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PAIR OF ITERATIVE ALGORITHM FOR SOLVING SPLIT INCLUSION PROBLEM ASSOCIATED TO CAYLEY'S OPERATOR IN HILBERT SPACES

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Abstract. The aspiration of the article is to find a solution of split inclusion problem associated to Cayley operator $\mathsf{C}_\lambda^{\mathsf{M}}$ in the framework of real Hilbert space and we employ a classical approach to develop an iterative algorithm for solving this particular inclusion problem. Under few reliable conditions, we state and prove a weak/strong convergence theorem for the proposed algorithm. In addition, we also present an application to the split feasibility problem and illustrate a numerical example in order to show that the algorithm we proposed is efficient and feasible.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Assume $\mathsf{D}:\mathsf{H}\to\mathsf{H}$ and $\mathsf{M}:\mathsf{H}\rightrightarrows\mathsf{H}$ be the single and multi-valued mappings respectively, then the Variational inclusion problem **(VIP)** is to obtain $x^*\in\mathsf{H}$ such that

$$0\in\mathsf{D}(x^*)+\mathsf{M}(x^*). \qquad (1.1)$$

"The forward-backward splitting algorithm and Douglas-Rachford algorithm have been proposed to solve Problem (1.1). Forward-backward splitting method has been proposed by Lions and Mercier [11], which is given by

$$x_{n+1}=(I-\lambda_n\mathsf{D})(I+\lambda_n\mathsf{M})x_n,$$

where $\lambda_n>0\ \forall n$ and $\mathsf{D}:\mathsf{H}\to\mathsf{H}$ is co-coercive operator. Mercier [11] had studied the convergence behavior of forward-backward method when M^{-1} is γ-strongly monotone with $\gamma>0$. They have proved that forward-backward algorithm converges weakly to the point in the solution set provided $\lambda_n<2\gamma$ is constant. In addition, if M is strongly monotone, then $\{x_n\}$ shows strong convergence to the unique solution of problem (1.1). Chen and Rockafellar [12] have also assumed the strong monotonicity of M to prove the strong convergence of forward-backward method which depends on Lipschitz constant and modulus of strong monotonicity.

In 2011, Moudafi [1] introduced the split variational inclusion problem **(SVIP)**: find $x^*\in\mathsf{H}$ such that

$$0\in\mathsf{M}_1(\mathrm{x}^*)\ \text{ and }\ 0\in\mathsf{M}_2(\mathsf{A}\mathrm{x}^*), \qquad (1.2)$$

where, $\mathsf{M}_1:\mathsf{H}_1\rightrightarrows\mathsf{H}_1$ and $\mathsf{M}_2:\mathsf{H}_2\rightrightarrows\mathsf{H}_2$ are multi-valued maximal monotone mappings and $\mathsf{A}:\mathsf{H}_1\to\mathsf{H}_2$ is a bounded linear operator. The solution set of the problem (1.2) is denoted by

$$\Delta:=\{\mathrm{x}^*\in\mathsf{H}_1:0\in\mathsf{M}_1(\mathrm{x}^*)\text{ and }0\in\mathsf{M}_2(\mathsf{A}\mathrm{x}^*)\}.$$

A monotone mapping M is said to be maximal if the graph of M, denoted as $G(\mathsf{M})$, is not properly contained in the graph of any other monotone mapping M, $G(\mathsf{M})=\{(x,y):y\in$

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$\mathsf{M}(x)\}$. It is well known that M is maximal if and only if for $(x,u) \in \mathsf{H}\times\mathsf{H}$, $\langle x-y, u-v\rangle \geq 0$ for all $(y,v) \in G(\mathsf{M})$ implies $u \in \mathsf{M}(x)$.

The resolvent operator $\mathsf{J}_\lambda^{\mathsf{M}}$ associated with M and λ is the mapping $\mathsf{J}_\lambda^{\mathsf{M}} : \mathsf{H} \to \mathsf{H}$ defined by

$$\mathsf{J}_\lambda^{\mathsf{M}}(x) = (I + \lambda\mathsf{M})^{-1}x, \quad x \in \mathsf{H}, \lambda > 0.$$

In 2012, Byrne et al.[2] studied the weak and strong convergence of the iterative methods for solving **(SVIP)**. To obtain the weak convergence, Byrne et al. introduced the following algorithm: for a given $x_0 \in \mathsf{H}_1$ and $\lambda > 0$, the sequence $\{x_n\}$ generated iteratively by the following scheme:

$$x_{n+1} = \mathsf{J}_\lambda^{\mathsf{M}_1}(I + \gamma\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{M}_2} - I)\mathsf{A})x_n, \quad \gamma \in \left(0, \frac{2}{\|\mathsf{A}^*\mathsf{A}\|}\right), \tag{1.3}$$

where, A^* is the adjoint of A, L is the spectral radius of $\mathsf{A}^*\mathsf{A}$ and $\gamma \in \left(0, \dfrac{2}{L}\right)$.

In 2001, a heavy ball method involved for studying maximal monotone operators is introduced by Alvarez and Attouch [5], where an inertial term was added. This procedure is called the inertial proximal point algorithm and it takes the shape

$$\begin{cases} x_0, x_1 \in \mathsf{H}_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = (I + \lambda_n\mathsf{M})^{-1}w_n, \end{cases} \tag{1.4}$$

they got the weak convergence for the mapping M, if $\{\lambda_n\}$ is nondecreasing and $\{\theta_n\} \subset [0,1)$

$$\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty.$$

In particular, the above condition on θ_n is true for $\theta_n < 1/3$.

Motivated and inspired by the work of Moudafi [1], Byrne et al.[2] and by the ongoing research in this direction [13, 14], we present a new split inclusion problem associated with Cayley operator, which is a generalization of the classic split inclusion problem which includes the generalized Cayley operator and the multi-valued mappings. Moreover, we proposed an iterative algorithms which converges weakly and strongly to some point of a solution set of the proposed problem.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y \in \mathsf{H}$ and $\alpha \in \mathbb{R}$, we have the following identities

$$(i)\ \|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.5}$$

$$(ii)\ \|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.6}$$

For every point $x \in \mathsf{H}$, there exists a unique point in C, denoted by $\mathsf{P}_C x$ such that $\|x - \mathsf{P}_C x\| \leq \|x - y\| \ \ \forall y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

Definition 2.1 A mapping $\mathsf{D} : \mathsf{H} \to \mathsf{H}$ is called nonexpansive if

$$\|\mathsf{D}(x) - \mathsf{D}(y)\| \leq \|x - y\| \ \ \forall x, y \in \mathsf{H}.$$

Definition 2.2 A mapping $\mathsf{D} : \mathsf{H} \to \mathsf{H}$ is called α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle \mathsf{D}(x) - \mathsf{D}(y), x - y \rangle \geq \alpha \|x - y\|^2 \qquad \forall x, y \in \mathsf{H}.$$

Definition 2.3 A mapping $\mathsf{D} : \mathsf{H} \to \mathsf{H}$ is called μ- inverse strongly monotone if there exists a constant $\mu > 0$ such that

$$\langle \mathsf{D}(x) - \mathsf{D}(y), x - y \rangle \geq \mu \|\mathsf{D}(x) - \mathsf{D}(y)\|^2 \qquad \forall x, y \in \mathsf{H}.$$

Definition 2.4 Let $\mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ be a multi-valued mapping, then it is said to be

(i) *monotone* if for all $x, y \in \mathsf{H}, u \in \mathsf{M}(x), v \in \mathsf{M}(y)$

$$\langle x - y, u - v \rangle \geq 0.$$

(ii) *strongly monotone* if for all $x, y \in \mathsf{H}, u \in \mathsf{M}(x), v \in \mathsf{M}(y)$, there exist $\theta > 0$ such that

$$\langle x - y, u - v \rangle \geq \theta \|x - y\|^2.$$

(iii) *maximal monotone* if M is monotone and $(\mathrm{I}+\lambda\ \mathsf{M})(\mathsf{H})=\mathsf{H}$ for all $\lambda > 0$, where I is the identity mapping on H.

Definition 2.5 Let $\mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ be a multi-valued mapping, then the resolvent operator is defined as:

$$\mathsf{J}_\lambda^{\mathsf{M}}(x) := (I + \lambda \mathsf{M})^{-1}(x), \forall x \in \mathsf{H},$$

for some $\lambda > 0$, where I stands for the identity operator on H.

