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STATISTICALLY MONOTONIC SEQUENCES OF BI-COMPLEX NUMBERS

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Abstract. In this article, we have introduced the concepts of statistically monotonic sequences of bi-complex numbers using the two types of partial order relations on the set of bi-complex numbers and have established some results. We have also investigated the concept of statistically monotonic sequence, statistically order convergence and statistically relatively uniform convergence in a bi-complex Riesz space and have studied some of their properties.

1. Introduction

Bi-complex Numbers: Segre [21] endeavored to create special algebras in a novel way. As elements of an infinite set of algebras, he conceptualized the commutative generalization of complex numbers as bi-complex numbers, tricomplex numbers, etc. Price [17] went on to develop function theory and bi-complex algebra after that. Price introduced the multicomplex spaces and functions. Functional analysis in bi-complex numbers, a substantially new subject, is not only relevant from a mathematical point of view, but also has significant applications in physics and engineering. Alpay et al.[1] developed a general theory of functional analysis with bi-complex scalars.

Sequence spaces play a central role in many areas of mathematics. The most popular sequence spaces are the spaces ℓ_p which consist of absolutely p-summable complex sequences having a lot of useful applications. Since they also have rich topological and geometric properties, researchers are motivated to use them to obtain new results in different sequence spaces. Recent works noted in [28, 24, 19, 18] are some examples on topological properties of some sequence spaces. Sager and Sağir [19] introduced bi-complex sequence spaces with Euclidean norm in the set of bi-complex numbers. The sets of real, complex, and bi-complex numbers are denoted by $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$, respectively.
Segre [21] defined the bi-complex number as:

$$\varpi = u_1 + i_2 v_1 = x_1 + i_1 y_1 + i_2 z_1 + i_1 i_2 t_1, u_1, v_1 \in \mathbb{C}_1 \text{ and } x_1, y_1, z_1, t_1 \in \mathbb{C}_0;$$

where the independent imaginary units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$.
The three types of conjugations on $\mathbb{C}_2$ are
(i) $\varpi^* = \bar{u_1} + i_2 \bar{v_1}$,
(ii) $\bar{\varpi} = u_1 - i_2 v_1$,
(iii) $\varpi^{'} = \bar{u_1} + i_2 \bar{v_1}$.
In $\mathbb{C}_2$ there are four idempotent elements which are $0_2 (0_2 = 0 + 0i_1 + 0i_2 + 0i_1 i_2), 1 + 0i_1 + 0i_2 + 0i_1 i_2, e_1 = \frac{1+i_1 i_2}{2}$ and $e_2 = \frac{1-i_1 i_2}{2}$. Out of which, e_1 and e_2 are nontrivial with $e_1 + e_2 = 1$ and $e_1 e_2 = 0$.
then the bi-complex number $\varpi = u_1 + i_2 v_1$ is said to be singular if and only If $|u_1^2 + v_1^2| = 0$.

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The bi-complex number $\varpi = u_1 + i_2 v_1$ can be uniquely expressed as the combination of e_1 and e_2, as follows:

$$\varpi = u_1 + i_2 v_1 = (u_1 - i_1 v_1)e_1 + (u_1 + i_1 v_1)e_2 = \nu_1 e_1 + \nu_2 e_2,$$

where $\nu_1 = (u_1 - i_1 v_1)$ and $\nu_2 = (u_1 + i_1 v_1)$.
The Euclidean norm $\|\cdot\|_{\mathbb{C}_2}$ on $\mathbb{C}_2$ is defined by

$$\begin{aligned}\|\varpi\|_{\mathbb{C}_2} &= \sqrt{x_1^2 + y_1^2 + z_1^2 + t_1^2} \\ &= \sqrt{|u_1|^2 + |v_1|^2} \\ &= \sqrt{\frac{|\nu_1|^2 + |\nu_2|^2}{2}}.\end{aligned}$$

If $\varpi_1, \varpi_2 \in \mathbb{C}_2$ then $\|\varpi_1 \varpi_2\|_{\mathbb{C}_2} \leq \sqrt{2}\|\varpi_1\|_{\mathbb{C}_2}\|\varpi_2\|_{\mathbb{C}_2}$ holds. Therefore $\mathbb{C}_2$ is modified Banach algebra.

