H_\sigma)} = \|f\|_{L^2(G)}.$$

The above theorem is known as the Plancherel theorem.

4. FOURIER ANALYSIS OF VECTOR-VALUED FUNCTIONS ON COMPACT GROUPS AND ASSIAMOUA SPACES

Let G be a compact group and let A be a complex Banach space. Let $L^1(G, A)$ denotes the set of A -valued functions on G which are integrable in the Bochner sense. The set $L^1(G, A)$ is endowed with the norm

$$\|f\|_{L^1(G, A)} = \int_G \|f(x)\|_A dx.$$

In 1989, in order to study the multipliers of $L^1(G, A)$ [2], Assiamoua introduced the Fourier-Stieltjes transform of vector measures on compact groups and consequently the Fourier transform of vector-valued functions on compact groups.

Let $f \in L^1(G, A)$. The Fourier transform of f has been interpreted for the first time by Assiamoua as the family $\widehat{f} = (\widehat{f}(\sigma))_{\sigma \in \widehat{G}}$ of sesquilinear maps $\widehat{f}(\sigma) : H_\sigma \times H_\sigma \rightarrow A$ defined by

$$\widehat{f}(\sigma)(\xi, \eta) = \int_G \langle \sigma(x)^* \xi, \eta \rangle_{H_\sigma} f(x) dx, \quad \xi, \eta \in H_\sigma. \quad (4.1)$$

The formula (4.1) is expressed here using our own notations (see [2, Section 4]). From now on, the formula (4.1) will be called Assiamoua's formula. The systematic study of

the main properties of this transformation was made by Assiamoua and Olubummo in [3]. To efficiently transcribe the results for the aforementioned study, some function spaces were introduced : the analogue of the spaces of Section 3. Let us briefly describe their construction.

For each $\sigma \in \widehat{G}$, fix once for all an orthonormal basis $(\xi_1^\sigma, \dots, \xi_{d_\sigma}^\sigma)$ of H_σ . Denote by $\mathcal{S}(H_\sigma \times H_\sigma, A)$ the set of sesquilinear mappings from $H_\sigma \times H_\sigma$ into A . The set $\mathcal{S}(H_\sigma \times H_\sigma, A)$ is endowed with the norm

$$\|\psi\|_{\mathcal{S}(H_\sigma \times H_\sigma, A)} = \sup\{\|\psi(\xi, \eta)\|_A : \|\xi\| \leq 1, \|\eta\| \leq 1\}.$$

Set

$$\mathcal{S}(\widehat{G}, A) = \prod_{\sigma \in \widehat{G}} \mathcal{S}(H_\sigma \times H_\sigma, A).$$

Consider the following spaces :

$$(1) \quad \mathcal{S}_p(\widehat{G}, A) = \left\{ \varphi \in \mathcal{S}(\widehat{G}, A) : \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \|\varphi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)\|_A^p < \infty \right\}, \quad 1 \leq p < \infty$$

equipped with the norm

$$\|\varphi\|_{\mathcal{S}_p(\widehat{G}, A)} = \left(\sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \|\varphi(\sigma)(\xi_j^\sigma, \xi_i^\sigma)\|_A^p \right)^{\frac{1}{p}},$$

$$(2) \quad \mathcal{S}_\infty(\widehat{G}, A) = \left\{ \varphi \in \mathcal{S}(\widehat{G}, A) : \sup_{\sigma \in \widehat{G}} \|\varphi(\sigma)\|_{\mathcal{S}(H_\sigma \times H_\sigma, A)} < \infty \right\} \text{ equipped with the norm}$$

$$\|\varphi\|_{\mathcal{S}_\infty(\widehat{G}, A)} = \sup_{\sigma \in \widehat{G}} \|\varphi(\sigma)\|_{\mathcal{S}(H_\sigma \times H_\sigma, A)},$$

$$(3) \quad \mathcal{S}_0(\widehat{G}, A) = \left\{ \varphi \in \mathcal{S}_\infty(\widehat{G}, A) : \forall \varepsilon > 0, \{\sigma \in \widehat{G} : \|\varphi(\sigma)\|_{\mathcal{S}(H_\sigma \times H_\sigma, A)} > \varepsilon\} \text{ is finite} \right\}.$$

Definition 4.1. The spaces $\mathcal{S}_p(\widehat{G}, A)$, $1 \leq p \leq \infty$ are called Assiamoua spaces.