Remark 2.1 *The resolvent operator* $\mathsf{J}_\lambda^{\mathsf{M}}$ *has the following properties:*

(i) *it is single-valued and nonexpansive, i.e.,*

$$\|\mathsf{J}_\lambda^{\mathsf{M}}(x) - \mathsf{J}_\lambda^{\mathsf{M}}(y)\| \leq \|x - y\|, \forall x, y \in \mathsf{H},$$

(ii) *it is 1-inverse strongly monotone, i.e,*

$$\|\mathsf{J}_\lambda^{\mathsf{M}}(x) - \mathsf{J}_\lambda^{\mathsf{M}}(y)\|^2 \leq \langle x - y, \mathsf{J}_\lambda^{\mathsf{M}}(x) - \mathsf{J}_\lambda^{\mathsf{M}}(y) \rangle, \forall x, y \in \mathsf{H}.$$

Definition 2.6 Let $\mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ be a multi-valued mapping and $\mathsf{J}_\lambda^{\mathsf{M}}$ be the resolvent operator associated with M, then the Cayley operator $\mathsf{C}_\lambda^{\mathsf{M}}$ is defined as:

$$\mathsf{C}_\lambda^{\mathsf{M}}(x) := (2\mathsf{J}_\lambda^{\mathsf{M}}(x) - I), \forall x \in \mathsf{H}.$$

Remark 2.2 *It can be easily seen that the Cayley operator* $\mathsf{C}_\lambda^{\mathsf{M}}$ *is 3-Lipschitz continuous.*

Let $\mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ be a multi-valued maximal monotone mapping, $\mathsf{J}_\lambda^{\mathsf{M}}$ be the resolvent operator and $\mathsf{C}_\lambda^{\mathsf{M}}$ be the Cayley operator associated with M, then the inclusion problem associated to Cayley's operator is to find $x \in \mathsf{H}$ such that

$$0 \in \mathsf{C}_\lambda^{\mathsf{M}}(x) + \mathsf{M}(x). \tag{2.7}$$

Lemma 2.1 [3] *Let* $\mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ *be a maximal monotone mapping and* $\mathsf{B} : \mathsf{H} \to \mathsf{H}$ *be a Lipschitz continuous mapping. Then a mapping* $\mathsf{B} + \mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ *is a maximal monotone mapping.*

In view of Remark 2.2 and Lemma 2.1, we can see that $\mathsf{C}_\lambda^{\mathsf{M}} + \mathsf{M} : \mathsf{H} \rightrightarrows \mathsf{H}$ is a maximal monotone mapping, where $\mathsf{C}_\lambda^{\mathsf{M}}$ is a Cayley operator. Now, we can easily define a new resolvent operator associated with the maximal monotone mapping $\mathsf{C}_\lambda^{\mathsf{M}} + \mathsf{M}$ as

$$\mathsf{J}_\lambda^{\mathsf{C}_\lambda^{\mathsf{M}}+\mathsf{M}}(x) := [I + \lambda(\mathsf{C}_\lambda^{\mathsf{M}} + \mathsf{M})]^{-1}(x),\ \forall\ x \in \mathsf{H}. \tag{2.8}$$

Notice that

$$\mathsf{J}_{\lambda}^{\mathsf{C}_{\lambda}^{\mathsf{M}}+\mathsf{M}}(x) = x$$
$$[I+\lambda(\mathsf{C}_{\lambda}^{\mathsf{M}}+\mathsf{M})]^{-1}(x) = x$$
$$x+\lambda(\mathsf{C}_{\lambda}^{\mathsf{M}}(x)+\mathsf{M}(x)) = x$$
$$0 \in \lambda(\mathsf{C}_{\lambda}^{\mathsf{M}}(x)+\mathsf{M}(x))$$
$$0 \in \mathsf{C}_{\lambda}^{\mathsf{M}}(x)+\mathsf{M}(x)$$

clearly, the fixed point of $\mathsf{J}_{\lambda}^{\mathsf{C}_{\lambda}^{\mathsf{M}}+\mathsf{M}}$ is a solution of the problem (2.7).

Remark 2.3 *From the definition* (2.1) *and* (2.3) *one can easily verify that the new resolvent operator* $\mathsf{J}_{\lambda}^{\mathsf{C}_{\lambda}^{\mathsf{M}}+\mathsf{M}}$ *is also nonexpansive and 1-inverse strongly monotone.*

Lemma 2.2 [10] *Let* $\mathsf{A}:\mathsf{H}_1\to\mathsf{H}_2$ *be a bounded linear operator with* $L=\|\mathsf{A}\|^2$ *and* $\mathsf{M}_2:\mathsf{H}_2\rightrightarrows\mathsf{H}_2$ *be a multi-valued maximal monotone mapping. Let* $\lambda>0$ *and* $\{\lambda_n\}$ *be a sequence of positive real numbers and define the operator* $U_n:\mathsf{H}_1\to\mathsf{H}_1$ *by*

$$U_n := I+\lambda_n\mathsf{A}^*(\mathsf{J}_{\lambda}^{\mathsf{M}_2}-I)\mathsf{A}.$$

Then for all $x\in\mathsf{H}_1$ *and* $p\in\mathsf{A}^{-1}(Fix(\mathsf{J}_{\lambda}^{\mathsf{M}_2})$ *we have*

$$\|U_nx-p\|^2\le\|x-p\|^2-\lambda_n(1-\lambda_nL)\|(I-\mathsf{J}_{\lambda}^{\mathsf{M}_2})\mathsf{A}x\|^2.$$

Lemma 2.3 [5] *Let* $\{\psi_n\},\{\delta_n\}$ *and* $\{\alpha_n\}$ *be the sequences in* $[0,+\infty)$ *such that* $\psi_{n+1}\le\psi_n+\alpha_n(\psi_n-\psi_{n-1})+\delta_n$ $\forall n\ge1$, $\sum_{n=1}^{+\infty}\delta_n<+\infty$ *and there exist a real number* α *with* $0\le\alpha_n\le\alpha<1$ *for all* $n\ge1$*. Then the following hold:*
(i) $\sum_{n=1}[\psi_n-\psi_{n-1}]_+<+\infty$ *where* $[t]_+=\max\{\mathrm{t},0\}$;
(ii) *there exists* $\psi^*\in[0,+\infty)$ *such that* $\lim_{n\to\infty}\psi_n=\psi^*$.

Lemma 2.4 [9] *Let* $\mathsf{M}:\mathsf{H}\rightrightarrows\mathsf{H}$ *be a set-valued maximal monotone mapping and* $\lambda>0$*. Then the following statements hold:*
(i) $\mathsf{J}_{\lambda}^{\mathsf{M}}$ *is a single-valued and firmly nonexpansive mappings;*
(ii) $Fix(\mathsf{J}_{\lambda}^{\mathsf{M}})=\mathsf{M}^{-1}(0)$;
(iii) $\|x-\mathsf{J}_{\lambda}^{\mathsf{M}}\|\le2\|x-\mathsf{J}_{\gamma}^{\mathsf{M}}\|$, $\forall 0<\lambda\le\gamma, x\in\mathsf{H}$;
(iv) $(I-\mathsf{J}_{\lambda}^{\mathsf{M}})$ *is firmly nonexapansive mapping;*
(v) *Suppose that* $\mathsf{M}^{-1}(0)\ne\phi$*. Then*
$\|\mathsf{J}_{\lambda}^{\mathsf{M}}(x)-z\|^2\le\|x-z\|^2-\|\mathsf{J}_{\lambda}^{\mathsf{M}}(x)-x\|^2$ *for all* $x\in\mathsf{H}$ *and* $z\in\mathsf{M}^{-1}(0)$;
and
$\langle x-\mathsf{J}_{\lambda}^{\mathsf{M}},\mathsf{J}_{\lambda}^{\mathsf{M}}-z\rangle\ge0$ *for all* $x\in\mathsf{H}$ *and* $z\in\mathsf{M}^{-1}(0)$.