Definition 1.1. [2] The i_1-partial order relation $\preceq_{i_1}$ on $\mathbb{C}_1$ is defined as follows;
For $u_1, v_1 \in \mathbb{C}_1$, $u_1 \preceq_{i_1} v_1$ if and only if $Re(u_1) \leq Re(v_1)$ and $Im(u_1) \leq Im(v_1)$.

Definition 1.2. [7] The i_2-partial order relation $\preceq_{i_2}$ on $\mathbb{C}_2$ is defined as follows;
Let $\varpi_1, \varpi_2 \in \mathbb{C}_2$, where $\varpi_1 = u_1 + i_2 v_1$ and $\varpi_2 = u_1^* + i_2 v_1^*$;
$\varpi_1 \preceq_{i_2} \varpi_2$ if and only if $u_1 \preceq_{i_1} u_1^*$ and $v_1 \preceq_{i_1} v_1^*$, i.e., $\varpi_1 \preceq_{i_2} \varpi_2$ if one of the following conditions is satisfied:
(i) $u_1 = u_1^*, v_1 = v_1^*$,
(ii) $u_1 \prec_{i_1} u_1^*, v_1 = v_1^*$,
(iii) $u_1 = u_1^*, v_1 \prec_{i_1} v_1^*$ and
(iv) $u_1 \prec_{i_1} u_1^*, v_1 \prec_{i_1} v_1^*$.
Particularly if $\varpi_1 \preceq_{i_2} \varpi_2$ and $\varpi_1 \neq \varpi_2$ we will write $\varpi_1 \precnsim_{i_2} \varpi_2$, i.e. one of (ii), (iii) and (iv) is satisfied and we will write $\varpi_1 \prec_{i_2} \varpi_2$ if only (iv) is satisfied.
For any two bi-complex numbers $\varpi_1, \varpi_2 \in \mathbb{C}_2$ can easily be verified the followings:
(i) $\varpi_1 \preceq_{i_2} \varpi_2 \implies \|\varpi_1\|_{\mathbb{C}_2} \leq \|\varpi_2\|_{\mathbb{C}_2}$,
(ii) $\|\varpi_1 + \varpi_2\|_{\mathbb{C}_2} \leq \|\varpi_1\|_{\mathbb{C}_2} + \|\varpi_2\|_{\mathbb{C}_2}$.

Definition 1.3. [3] The Id-partial order relation $\preceq_{i_{Id}}$ on $\mathbb{C}_2$ is defined as follows;
Let $\varpi_1, \varpi_2 \in \mathbb{C}_2$, where $\varpi_1 = u_1 + i_2 v_1 = \nu_1 e_1 + \nu_2 e_2$ and $\varpi_2 = u_1^* + i_2 v_1^* = \nu_1^* e_1 + \nu_2^* e_2$;
$\varpi_1 \preceq_{i_{Id}} \varpi_2$ if and only if $\nu_1 \preceq_{i_1} \nu_1^*$, $\nu_2 \preceq_{i_1} \nu_2^*$ on $\mathbb{C}_1$. i.e., $\varpi_1 \preceq_{i_{Id}} \varpi_2$ if one of the following conditions is satisfied:
(i) $\nu_1 = \nu_1^*, \nu_2 = \nu_2^*$,
(ii) $\nu_1 \prec_{i_1} \nu_1^*, \nu_2 = \nu_2^*$,
(iii) $\nu_1 = \nu_1^*, \nu_2 \prec_{i_1} \nu_2^*$ and
(iv) $\nu_1 \prec_{i_1} \nu_1^*, \nu_2 \prec_{i_1} \nu_2^*$.
Particularly if $\varpi_1 \preceq_{i_{Id}} \varpi_2$ and $\varpi_1 \neq \varpi_2$, we will write $\varpi_1 \precnsim_{i_{Id}} \varpi_2$ i.e. one of (ii), (iii) and (iv) is satisfied and we will write $\varpi_1 \prec_{i_{Id}} \varpi_2$ if only (iv) is satisfied.
For any two bi-complex numbers $\varpi_1, \varpi_2 \in \mathbb{C}_2$ we can verify the followings:
(i) $\varpi_1 \preceq_{i_{Id}} \varpi_2 \implies \|\varpi_1\|_{\mathbb{C}_2} \leq \|\varpi_2\|_{\mathbb{C}_2}$.