Using the Assiamoua spaces, some properties of the Fourier transform read as follows [3].

Theorem 4.2. Let G be a compact group and A a complex Banach space. The mapping $f \mapsto \widehat{f}$ from $L^1(G, A)$ into $\mathcal{S}_\infty(\widehat{G}, A)$ is linear, injective and continuous.

The following theorem contains the analogue of the Riemann-Lebesgue lemma.

Theorem 4.3. Let G be a compact group and A a complex Banach space. If $f \in L^1(G, A)$, then $\widehat{f} \in \mathcal{S}_0(\widehat{G}, A)$. Moreover, $\widehat{L^1(G, A)}$ is dense in $\mathcal{S}_0(\widehat{G}, A)$ where $\widehat{L^1(G, A)}$ is the image of $L^1(G, A)$ by the Fourier transform.

Theorem 4.4. Let G be a compact group. If A is a Hilbert space, then the Fourier transformation $f \mapsto \widehat{f}$ is an isometry from $L^2(G, A)$ onto $\mathcal{S}_2(\widehat{G}, A)$. Moreover, if $f \in L^2(G, A)$ then

$$f(x) = \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \widehat{f}(\sigma) \langle \sigma(x) \xi_j^\sigma, \xi_i^\sigma \rangle, \quad x \in G.$$

5. TENSOR PRODUCT APPROACH TO ASSIAMOUA SPACES

Using the tensor product theory, it is possible to construct the Assiamoua spaces of Section 4 in an equivalent way. Denote by $\overline{H}_\sigma$ the conjugate of the Hilbert space H_σ . The tensor product space $H_\sigma \otimes \overline{H}_\sigma$ is equipped with the projective tensor norm. Since it is of finite dimension (its dimension is d_σ^2) then it is complete.

One may interpret the Fourier transform as the family $(\widehat{f}(\sigma))_{\sigma \in \widehat{G}}$ of linear operators from $H_\sigma \otimes \overline{H}_\sigma$ into A defined by (see [11]) :

$$\widehat{f}(\sigma)(\xi \otimes \eta) = \int_G \langle \sigma(x)^* \xi, \eta \rangle_{H_\sigma} f(x) dx, \quad \xi \in H_\sigma, \eta \in \overline{H}_\sigma. \quad (5.2)$$

The Assiamoua spaces as defined in Section 4 are slightly modified.

Denote by $\mathfrak{B}(H_\sigma \otimes \overline{H}_\sigma, A)$ the space of bounded linear operators from $H_\sigma \otimes \overline{H}_\sigma$ into A equipped with the norm

$$\|\psi\|_{\mathfrak{B}(H_\sigma \otimes \overline{H}_\sigma, A)} = \sup\{\|\psi(\xi \otimes \eta)\|_A : \|\xi\|_{H_\sigma} \leq 1, \|\eta\|_{\overline{H}_\sigma} \leq 1\}.$$

Now, set

$$\mathfrak{B}(\widehat{G}, A) = \prod_{\sigma \in \widehat{G}} \mathfrak{B}(H_\sigma \otimes \overline{H}_\sigma, A).$$

The space $\mathcal{S}_p(\widehat{G}, A)$ becomes

$$\mathfrak{B}_p(\widehat{G}, A) = \left\{ \varphi \in \mathfrak{B}(\widehat{G}, A) : \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \|\varphi(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma)\|_A^p < \infty \right\}, \quad 1 \leq p < \infty$$

equipped with the norm

$$\|\varphi\|_{\mathfrak{B}_p(\widehat{G}, A)} = \left(\sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \|\varphi(\sigma)(\xi_j^\sigma \otimes \xi_i^\sigma)\|_A^p \right)^{\frac{1}{p}}$$

and $\mathcal{S}_\infty(\widehat{G}, A)$ becomes

$$\mathfrak{B}_\infty(\widehat{G}, A) = \left\{ \varphi \in \mathfrak{B}(\widehat{G}, A) : \sup_{\sigma \in \widehat{G}} \|\varphi(\sigma)\|_{\mathfrak{B}(H_\sigma \otimes \overline{H}_\sigma, A)} < \infty \right\}$$

equipped with the norm

$$\|\varphi\|_{\mathfrak{B}_\infty(\widehat{G}, A)} = \sup_{\sigma \in \widehat{G}} \|\varphi(\sigma)\|_{\mathfrak{B}(H_\sigma \otimes \overline{H}_\sigma, A)}.$$