Lemma 2.5 [4] *Let* H_1 *and* H_2 *be two real Hilbert spaces,* $\mathsf{A}:\mathsf{H}_1\to\mathsf{H}_2$ *be a bounded linear operator and* A^* *be adjoint of* A*, and let* $\beta>0$ *be fixed. Let* $\mathsf{M}:\mathsf{H}\rightrightarrows\mathsf{H}$ *be a set-valued maximal monotone mapping, and let* $\mathsf{J}_{\lambda}^{\mathsf{M}}$ *be a resolvent mapping of* M*.Then*

$$\|(I-\mathsf{J}_{\lambda}^{\mathsf{M}})\mathsf{A}x-(I-\mathsf{J}_{\lambda}^{\mathsf{M}})\mathsf{A}y\|^2\le\langle\mathsf{A}^*(I-\mathsf{J}_{\lambda}^{\mathsf{M}})\mathsf{A}x-\mathsf{A}^*(I-\mathsf{J}_{\lambda}^{\mathsf{M}})\mathsf{A}y,x-y\rangle,$$

for all $x,y\in\mathsf{H}_1$.

Lemma 2.6 [6] *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that there exists a subsequence* $\{a_{n_j}\}$ *of* $\{a_n\}$ *such that* $a_{n_j}<a_{n_j+1}$ *for all* $j\in\mathbb{N}$*. Then there exists a non-decreasing sequence* $\{m_k\}$ *such that* $\lim_{n\to\infty}m_k=\infty$ *and the following properties are satisfied by all (sufficiently large) number* $k\in\mathbb{N}$:

$$a_{m_k}\le a_{m_k+1}\ \ \text{and}\ \ a_k\le a_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1,2,3,...,k\}$ *such that* $a_n<a_{n+1}$.

Lemma 2.7 [7] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n$$

where $\{\alpha_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence such that
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \sup b_n \leq 0$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 [8] *Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*
(i) *for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
(ii) *every sequentially weak cluster point of $\{x_n\}$ is in C.*
Then $\{x_n\}$ converges weakly to a point in C.

3. Main Result

Let H_1 and H_2 be two real Hilbert spaces and $\mathsf{B}_1 : \mathsf{H}_1 \rightrightarrows \mathsf{H}_1$, $\mathsf{B}_2 : \mathsf{H}_2 \rightrightarrows \mathsf{H}_2$ be the multi-valued maximal monotone mappings, where $\mathsf{B}_1 := \mathsf{C}_\lambda^{\mathsf{M}_1} + \mathsf{M}_1$ and $\mathsf{B}_2 := \mathsf{C}_\lambda^{\mathsf{M}_2} + \mathsf{M}_2$ associated with multi-valued mappings M_1 and M_2 respectively. Consider $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ be bounded linear operator. Then Split Inclusion Problem (**SIP**) associated to Cayley operator is to find $x^* \in \mathsf{H}_1$ such that

$$0 \in \mathsf{B}_1(x^*) \tag{3.9}$$

and $y^* = \mathsf{A}x^*$ solves

$$0 \in \mathsf{B}_2(y^*). \tag{3.10}$$

The solution set of (**SIP**) is defined by $\Omega := \{x^* \in \mathsf{H}_1 : 0 \in \mathsf{B}_1(x^*) \text{ and } \mathrm{y}^* = \mathsf{A}\mathrm{x}^* \text{ such that } 0 \in \mathsf{B}_2(\mathrm{y}^*)\}$.

A classical approach for solving (**SIP**)(3.9)-(3.10) is an iterative method, which involves the resolvent operator associated with the maximal monotone operator. For a given $x_0 \in \mathsf{H}_1$ and $\lambda > 0$, compute

$$x_{n+1} = \mathsf{J}_\lambda^{\mathsf{B}_1}(x_n + \gamma \mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}x_n), \tag{3.11}$$

where A^* is the adjoint of A, L is the spectal radius of the $\mathsf{A}^*\mathsf{A}$ and $\gamma \in \left(0, \frac{1}{L}\right)$.

First, we establish a weak convergence theorem for solving (**SIP**)(3.9)-(3.10).

Theorem 3.1 *Let H_1 and H_2 be two real Hilbert spaces and $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ be a bounded linear operator, and A^* be the adjoint of A. Let $\mathsf{B}_1 : \mathsf{H}_1 \rightrightarrows \mathsf{H}_1$ and $\mathsf{B}_2 : \mathsf{H}_2 \rightrightarrows \mathsf{H}_2$ be two set-valued maximal monotone mappings and Ω be the solution set of (**SIP**)(3.9) – (3.10) with $\Omega \neq \phi$. Assume that the sequence $\{\theta_n\}$ is non-decreasing such that $0 \leq \theta_n \leq \theta < 1$. Let $\lambda > 0$ and $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < (\frac{1}{L})$, where $L := \|\mathsf{A}\|^2$ and the sequence $\{\alpha_n\}$ is non-decreasing such that*

$$\alpha \leq \alpha_n \leq \frac{1}{1+\theta+\delta} \tag{3.12}$$

for some $\delta > 0$ and $\alpha > 0$.
Let $\{x_n\}$ be a sequence in H_1 defined by

$$\begin{cases} x_0, x_1 \in \mathsf{H}_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \mathsf{J}_\lambda^{\mathsf{B}_1}(I + \lambda_n \mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A})w_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n. \end{cases}$$

Then, the sequence $\{x_n\}$ converges weakly to an element of Ω.

Proof. Fix $p \in \Omega$. Since $\lambda_n \in [a, b] \subset \left(0, \frac{1}{L}\right)$, by Lemma 2.2, we have

$$\|y_n - p\| \leq \|w_n - p\|. \tag{3.13}$$

Using (2.6) and (3.13) we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n y_n - p\|^2 \\ &= \|(1-\alpha_n)(x_n - p) + \alpha_n(y_n - p)\|^2 \\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 - (1-\alpha_n)\alpha_n\|y_n - x_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|w_n - p\|^2 - (1-\alpha_n)\alpha_n\|y_n - x_n\|^2 \end{aligned} \tag{3.14}$$

on the other hand, we also have

$$y_n - x_n = \frac{1}{\alpha_n}(x_{n+1} - x_n). \tag{3.15}$$

Combining (3.14) and (3.15) we get

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|w_n - p\|^2 - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2. \tag{3.16}$$

By the definition of w_n, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|(1+\theta_n)(x_n - p) - \theta_n(x_{n-1} - p)\|^2 \\ &= (1+\theta_n)\|x_n - p\|^2 - \theta_n\|x_{n-1} - p\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 \end{aligned} \tag{3.17}$$

combining (3.16) with (3.17) we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq& (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(1+\theta_n)\|x_n - p\|^2 - \alpha_n\theta_n\|x_{n-1} - p\|^2 \\ &+ \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2 \\ =& (1+\alpha_n\theta_n)\|x_n - p\|^2 - \alpha_n\theta_n\|x_{n-1} - p\|^2 \\ &+ \alpha_n\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 - \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2 \qquad (3.18) \\ =& (1+\gamma_n)\|x_n - p\|^2 - \gamma_n\|x_{n-1} - p\|^2 + \mu_n\|x_n - x_{n-1}\|^2 \\ &- \frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2, \qquad (3.19)\end{aligned}$$

where, $\gamma_n := \alpha_n\theta_n$ and $\mu_n := \alpha_n\theta_n(1+\theta_n)$.