Statistical Convergence: Nowadays, the theory of statistical convergence is an active area of research. Fast [14], Buck [5] and Schoenberg [20] independently defined statistical convergence as a generalization of ordinary convergence for real or complex sequences.

After that it has been studied by many researchers like Fridy[15], Salat [22] Tripathy[27]. Recently Das, Tripathy [8] studied on λ^2-statistical convergengence on complex uncertain sequences, Das et. al [9, 10, 11] studied statistically convergent on different aspect. Nath et al. [13] studied on statistically pre-Cauchy sequences of complex uncertain variables and Bera, Tripathy [3, 4] studied on statistical convergence in bi-complex valued metric space, statistical bounded sequences of bi-complex numbers. Statistical convergence has been discussed over several abstract spaces, including Banach spaces [6], complex uncertain spaces, and fuzzy number spaces etc.

A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function on E. A sequence of bi-complex number (ϖ_k) is said to be statistically convergent to $\varpi \in \mathbb{C}_2$ if for every $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N} : \|\varpi_k - \varpi\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

We write as $stat - \lim \varpi_k = \varpi$. If $\varpi = 0_2$, then (ϖ_k) is statistically null sequence. A sequence of bi-complex number (ϖ_k) is said to be statistically bounded if there exists $M > 0$ such that

$$\delta(\{k \in \mathbb{N} : \|\varpi_k\|_{\mathbb{C}_2} \geq M\}) = 0.$$

In this context, we focus to establish basic definitions and achieve basic results. Some research has been conducted on statistically convergent sequences of bi-complex numbers. This motivated us to investigate the statistical monotonicity of sequences of bi-complex numbers. Throughout this article we use two types of partial order relations on $\mathbb{C}_2$. Since there are no connection between this two types of relations, although we get almost same results over these two types of partial orders.

Lemma 1.4. [27] *(i) Let $(\varpi_k) \in \omega^*$, then (ϖ_k) is said to be statistically convergent to ϖ if and only if there exists a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and (ϖ_{k_n}) is convergent to ϖ.*
(ii) A sequence (ϖ_k) is said to be statistically bounded if there exists a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and (ϖ_{k_n}) is bounded.

2. Statistical Monotonicity of a Sequence of Bi-complex Numbers

Tripathy[25] rectified the notion of statistical monotone sequence introduced by Fridy in [15] for real sequences. In this section we shall extend it to sequence of bi-complex numbers using two types of partial orders.

Definition 2.1. A sequence (ϖ_k) of bi-complex numbers is said to be statistically monotonic increasing w.r.t. i_2-partial order if there exists a subset $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and $\varpi_{k_n} \preceq_{i_2} \varpi_{k_{n+1}}$, for all $n \in \mathbb{N}$. In this case, we write $\varpi_n \uparrow_{i_2}$
Similarly we can define statistical monotonic decreasing w.r.t. i_2-partial order. In this case, we write $\varpi_n \downarrow_{i_2}$. A sequence (ϖ_k) of bi-complex numbers is said to be statistical monotone w.r.t. i_2-partial order if it is either $\varpi_n \uparrow_{i_2}$ or $\varpi_n \downarrow_{i_2}$.

Definition 2.2. A sequence (ϖ_k) of bi-complex numbers is said to be statistically monotonic increasing w.r.t. i_{Id}-partial order if there exists a subset $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and $\varpi_{k_n} \preceq_{i_{Id}} \varpi_{k_{n+1}}$, for all $n \in \mathbb{N}$. In this case, we write $\varpi_n \uparrow_{i_{Id}}$.
Similarly we can define statistical monotonic decreasing w.r.t. i_{Id}-partial order. In this case, we write $\varpi_n \downarrow_{i_{Id}}$ respectively. A sequence (ϖ_k) of bi-complex numbers is said to be statistical monotone w.r.t. i_{Id}-partial order if it is either $\varpi_n \uparrow_{i_{Id}}$ or $\varpi_n \downarrow_{i_{Id}}$.