One can see that the correspondance $\mathcal{S}_p(\widehat{G}, A) \leftrightarrow \mathfrak{B}_p(\widehat{G}, A)$ is one-to-one. In fact, if we denote by $B(H_\sigma \times \overline{H}_\sigma, A)$ the space of bilinear maps from $H_\sigma \times \overline{H}_\sigma$ into A , then we have the following identification by the effect of linearization of bilinear maps :

$$\mathcal{S}(H_\sigma \times H_\sigma, A) \simeq B(H_\sigma \times \overline{H}_\sigma, A) \simeq \mathfrak{B}(H_\sigma \otimes \overline{H}_\sigma, A).$$

The following definition is equivalent to Definition 4.1.

Definition 5.1 (Alternative definition of Assiamoua spaces). We call Assiamoua spaces the spaces $\mathfrak{B}_p(\widehat{G}, A)$, $1 \leq p \leq \infty$.

The well known properties of the Fourier analysis of vector-valued functions on compact groups can be transcribed using the spaces $\mathfrak{B}_p(\widehat{G}, A)$. For instance, The Fourier transformation $f \mapsto \widehat{f}$ is an isometry from $L^2(G, A)$ onto $\mathfrak{B}_2(\widehat{G}, A)$.

6. TOPOLOGICAL PROPERTIES OF ASSIAMOUA SPACES

The Assiamoua spaces $\mathcal{S}_p(\widehat{G}, A)$ were studied from the topological point of view. It had been found out that they have interesting topological properties. The detailed proofs can be found in [3, 12]. Let us mention some of the results.

Theorem 6.1. *The spaces $\mathcal{S}_p(\widehat{G}, A)$, $1 \leq p \leq \infty$, are Banach spaces.*

The case $p = \infty$ was studied in [3] and the case $1 \leq p < \infty$ was studied in [12].

Theorem 6.2. [3] *If A is a Hilbert space, then so is $\mathcal{S}_2(\widehat{G}, A)$.*

In this case, the inner product in $\mathcal{S}_2(\widehat{G}, A)$ is defined as follows : for $\varphi_1, \varphi_2 \in \mathcal{S}_2(\widehat{G}, A)$,

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{S}_2(\widehat{G}, A)} = \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \langle \varphi_1(\sigma)(\xi_j^\sigma, \xi_i^\sigma), \varphi_2(\sigma)(\xi_j^\sigma, \xi_i^\sigma) \rangle_A. \quad (6.3)$$

Theorem 6.3. *The topological dual of the space $\mathcal{S}_p(\widehat{G}, A)$, $1 \leq p < \infty$ is $\mathcal{S}_q(\widehat{G}, A^*)$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$ and A^* is the topological dual of A .*

The case $p = 1$ was proved in [13] when studying the Fourier algebra of vector-valued functions on compact groups. The case $1 < p < \infty$ was proved in [12]. In both cases, the results are achieved by showing that the following mapping is a linear surjective isometry :

$$T : \mathcal{S}_q(\widehat{G}, A^*) \longrightarrow \left(\mathcal{S}_p(\widehat{G}, A) \right)^*$$

defined by

$$\langle T\theta, \varphi \rangle = \sum_{\sigma \in \widehat{G}} d_\sigma \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \langle \theta(\sigma)(\xi_j^\sigma, \xi_i^\sigma), \varphi(\sigma)(\xi_j^\sigma, \xi_i^\sigma) \rangle,$$

$\theta \in \mathcal{S}_q(\widehat{G}, A^*)$, $\varphi \in \mathcal{S}_p(\widehat{G}, A)$ and the brackets $\langle \cdot, \cdot \rangle$ of the left and right sides are duality brackets.