Put $\gamma_n := \|x_n - p\|^2 - \gamma_n\|x_{n-1} - p\|^2 + \mu_n\|x_n - x_{n-1}\|^2$. By the sequences $\{\alpha_n\}$, $\{\theta_n\}$ are non-decreasing we have the sequence$\{\gamma_n\}$ is non-decreasing. This implies that

$$\begin{aligned}\gamma_{n+1} - \gamma_n &= \|x_{n+1} - p\|^2 - (1+\gamma_{n+1})\|x_n - p\|^2 + \gamma_n\|x_{n-1} - p\|^2 \\ &\quad + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2 \\ &\leq \|x_{n+1} - p\|^2 - (1+\gamma_n)\|x_n - p\|^2 + \gamma_n\|x_{n-1} - p\|^2 \\ &\quad + \mu_{n+1}\|x_{n+1} - x_n\|^2 - \mu_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\begin{aligned}\gamma_{n+1} - \gamma_n &\leq -\frac{1-\alpha_n}{\alpha_n}\|x_{n+1} - x_n\|^2 + \mu_{n+1}\|x_{n+1} - x_n\|^2 \\ &= -\Big(\frac{1-\alpha_n}{\alpha_n} - \mu_{n+1}\Big)\|x_{n+1} - x_n\|^2. \end{aligned} \tag{3.21}$$

Thanks to (3.12), we get

$$\frac{1-\alpha_n}{\alpha_n} - \mu_{n+1} = \frac{1}{\alpha_n} - 1 - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1})$$
$$\geq 1+\theta+\delta-1-\frac{1}{1+\theta+\delta}(\theta^2+\theta)$$
$$= \delta + \frac{\theta\delta}{1+\theta+\delta} \geq \delta. \tag{3.22}$$

Combining (3.21) and (3.22) we get

$$\gamma_{n+1} - \gamma_n \leq -\delta\|x_{n+1}-x_n\|^2 \leq 0. \tag{3.23}$$

This implies that the sequence $\{\gamma_n\}$ is nonincreasing. On the other hand, we have

$$\gamma_n = \|x_n - p\|^2 - \gamma_n\|x_{n-1}-p\|^2 + \mu_n\|x_n - x_{n-1}\|^2$$
$$\geq \|x_n - p\|^2 - \gamma_n\|x_{n-1}-p\|^2. \tag{3.24}$$

We have

$$\gamma_n = \alpha_n\theta_n < \frac{\theta}{1+\theta+\delta} =: \gamma < 1. \tag{3.25}$$

It implies from (3.24) and (3.25) that

$$\|x_n - p\|^2 \leq \gamma_n\|x_{n-1}-p\|^2 + \gamma_n$$
$$\leq \gamma\|x_{n-1}-p\|^2 + \gamma_1$$
$$\leq \cdots \leq \gamma^n\|x_0-p\|^2 + \gamma_1(\gamma^{n-1}+ ... +1)$$
$$\leq \gamma^n\|x_0-p\|^2 + \frac{\gamma_1}{1-\gamma}. \tag{3.26}$$

We also have

$$\gamma_{n+1} = \|x_{n+1}-p\|^2 - \gamma_{n+1}\|x_n - p\|^2 + \mu_{n+1}\|x_{n+1}-x_n\|^2$$
$$\geq -\gamma_{n+1}\|x_n-p\|^2. \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$-\gamma_{n+1} \leq \gamma_{n+1}\|x_n-p\|^2 \leq \gamma\|x_n-p\|^2 \leq \gamma^{n+1}\|x_0-p\|^2 + \frac{\gamma\gamma_1}{1-\gamma}.$$

Thanks to (3.23) we obtain

$$\delta\sum_{n=1}^{k}\|x_{n+1}-x_n\|^2 \leq \gamma_1 - \gamma_{k+1} \leq \gamma^{k+1}\|x_0-p\|^2 + \frac{\gamma_1}{1-\theta}$$
$$\leq \|x_0-p\|^2 + \frac{\gamma_1}{1-\theta}.$$

This implies

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 < +\infty. \tag{3.28}$$

Therefore, we obtain $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Since (3.18) we get

$$\|x_{n+1}-p\|^2 \leq (1+\gamma_n)\|x_n-p\|^2 - \gamma_n\|x_{n-1}-p\|^2 + 2\theta\|x_n-x_{n-1}\|^2. \tag{3.29}$$

By (3.28), (3.29) and Lemma 2.5 we have

$$\lim_{n\to\infty}\|x_n-p\|^2 = l, \tag{3.30}$$

and by (3.17) we obtain

$$\lim_{n\to\infty} \|w_n - p\|^2 = l. \tag{3.31}$$

From the definition of w_n we have

$$\lim_{n\to\infty} \|w_n - x_n\| \leq \theta \lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{3.32}$$

It follows from (3.14) that

$$\|x_{n+1} - p\|^2 \leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n \|y_n - p\|^2. \tag{3.33}$$

This implies that

$$\|y_n - p\|^2 \geq \frac{\|x_{n+1} - p\|^2 - \|x_n - p\|^2}{\alpha_n} + \|x_n - p\|^2. \tag{3.34}$$

Since $\{\alpha_n\}$ is bounded, it implies from (3.30) and (3.34) that

$$\lim_{n\to\infty} \|y_n - p\|^2 \geq \lim_{n\to\infty} \|x_n - p\|^2 = l. \tag{3.35}$$

By Lemma 2.2 we get

$$\lim_{n\to\infty} \|y_n - p\|^2 \leq \lim_{n\to\infty} \|w_n - p\|^2 = l. \tag{3.36}$$

Combining (3.35) and (3.36) we obtain

$$\lim_{n\to\infty} \|y_n - p\|^2 = l. \tag{3.37}$$

By Lemma 2.2 we get

$$\lambda_n(1-\lambda_n L)\|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|^2 \leq \|w_n - p\|^2 - \|y_n - p\|^2 \to 0. \tag{3.38}$$

From $\lambda_n \in [a,b] \subset \left(0, \frac{1}{L}\right)$, we get $\lim_{n\to\infty} \|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\| = 0$. We have

$$\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\| \leq \|\mathsf{A}^*\|\|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\| \to 0. \tag{3.39}$$

Since the fact that $\mathsf{J}_\lambda^{\mathsf{B}_2}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|y_n - p\|^2 =& \|\mathsf{J}_\lambda^{\mathsf{B}_1}(w_n + \lambda_n \mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n) - p\|^2 \\
=& \|\mathsf{J}_\lambda^{\mathsf{B}_1}(w_n + \lambda_n \mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n) - \mathsf{J}_\lambda^{\mathsf{B}_2}(p)\|^2 \\
\leq& \langle y_n - p, w_n + \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n - p\rangle \\
=& \frac{1}{2}\Big\{\|y_n - p\|^2 + \|w_n + \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n - p\|^2 \\
&- \|y_n - p - [w_n + \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n - p]\|^2\Big\} \\
=& \frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 + \frac{1}{2}\lambda_n^2\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\|^2 \\
&+ \langle w_n - p, \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\rangle - \frac{1}{2}\|y_n - w_n\|^2 \\
&- \frac{1}{2}\lambda_n^2\|A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\|^2 + \langle y_n - w_n, \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\rangle \\
=& \frac{1}{2}\|y_n - p\|^2 + \frac{1}{2}\|w_n - p\|^2 - \frac{1}{2}\|y_n - w_n\|^2 \\
&+ \langle y_n - p, \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|y_n - p\|^2 \leq& \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - p, \lambda_n A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\rangle \\
\leq& \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\lambda_n\|y_n - p\|\|A^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)Aw_n\|.
\end{aligned} \tag{3.40}$$

It follows from (3.33) and (3.40) that

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq &(1-\alpha_n)\|x_n-p\|^2+\alpha_n\|w_n-p\|^2-\alpha_n\|y_n-w_n\|^2\\ &+2\lambda_n\alpha_n\|y_n-p\|\|A^*(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)Aw_n\|.\end{aligned}$$

This implies that

$$\begin{aligned}\alpha_n\|y_n-w_n\|^2 \leq &\|x_n-p\|^2-\|x_{n+1}-p\|^2\|+\alpha_n(\|w_n-p\|^2-\|x_n-p\|^2)\\ &+2\lambda_n\alpha_n\|y_n-p\|\|A^*(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)Aw_n\|.\end{aligned} \tag{3.41}$$

Combining (3.30),(3.31),(3.39) and (3.41) we get

$$\lim_{n\to\infty}\|y_n-w_n\|=0. \tag{3.42}$$

Now, we show that the sequence $\{x_n\}$ converges weakly to an element of Ω. Indeed, since $p\in\Omega$, it follows that $p\in\mathsf{B}_1^{-1}(0)$ and that is $p\in\mathrm{Fix}(\mathsf{J}_\lambda^{\mathsf{B}_1})$. By Lemma 2.3 (v) we get

$$\langle\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-w_n+y_n,y_n-p\rangle\leq 0. \tag{3.43}$$

It also follows that $\mathsf{A}p\in\mathsf{B}_2^{-1}(0)$, thus $\mathsf{A}p\in\mathrm{Fix}(\mathsf{J}_\lambda^{\mathsf{B}_2})$. This implies that $\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}p=0$. By Lemma 2.4 we have

$$\langle\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n-\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}p,y_n-p\rangle\geq\|(I-\mathsf{J}_\lambda^{\mathsf{B}_2})Ay_n\|^2. \tag{3.44}$$

It follows from (3.43) and (3.44) that

$$\begin{aligned}\lambda_n\|\mathsf{A}y_n-\mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}y_n\|^2 \leq &\langle\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n,y_n-p\rangle\\ \leq&\langle w_n-y_n-\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n\\ &\quad+\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n,y_n-p\rangle\\ &\quad+\langle\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-w_n+y_n,y_n-p\rangle\\ \leq&\langle w_n-y_n-\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n\\ &\quad+\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n,y_n-p\rangle\\ \leq&\|w_n-y_n-\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n\\ &\quad+\lambda_n\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\|\|y_n-p\|\\ \leq&(\|w_n-y_n\|+b\|\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n\\ &\quad-\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\|)\|y_n-p\|.\end{aligned} \tag{3.45}$$

On the other hand, using Lemma 2.3 (iv) we have

$$\begin{aligned}\|\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\|^2 =&\langle\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n,\\ &\ \mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\rangle\\ =&\langle\mathsf{A}\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n,\\ &\ (I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\rangle\\ \leq&\|\mathsf{A}\mathsf{A}^*\|\|(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\|^2\\ \leq&L\|\mathsf{A}w_n-\mathsf{A}y_n\|^2\\ \leq&L^2\|w_n-y_n\|^2.\end{aligned}$$

This implies that

$$\|\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n-\mathsf{A}^*(I-\mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\|\leq L\|w_n-y_n\|. \tag{3.46}$$

It follows from (3.45) and (3.46) that

$$\lambda_n\|\mathsf{A}y_n - \mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}y_n\| \leq (1+bL)\|w_n - y_n\|\|y_n - p\|.$$

It follows from $\lambda_n \geq a > 0$ and (3.42) that

$$\lim_{n\to\infty} \|\mathsf{A}y_n - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}y_n\| = 0. \tag{3.47}$$

We also have

$$\begin{aligned}\|\mathsf{A}w_n - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}w_n\| \leq &\|\mathsf{A}w_n - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}w_n - \mathsf{A}y_n + \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}w_n\| + \|\mathsf{A}y_n - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}y_n\| \\ \leq &2\|\mathsf{A}\|\|w_n - y_n\| + \|\mathsf{A}y_n - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}y_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|\mathsf{A}w_n - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}w_n\| = 0. \tag{3.48}$$

By the definition of of y_n and $\mathsf{J}_\lambda^{\mathsf{B}_1}$ is firmly nonoexpansive, we have

$$\begin{aligned}\|y_n - \mathsf{J}_\lambda^{\mathsf{B}_1}w_n\| = &\|\mathsf{J}_\lambda^{\mathsf{B}_1}(w_n - \lambda_n\mathsf{A}^*(I - \mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n) - \mathsf{J}_\lambda^{\mathsf{B}_1}w_n\| \\ \leq &\lambda_n\|\mathsf{A}^*\|\|(I - \mathsf{J}_\lambda^{\mathsf{B}_2})\mathsf{A}w_n\|.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|y_n - \mathsf{J}_\lambda^{\mathsf{B}_1}w_n\| = 0 \tag{3.49}$$

Therefore, from (3.42) and (3.49) we get

$$\lim_{n\to\infty} \|w_n - \mathsf{J}_\lambda^{\mathsf{B}_1}w_n\| = 0 \tag{3.50}$$

Since $\{x_n\}$ is bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in \mathsf{H}_1$ such that $\{x_{n_k}\} \rightharpoonup q$. By (3.32) we get $w_{n_k} \rightharpoonup q$. Since A is bounded linear operator, it follows that $\mathsf{A}w_{n_k} \rightharpoonup \mathsf{A}q$. By (3.50) and Lemma 2.3 (i) we get $q \in Fix(\mathsf{J}_\lambda^{\mathsf{B}_1})$. By (3.48) and Lemma 2.3 (i) we have $\mathsf{A}q \in Fix(\mathsf{J}_\lambda^{\mathsf{B}_1})$.
Therefore, we proved that:
(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$;
(ii) If $x_{n_k} \rightharpoonup q$ then $q \in \Omega$.
By Lemma 2.6, we get $\{x_n\}$ converges weakly to an element of Ω. □

Next, we establish strong convergence theorem for solving (**SIP**)(3.9)-(3.10).

Theorem 3.2 *Let H_1 and H_2 be two real Hilbert spaces and $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ be a bounded linear operator and A^* be the adjoint of A. Let $\mathsf{B}_1 : \mathsf{H}_1 \rightrightarrows \mathsf{H}_1$ and $\mathsf{B}_2 : \mathsf{H}_2 \rightrightarrows \mathsf{H}_2$ be two set-valued maximal monotone mappings and Ω be the solution set of (**SIP**)(3.9)-(3.10). with $\Omega \neq \phi$. Assume that the sequence $\{\theta_n\}$ is sequence in $[0,\theta]$ for some $\theta > 0$. Let $\lambda > 0$ and $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < (\frac{1}{L})$, where $L := \|\mathsf{A}\|^2$. Let $f : \mathsf{H}_1 \to \mathsf{H}_1$ be a contraction mapping with contraction parameter $\kappa \in [0,1)$. Assume that $\{\alpha_n\}$ is a sequence in $(0,1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ be a sequence in H_1 defined by

$$\begin{cases} x_0, x_1 \in \mathsf{H}_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \mathsf{J}_\lambda^{\mathsf{B}_1}(I + \lambda_n\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A})w_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n \end{cases} \tag{3.51}$$

Assume that the sequence $\{\theta_n\}$ is chosen such that satisfying the following condition

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|=0. \tag{3.52}$$

Then the sequence $\{x_n\}$ generated by (3.51) converges strongly to an element of $p\in\Omega$, where $p=P_\Omega\circ f(p)$.

Proof. **Claim 1**. The sequence $\{x_n\}$ is bounded. Indeed, by Lemma 2.2, we have

$$\|y_n-p\|^2\leq\|w_n-p\|^2-\lambda_n(1-\lambda_nL)\|(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)\mathsf{A}w_n\|^2. \tag{3.53}$$

Therefore

$$\|y_n-p\|\leq\|w_n-p\| \tag{3.54}$$

From the definition of w_n we get

$$\begin{aligned}\|w_n-p\|&=\|x_n+\theta_n(x_n-x_{n-1})-p\|\\&\leq\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|\\&=\|x_n-p\|+\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\alpha_n.\end{aligned} \tag{3.55}$$

By condition $(\theta_n/\alpha_n)\|x_n-x_{n-1}\|\to0$, there exist a constant $M_1>0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq M_1.\quad\forall n \tag{3.56}$$

Combining (3.54), (3.55) and (3.56) we obtain

$$\|y_n-p\|\leq\|w_n-p\|\leq\|x_n-p\|+\alpha_nM_1. \tag{3.57}$$

From the definition of $\{x_n\}$ we get

$$\begin{aligned}\|x_{n+1}\|&=\|\alpha_nf(x_n)+(1-\alpha_n)y_n-p\|\\&=\|\alpha_nf(x_n-p)+(1-\alpha_n)(y_n-p)\|\\&\leq\alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|(y_n-p)\|\\&\leq\alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|(y_n-p)\|\\&\leq\alpha_n\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|(y_n-p)\|.\end{aligned} \tag{3.58}$$

Substituting (3.57) into (3.58) we obtain

$$\begin{aligned}\|\mathrm{x_{n+1}-p}\|&\leq(1-(1-\kappa)\beta_n)\|x_n-p\|+\beta_nM_1+\beta_n\|f(p)-p\|\\&=(1-(1-\kappa)\beta_n)\|x_n-p\|+(1-\kappa)\beta_n\frac{M-1+\|f(p)-p\|}{1-\kappa}\\&\leq\max\{\|\mathrm{x_n-p}\|,\frac{\mathrm{M_1+\|f(p)-p\|}}{1-\kappa}\}\\&\leq\ldots\leq\max\{\|\mathrm{x_0-p}\|,\frac{\mathrm{M_1+\|f(p)-p\|}}{1-\kappa}\}\end{aligned}$$

Thsi implies $\{x_n\}$ is bounded. We also get $\{y_n\},\{f(x_n)\},\{w_n\}$ are bounded.
Claim 2.