7. THE SOBOLEV SPACES THAT ASSIAMOUA COULD HAVE CONSTRUCTED

When we were writing the manuscript [14] on Sobolev spaces of Bessel potential type consisting of vector-valued functions on a compact group, we told ourself that Assiamoua could have written on the subject in 1989 because he had all the necessary ingredients at his disposal in the paper [3] : his elegant interpretation of the Fourier transform for vector-valued functions, a Fourier inversion formula and an analogue of the Plancherel theorem. All he had to do was to put them together to complete the puzzle. Unfortunately, he did not do so probably because he was not interested in Sobolev spaces at that time. Moreover, as far as we know, it was in 2014 that was published the first article dealing with Sobolev spaces defined on abstract locally compact abelian groups using the Fourier transformation without taking into account the weak derivation[7].

The Sobolev spaces $H_\gamma^s(G, A)$, constructed in [14], rests on the Assiamoua space $\mathcal{S}_2(\widehat{G}, A)$. Let $s \in [0, \infty)$ and consider the map $\gamma : \widehat{G} \longrightarrow [0, \infty)$. Then, the Sobolev space $H_\gamma^s(G, A)$ is defined as

$$H_\gamma^s(G, A) = \left\{ f \in L^2(G, A) : (1 + \gamma^2)^{\frac{s}{2}} \widehat{f} \in \mathcal{S}_2(\widehat{G}, A) \right\},$$

endowed with the norm

$$\|f\|_{H_\gamma^s(G,A)} = \left(\sum_{\sigma \in \widehat{G}} d_\sigma (1 + \gamma(\sigma)^2)^s \sum_{i=1}^{d_\sigma} \sum_{j=1}^{d_\sigma} \|\widehat{f}(\sigma)(\xi_j^\sigma, \xi_i^\sigma)\|_A^2 \right)^{\frac{1}{2}}.$$

Using properties of the Fourier transform (mainly the inversion formula and the Plancherel type theorem) together with those of the space $\mathcal{S}_2(\widehat{G}, A)$, we obtain results, some of which are recorded in the following theorem [14]:

Theorem 7.1. *Let G be a compact group. Let A be a Banach space.*

- (1) *The space $H_\gamma^s(G, A)$ is a Banach space.*
- (2) *The Sobolev space $H_\gamma^s(G, A)$ is continuously embedded in the space $L^2(G, A)$ with*

$$\|f\|_{L^2(G,A)} \leq \|f\|_{H_\gamma^s(G,A)}.$$

- (3) *If $\sum_{\sigma \in \widehat{G}} \frac{d_\sigma^3}{(1 + \gamma(\sigma)^2)^s} < \infty$, then the Sobolev space $H_\gamma^s(G, A)$ is continuously embedded in the space $\mathcal{C}(G, A)$, where $\mathcal{C}(G, A)$ is the space of A -valued continuous functions on G .*

8. SOME IDEAS TO MOVE FURTHER

In this section, we discuss some ideas in a questioning form for future research related to the Assiamoua spaces $\mathcal{S}_p(\widehat{G}, A)$, $1 \leq p \leq \infty$.

- (1) The space $\mathcal{S}_p(\widehat{G}, A)$ is a Banach space. However, it can possess other interesting properties which can be investigated.

Question 1. *Does it have geometric properties such as the Radon-Nikodym property, the Dunford-Pettis property or the Schur property?*

- (2) The Plancherel type result of Theorem 4.4 is obtained under the assumption that A is a Hilbert space.

Question 2. *Is it possible to achieve the same result without the Hilbert space structure assumption on A ?*

- (3) The Pego theorem characterizes the precompact subsets of $L^2(\mathbb{R}^n)$ through the Fourier transform. The analogue of the Pego theorem on compact groups was proved in [10] via the Fourier analysis of complex-valued functions on compact groups.

Question 3. *Can we prove the analogue of the Pego theorem for $L^2(G, A)$ using the Fourier transform of vector-valued functions and some properties of the spaces $\mathcal{S}_p(\widehat{G}, A)$?*

- (4) The Rellich-Kondrachov theorem provides conditions under which a Sobolev space is compactly embedded in a Lebesgue space.