$$(1-\alpha_n)\lambda_n(1-\lambda_nL)\|((\mathsf{J}_\lambda^{\mathsf{B}_2}-I))\mathsf{A}w_n\|\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_4,$$

for some $M_4 > 0$. indeed, we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)y_n - p\|^2 \\
&\leq \|\alpha_n f(x_n) - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n(\|f(x_n) - f(p)\| + \|f(p) - p\|)^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n(\kappa\|x_n - p\| + \|f(p) - p\|)^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \alpha_n(2\|x_n - p\|.\|f(p) - p\| + \|f(p) - p\|) + \|f(p) - p\|^2 \\
&\quad + (1-\alpha_n)\|y_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 + \alpha_n M_2, \qquad (3.59)
\end{aligned}
$$

for some $M_2 > 0$. Substituting (3.53) into (3.59) we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 \\
& - (1-\alpha_n)\lambda_n(1-\lambda_n L)\|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|^2 + \beta_n M_2. \qquad (3.60)
\end{aligned}
$$

It implies from (3.57) that

$$
\begin{aligned}
\|w_n - p\|^2 \leq & (\|x_n - p\| + \alpha_n M_1)^2 \\
& = \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\
\leq & \|x_n - p\|^2 + \alpha_n M_3, \qquad (3.61)
\end{aligned}
$$

for some $M_3 > 0$. Combining (3.60) and (3.61) we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
& + \alpha_n M_3 - (1-\alpha_n)\lambda_n(1-\lambda_n L)\|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|^2 + \alpha_n M_2 \\
= & \|x_n - p\|^2 + \alpha_n M_3 - (1-\alpha_n)\lambda_n(1-\lambda_n L)\|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|^2 \\
& + \alpha_n M_2.
\end{aligned}
$$

This implies that

$$
(1-\alpha_n)\lambda_n(1-\lambda_n L)\|(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|^2 \leq \|x_{n+1} - q\|^2 - \|x_n - q\|^2 + \alpha_n M_4,
$$

where $M_4 := M_2 + M_3$.

Claim 3.

$$
\begin{aligned}
(1-\alpha_n)\|y_n - w_n\|^2 \leq & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4 \\
& + 2(1-\alpha_n)\lambda_n\|y_n - p\|.\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|.
\end{aligned}
$$

Indeed, according to (3.40) we have

$$
\|y_n - p\|^2 \leq \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\lambda_n\|y_n - p\|.\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\|. \qquad (3.62)
$$

Combining (3.59) and (3.62) we get

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|w_n - p\|^2 - (1-\alpha_n)\|y_n - w_n\|^2 \\
& + 2\lambda_n(1-\alpha_n)\|y_n - p\|.\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\| + \alpha_n M_2. \qquad (3.63)
\end{aligned}
$$

Substituting (3.61) into (3.63) we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq & \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 + \alpha_n M_3 \\
& - (1-\alpha_n)\|y_n - w_n\|^2 + 2\lambda_n(1-\alpha_n)\|y_n - p\| \\
& \times \|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\| + \alpha_n M_2 \\
\leq & \|x_n - p\|^2 - (1-\alpha_n)\|y_n - w_n\|^2 \\
& + 2(1-\alpha_n)\lambda_n\|y_n - p\|.\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2} - I)\mathsf{A}w_n\| + \alpha_n M_4.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(1-\alpha_n)\|y_n-w_n\|^2 \leq & \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4 \\
& +2(1-\alpha_n)\lambda_n\|y_n-p\|.\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)\mathsf{A}w_n\|+\alpha_n M_4.
\end{aligned}
$$

Claim 4.

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq & (1-(1-\kappa)\alpha_n)\|x_n-p\|^2+(1-\kappa)\alpha_n \\
& \times\left[\frac{2}{1-\kappa}\langle f(p)-p, x_{n+1}-p\rangle+\frac{\theta_n}{\alpha_n}.\|x_n-x_{n-1}\|.\frac{M}{1-\kappa}\right],
\end{aligned}
$$

for some $M>0$. Indeed, we have

$$
\begin{aligned}
\|w_n-p\|^2 &= \|x_n+\theta_n(x_n-x_{n-1}-p)\|^2 \\
&= \|x_n-p\|^2+2\theta_n\langle x_n-p, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2 \\
&\leq \|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2 \\
&= \|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|\Big[2\|x_n-p\|+\theta_n\|x_n-x_{n-1}\|\Big] \\
&\leq \|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|M, \qquad (3.64)
\end{aligned}
$$

for $M>0$. Using (2.5) we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)y_n-p\|^2 \\
&= \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(y_n-p)+\alpha_n(f(p)-p)\|^2 \\
&\leq \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(y_n-p)\|^2 \\
&\quad +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \alpha_n\|f(x_n)-f(p)\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\quad +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \alpha_n\kappa^2\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\quad +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
&\leq \alpha_n\kappa\|x_n-p\|^2+(1-\alpha_n)\|w_n-p\|^2+2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle. \qquad (3.65)
\end{aligned}
$$

Substituting (3.64) into (3.65) we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \leq & (1-(1-\kappa)\alpha_n)\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|M \\
& +2\alpha_n\langle f(p)-p, x_{n+1}-p\rangle \\
= & (1-(1-\kappa)\alpha_n)\|x_n-p\|^2+(1-\kappa)\alpha_n \\
& \times\left[\frac{2}{1-\kappa}\leq f(p)-p, x_{n+1}-p\rangle+\frac{\theta_n}{\alpha_n}.\|x_n-x_{n-1}\|.\frac{M}{1-\kappa}\right].
\end{aligned}
$$

Claim 5. The sequence $\{\|x_n-p\|\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n-p\|^2\}$.

Case 1. There exists $N\in\mathbb{N}$ such that $\|x_{n+1}-p\|^2\leq\|x_n-p\|^2$ $\forall n\geq N$. This implies that $\lim_{n\to\infty}\|x_n-p\|$ exists and according to Claim 2 we obtain

$$
\lim_{n\to\infty}\|(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)\mathsf{A}w_n\|=0. \qquad (3.66)
$$

According to Claim 3 and (3.66) we get

$$
\lim_{n\to\infty}\|y_n-w_n\|=0. \qquad (3.67)
$$

We show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. Indeed, we have

$$\|x_{n+1}-y_n\| = \alpha_n \|y_n - f(x_n)\| \to 0, \tag{3.68}$$

and

$$\|x_n - w_n\| = \theta_n \|x_n - x_{n-1}\| = \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|.\alpha_n \to 0. \tag{3.69}$$

It implies from (3.67),(3.68) and (3.69) that

$$\|x_{n+1}-x_n\| \le \|x_{n+1}-y_n\| + \|y_n - w_n\| + \|w_n - x_n\| \to 0. \tag{3.70}$$

Since the sequence $\{x_n\}$ is bounded, it implies that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that weak convergence to some $z \in \mathbb{H}$ such that

$$\lim_{n\to\infty} \sup\langle f(p)-p, x_n - p\rangle = \lim_{n\to\infty}\langle f(p)-p, x_{n_k}-p\rangle = \langle f(p)-p, z-p\rangle. \tag{3.71}$$

We will show that $z \in \Omega$. Indeed, thanks to (3.48) and (3.49), we obtain

$$\lim_{\mathrm{n}\to\infty} \|\mathsf{A}\mathrm{w_n} - \mathsf{J}_\lambda^{\mathsf{B}_2}\mathsf{A}\mathrm{w_n}\| = 0 \quad \text{and} \quad \lim_{\mathrm{n}\to\infty}\|\mathrm{w_n} - \mathsf{J}_\lambda^{\mathsf{B}_1}\mathrm{w_n}\| = 0. \tag{3.72}$$

By (3.69) we get $w_{n_k} \rightharpoonup z$. Since A is a bounded linear operator, it follows that $\mathsf{A}w_{n_k} \rightharpoonup \mathsf{A}z$. By (3.72) and Lemma 2.6 (i) we get $z \in Fix(\mathsf{J}_\lambda^{\mathsf{B}_1})$ and $\mathsf{A}z \in Fix(\mathsf{J}_\lambda^{\mathsf{B}_2})$.
Therefore $z \in \Omega$.
Since (3.71) and $\mathrm{p} = \mathrm{P}_\Omega \circ \mathrm{f(p)}$, we have

$$\lim_{n\to\infty} \sup\langle f(p)-p, x_n - p\rangle = \langle f(p)-p, z-p\rangle \tag{3.73}$$

Combining (3.70) and (3.73) we obtain

$$\lim_{n\to\infty} \sup\langle f(p)-p, x_{n+1}-p\rangle \le \lim_{n\to\infty}\sup\langle f(p)-p, x_n-p\rangle = \langle f(p)-p, z-p\rangle. \tag{3.74}$$

Using Lemma 2.5 and (3.74), the restriction $\lim_{n\to\infty}\left(\frac{\theta_n}{\alpha_n}\right)\|x_n - x_{n-1}\| = 0$ and Claim 4 we get $\lim_{n\to\infty}\|x_n - p\| = 0$.

Case 2. There exist a subsequernce $\{\|x_{n_j}-p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j}-p\|^2 < \|x_{n_j+1}\|^2$ for all j$\in \mathbb{N}$. In this case, it follows from Lemma 2.5 that there exists a non-decreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} m_k = \infty$ and the following inequalities hold for all k$\in \mathbb{N}$:

$$\|\mathrm{x_{m_k}} - \mathrm{p}\|^2 \le \|\mathrm{x_{m_k+1}} - \mathrm{p}\|^2 \quad \text{and} \quad \|\mathrm{x_k} - \mathrm{p}\|^2 \le \|\mathrm{x_{m_k}} - \mathrm{p}\|^2. \tag{3.75}$$

According to claim 2, we have

$$\begin{aligned}(1-&\alpha_{m_k})\lambda_{m_k}(1-\lambda_{m_k}L)\|((\mathsf{J}_\lambda^{\mathsf{B}_2}-I))\mathsf{A}w_{m_k}\| \\ &\le \|x_{m_k}-p\|^2 - \|x_{m_k+1}-p\|^2 + \alpha_{m_k}M_4 \\ &\quad a_{m_k}M_4.\end{aligned}$$

Therefore, we obtain

$$\lim_{k\to\infty}\|(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)\mathsf{A}w_{m_k}\| = 0. \tag{3.76}$$

According to claim 3, we have

$$\begin{aligned}(1-\alpha_{m_k})\|y_{m_k}-w_{m_k}\|^2 \le &\|x_{m_k}-p\|^2 - \|x_{m_k+1}-p\|^2 + \alpha_{m_k}M_4 \\ &+ 2(1-\alpha_{m_k})\lambda_{m_k}\|y_{m_k}-p\|.\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)\mathsf{A}w_{m_k}\| \\ &\alpha_{m_k}M_4 + 2(1-\alpha_{m_k})\lambda_{m_k}\|y_{m_k}-p\|. \\ &\times\|\mathsf{A}^*(\mathsf{J}_\lambda^{\mathsf{B}_2}-I)\mathsf{A}w_{m_k}\|.\end{aligned}$$

Therefore, we obtain

$$\lim_{k\to\infty} \|y_{m_k} - w_{m_k}\| = 0.$$

Using the same arguments as in the proof of Case 1 we obtain

$$\|\mathrm{x}_{\mathrm{m_k}+1} - \mathrm{x}_{\mathrm{m_k}}\| \to 0 \quad \text{and} \quad \lim_{\mathrm{n}\to\infty} \sup\langle \mathrm{f(p)} - \mathrm{p}, \mathrm{x}_{\mathrm{m_k}} - \mathrm{p}\rangle \leq 0.$$

According to claim 4 we have

$$\begin{aligned}\|x_{m_k+1} - p\|^2 \leq& (1-(1-\kappa)\alpha_{m_k})\|x_{m_k} - p\|^2 + (1-\kappa)\alpha_{m_k} \\ &\times \Big[\frac{2}{1-\kappa}\langle f(p) - p, x_{m_k+1} - p\rangle + \frac{\theta_{m_k}}{\alpha_{m_k}}.\|x_{m_k} - x_{m_k-1}\|.\frac{M}{1-\kappa}\Big]. \quad (3.77)\end{aligned}$$

From (3.75) and (3.77) we obtain

$$\|x_{m_k+1} - p\|^2 \leq \Big[\frac{2}{1-\kappa}\langle f(p) - p, x_{m_k+1} - p\rangle + \frac{\theta_{m_k}}{\alpha_{m_k}}.\|x_{m_k} - x_{m_k-1}\|.\frac{M}{1-\kappa}\Big].$$

Therefore,

$$\lim_{k\to\infty} \sup \|x_{m_k+1} - p\| \leq 0. \quad (3.78)$$

Combining (3.75) and (3.78) $\lim_{k\to\infty} \|x_k - p\| \leq 0$, that is $x_k \to p$. The proof is completed. □

4. Application to the split feasibility problem

Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed covex subsets of real Hilbert spaces H_1 and H_2, respectively.The split feasibility problem is formulated as:

$$\text{(SEP) Find } x^* \in \mathcal{C} \text{ such that } \mathsf{A}x^* \in \mathcal{Q},$$

where $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ is a bounded linear operator. Let Δ_{SEP} be the solution set of problem (SEP). In 1994, Censor and Elfving [9] first introduced the SFP in finite-dimensional Hilbert spaces for modelling inverse problems which arise from phase retrievals and in medical image reconstruction. It has been found that the SFP can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning.

Let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty)$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in \mathsf{H} : f(x) - f(y) \leq \langle z, x - y\rangle \forall y \in \mathsf{H}\}$$

for all $x \in \mathsf{H}$. Let $\mathcal{C}$ be a nonempty closed convex subset of real Hilbert space H and $i_{\mathcal{C}}$ be the indicator function of $\mathcal{C}$, that is,

$$i_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if} \quad x \in \mathcal{C}; \\ \infty & \text{if} \quad x \notin \mathcal{C}. \end{cases}$$

Further, we define the normal cone $N_{\mathcal{C}}u$ of $\mathcal{C}$ at $u \in \mathcal{C}$ as follows:

$$N_{\mathcal{C}}u = \{z \in \mathsf{H} : \langle z, v - u\rangle \leq 0 \quad \forall v \in \mathcal{C}\}.$$

We known that $i_{\mathcal{C}}$ is a proper, lower semicontinuous and convex function on H. Thus, the subdifferential $\partial i_{\mathcal{C}}$ of $i_{\mathcal{C}}$ is a maximal monotone operator. So, we can define the resolvent $\mathsf{J}_\lambda^{\partial i_{\mathcal{C}}}$ of $\partial i_{\mathcal{C}}$ for each $\lambda > 0$, that is

$$\mathsf{J}_\lambda^{\partial i_{\mathcal{C}}} x = (I + \lambda\partial i_{\mathcal{C}})^{-1}x$$

for all $x \in \mathsf{H}$. Furthermore, for each $x \in \mathcal{C}$ we have

$$\begin{aligned}\partial i_{\mathcal{C}}x &= \{z \in \mathsf{H} : i_{\mathcal{C}} + \langle z, y - x\rangle \leq i_{\mathcal{C}}y \ \ \forall y \in \mathsf{H}\} \\ &= \{z \in \mathsf{H} : \langle z, y - x\rangle \leq 0 \ \ \forall y \in \mathcal{C}\} \\ &= N_{\mathcal{C}}x.\end{aligned}$$

Therefore, for each $\alpha > 0$, we derive

$$\begin{aligned} y = \mathsf{J}_\lambda^{\partial i_C} x &\iff x \in y + \lambda \partial i_{\mathcal{C}} y \iff x - y \in \lambda \partial i_{\mathcal{C}} y \\ &\iff \langle x - y, z - y \rangle \leq 0 \quad \forall z \in \mathcal{C} \\ &\iff y = P_{\mathcal{C}} x. \end{aligned}$$

Now applying Theorem 3.1 and Theorem 3.2 we obtain the following results.