Question 4. *Can we investigate the analogue of the Rellich-Kondrachov theorem for the Sobolev space $H_\gamma^s(G, A)$?*

- (5) We may define and study the Sobolev spaces of Section 7 in a more general setting. For instance, we can consider the Sobolev spaces $H_\gamma^{s,p}(G, A)$, $1 \leq p \leq \infty$ defined by

$$H_\gamma^{s,p}(G, A) = \left\{ f \in L^2(G, A) : (1 + \gamma^2)^{\frac{s}{2}} \widehat{f} \in \mathcal{S}_p(\widehat{G}, A) \right\}, \quad 1 \leq p \leq \infty.$$

Question 5. *What are the main properties of the Sobolev spaces $H_\gamma^{s,p}(G, A)$ and what are their connections with other function spaces?*

9. CONCLUSION

In this article, we have briefly described the Fourier analysis of complex-valued functions and vector-valued functions on compact groups. We have highlighted a family of function spaces coined as Assiamoua spaces which play a fundamental role in the analysis of vector-valued functions and we have exhibited some of their major properties. Finally, we have provided a list of questions which may interest some researchers in the field of harmonic analysis or functional analysis in general.

REFERENCES

- [1] Y. I. Akakpo, M. N. Hounkonnou, K. Enakoutsa and V. S. K. Assiamoua, *Development of a Fourier-Stieltjes transform using an induced representation on locally compact groups*, Afr. Mat. **35** (2024), article 68, doi.org/10.1007/s13370-024-01199-0.
- [2] V. S. K. Assiamoua, *$L_1(G, A)$ -multipliers*, Acta Sci. Math.(Szeged) **53** (1989), 309–3018, pub.acta.hu/acta/viewArticle.action.
- [3] V. S. K. Assiamoua and A. Olubummo, *Fourier-Stieltjes transforms of vector-valued measures on compact groups*, Acta Sci. Math.(Szeged) **53** (1989), 301–307, pub.acta.hu/acta/viewArticle.action.
- [4] G. B. Folland, *Fourier analysis and its applications*, Pure and Applied Undergraduate Texts, **4**, AMS, 1992, bookstore.ams.org/view?ProductCode=AMSTEXT/4.
- [5] G. B. Folland, *A course in abstract harmonic analysis*, CRC Press, 1995, x+433.
- [6] J. Fourier, *Théorie analytique de la chaleur*, Didot, 1822, x+639.
- [7] P. Górká, T. Kostrzewa and E. G. Reyes, *Sobolev spaces on locally compact abelian groups : compact embeddings and local spaces*, J. Funct. Spaces (2014), Article ID 404738, 6 pages, doi.org/10.1155/2014/404738.
- [8] L. Grafakos, *Classical Fourier analysis*, (third edition), Graduate Texts in Mathematics, Springer, New York, 2014, x+638, doi.org/10.1007/978-1-4939-1194-3.
- [9] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. II, Grundlehren Math. Wiss. 152, Springer Verlag-Berlin, Heidelberg, 1970, x+774, doi.org/10.1007/978-3-662-26755-4.
- [10] M. Kumar, *Pego theorem on compact groups*, Pacific J. Math. **328** (2024), no. 1, 137–143, doi.org/10.2140/pjm.2024.328.137.
- [11] Y. Mensah, *Facts about the Fourier-Stieltjes transform of vector measures on compact groups*, Int. J. Anal. and Appl. **2** (2013), no. 1, 19–25, www.etamaths.com/index.php/ijaa/article/view/80.
- [12] Y. Mensah, V. S. K. Assiamoua, *On spaces of Fourier-Stieltjes transform of vector measures on compact groups*, Math. Sci. **4** (2010), no. 1, 1–8, www.sid.ir/filesserver/je/1010720100101.
- [13] Y. Mensah, V. S. K. Assiamoua, *The dual of the Fourier algebra $A_1(G, A)$ of vector valued functions on compact groups*, Afr. Diaspora J. Math. **8** (2009), no. 1, 28–34.
- [14] Y. Mensah, *Sobolev spaces of vector-valued functions on compact groups*, 2024. doi.org/10.48550/arXiv.2406.1019.

Yaogan Mensah: mensahyaogan2@gmail.com

Department of Mathematics, University of Lomé, POB 1515 Lomé 1, Togo.