Theorem 4.1 *Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convexsubsets Hilbert spaces H_1 and H_2, respectively and $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ be a bounded linear operator, and A^* be the adjoint of A. Let Δ_{SEP} be the solution set of problem (SEP) with $\Delta_{SEP} \neq \phi$. Assume that the sequence $\{\theta_n\}$ is non-decreasing such that $0 \leq \theta_n \leq \theta < 1$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < (\frac{1}{L})$, where $L := \|\mathsf{A}\|^2$ and the sequence $\{\alpha_n\}$ is non-decreasing such that*

$$\alpha \leq \alpha_n \leq \frac{1}{1 + \theta + \delta} \tag{4.79}$$

for some $\delta > 0$ and $\alpha > 0$.
Let $\{x_n\}$ be a sequence in H_1 defined by

$$\begin{cases} x_0, x_1 \in \mathsf{H}_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_{\mathcal{C}}(I + \lambda_n \mathsf{A}^*(P_{\mathcal{Q}} - I)\mathsf{A})w_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n. \end{cases}$$

Then, the sequence $\{x_n\}$ converges weakly to an element of Δ_{SEP}.

Theorem 4.2 *Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2 and $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ be a bounded linear operator and A^* be the adjoint of A.Let Δ_{SEP} be the solution set of problem (SEP) with $\Delta_{SEP} \neq \phi$. Assume that the sequence $\{\theta_n\}$ is sequence in $[0, \theta]$ for some $\theta > 0$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < (\frac{1}{L})$, where $L := \|\mathsf{A}\|^2$. Let $f : \mathsf{H}_1 \to \mathsf{H}_1$ be a contaction mapping with contraction parameter $\kappa \in [0, 1)$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty.$$

Let $\{x_n\}$ be a sequence in H_1 defined by

$$\begin{cases} x_0, x_1 \in \mathsf{H}_1, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_{\mathcal{C}}(I + \lambda_n \mathsf{A}^*(P_{\mathcal{Q}} - I)\mathsf{A})w_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n \end{cases} \tag{4.80}$$

Assume that the sequence $\{\theta_n\}$ is chosen such that satisfying the following condition

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0. \tag{4.81}$$

Then the sequence $\{x_n\}$ generated by (4.80) converges strongly to an element of $p \in \Delta_{SEP}$, where $p = P_{\Delta_{SEP}} \circ f(p)$."

5. Numerical illustration

In this section, we consider a numerical example to illustrate the convergence of the algorithms.

Example 5.1. Let $\mathcal{H} = \mathbb{R}$, the set of real numbers and $f : \mathbb{R} \to \mathbb{R}$ be contraction mapping and $\mathcal{M}_1, \mathcal{M}_2 : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued mappings. Let $f(x) := \frac{x}{10}$ and $\mathcal{M}_1 := \{\frac{x}{5}\}$ and $\mathcal{M}_2 := \{\frac{x}{7}\}\ \forall x \in \mathbb{R}$, $\alpha_n := \frac{1}{2n}$, $\lambda_n := \frac{1}{n+3}$ and $\theta_n := \frac{1}{(n+1)^2}$, then we calculate resolvent operator $\mathsf{J}_\lambda^{\mathcal{M}}$ and Cayley operator $\mathsf{C}_\lambda^{\mathcal{M}}$ for $\lambda = 1$ as follows:

$$\mathsf{J}_\lambda^{\mathcal{M}_1}(x) = (I + \lambda \mathcal{M}_1)^{-1}(x) = \frac{5x}{6},$$

$$\mathsf{C}_\lambda^{\mathcal{M}_1}(x) = (2\mathsf{J}_\lambda^{\mathcal{M}_1}(x) - I) = \frac{2x}{3},$$

$$\mathsf{J}_\lambda^{\mathsf{B}_1}(x) = \frac{15x}{28}.$$

$$\mathsf{J}_\lambda^{\mathcal{M}_2}(x) = (I + \lambda \mathcal{M}_2)^{-1}(x) = \frac{7x}{8},$$

$$\mathsf{C}_\lambda^{\mathcal{M}_2}(x) = (2\mathsf{J}_\lambda^{\mathcal{M}_2}(x) - I) = \frac{3x}{4},$$

$$\mathsf{J}_\lambda^{\mathsf{B}_2}(x) = \frac{-25x}{81}.$$

All the assumptions of Theorem 3.1 and Theorem 3.2 are satisfied and the iterative sequence $\{x_n\}$ generated in the above algorithms is converges weakly to $p = 0$ (shows in Fig.1) and converges strongly to $p = 0$ (shows in Fig.2) for different initial values.

6. Table

TABLE 1. Numerical results for two different initial values $x_0 = 1, x_1 = 2$ and $x_0 = -1, x_1 = -2$.

Number of iterations (n)	$\mathbf{x_n}$ $\mathbf{x_0 = -1, x_1 = -2}$	$\mathbf{x_n}$ $\mathbf{x_0 = 1, x_1 = 2}$
1.	-1.00000	1.00000
2.	-2.00000	2.000000
6.	0.35303	-0.003609
9.	0.017780	-0.017792
15.	0.001278	0.001938
19.	-0.000215	0.000224
23.	0.00003840	0.000130
27.	-0.000007	0.000054
31.	0.00000234	0.000002
34.	-0.0000000	0.000001
35.	0.0000000	0.000000

7. figures

All of the codes have been developed in MATLAB R2021a for simplicity. We've tried for different initial points and found that the sequence $\{x_n\}$converges to the solution of the problem in each case. Graphs of convergence is depicted in the Figures below.

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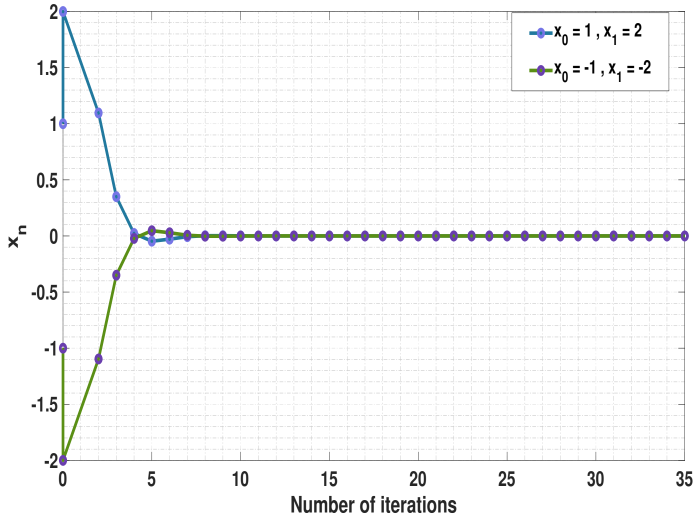

FIGURE 1. Weak convergence of sequence $\{x_n\}$ for initial values $x_0 = -1, x_1 = -2$ and $x_0 = 1, x_1 = 2$.

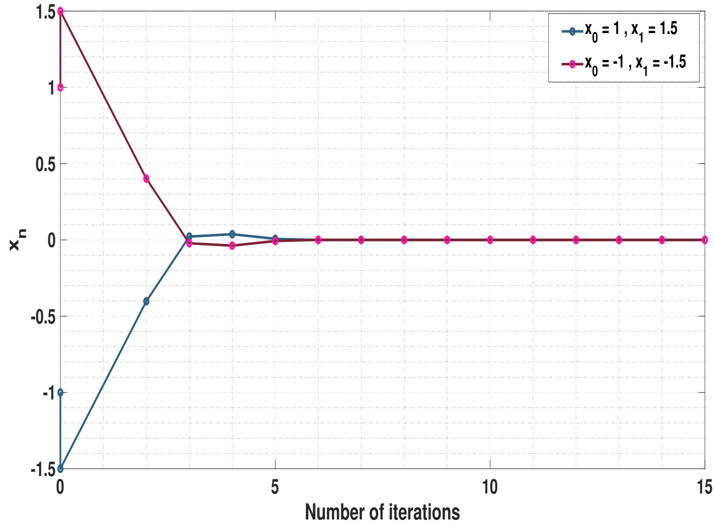

FIGURE 2. Strong convergence of sequence $\{x_n\}$ for initial values $x_0 = 1, x_1 = 1.5$ and $x_0 = -1, x_1 = -1.5$.

9. CONCLUSION

In this paper, we solved a split inclusion problem associated to Cayley's operator by using some classical approach of generating sequences and introduce two new algorithms in real Hilbert space. Also, weak and strong convergence theorems are established under standard assumptions imposed on operators, parameters and mappings. Finally, a numerical experiment has also been performed to illustrate the convergence of proposed algorithms.

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