Theorem 2.3. *If a sequence (ϖ_k) of bi-complex numbers is monotonic, then it is statistically monotonic w.r.t. i_2(or i_{Id})-partial order. But the converse part is not true in general.*

Proof. Let (ϖ_k) be a monotonic increasing w.r.t i_2-partial order, then $\varpi_{k_n} \preceq_{i_{Id}} \varpi_{k_{n+1}}$, for all $n \in \mathbb{N}$. Since $\delta(N) = 1$, so (ϖ_k) is a statistically monotonic increasing w.r.t i_2-partial order. Similarly, if (ϖ_k) is a monotonic decreasing w.r.t i_2-partial order, then we can easily varified that (ϖ_k) is a statistically monotonic decreasing w.r.t i_2-partial order. Hence,(ϖ_k) is a statistically monotonic increasing w.r.t.i_2-partial order.
In similar way it can be proved for i_{Id}-partial order. □

Example 2.4. Consider a sequence of bi-complex numbers (ϖ_k) defined by

$$\varpi_k = \begin{cases} 1 + i_1 + i_2 + i_1 i_2, & \text{for } k = i^2, i \in \mathbb{N}; \\ k + i_1 k + i_2 k + i_1 i_2 k, & \textit{otherwise}. \end{cases}$$

Let $S = \{k : k \neq i^2, i \in \mathbb{N}\}$, then $\delta(S) = 1$ and $\varpi_{k_n} \preceq_{i_2} \varpi_{k_{n+1}}, n \in \mathbb{N}$. Therefore (ϖ_k) is statistically monotone with respect to i_2-partial order, although (ϖ_k) is not monotone with respect to i_2-partial order as $\varpi_4 \preceq_{i_2} \varpi_2$.

Theorem 2.5. *Let $(\varpi_k) \in \omega^*$. If*
(i) (ϖ_k) is statistically monotonic increasing w.r.t. i_2-partial order then

$$\delta(\{k \in \mathbb{N} : \varpi_{k+1} \preceq_{i_2} \varpi_k\}) = 0.$$

(ii) (ϖ_k) is statistically monotonic decreasing w.r.t. i_2-partial order then

$$\delta(\{k \in \mathbb{N} : \varpi_k \preceq_{i_2} \varpi_{k+1}\}) = 0.$$

Proof. (i) Since, (ϖ_k) is statistically monotonic increasing w.r.t. i_2-partial order then there exists such a set $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\} \subset \mathbb{N}$ such that $\delta(S) = 1$ and $\varpi_{k_n} \preceq_{i_2} \varpi_{k_{n+1}}$, for all $n \in \mathbb{N}$.
Now, $\{k \in K : \varpi_k \preceq_{i_2} \varpi_{k+1}\} \subseteq \{k \in \mathbb{N} : \varpi_k \preceq_{i_2} \varpi_{k+1}\}$.
then $\delta(\{k \in \mathbb{N} : \varpi_k \preceq_{i_2} \varpi_{k+1}\}) = 1$. Which implies $\delta(\{k \in \mathbb{N} : \varpi_{k+1} \preceq_{i_2} \varpi_k\}) = 0$.
Similarly, we can prove the second part. □

Remark 2.6. The converse of the previously mentioned theorem is not true in general.

Example 2.7. Consider a sequence (ϖ_k) of bi-complex numbers defined by

$$\varpi_k = \begin{cases} e_1 + e_2, & \text{for } 2^i \leq k < 2^{i+1} - 1, i = 2n, n \in \mathbb{N}; \\ e_1 e_2, & \textit{otherwise}. \end{cases}$$

Then, $\delta(\{k \in \mathbb{N} : \varpi_{k+1} \preceq_{i_2} \varpi_k\}) = \delta(\{4, 16, 64, ...\}) = 0$, but there exists no subset $S = \{k_1 < k_2 < ..\}$ such that $\delta(S) = 1$ and (ϖ_{k_n}) is monotonic increasing w.r.t i_2.

Theorem 2.8. *Let $(\varpi_k) \in \omega^*$. If*
(i) (ϖ_k) is statistically monotonic increasing w.r.t. i_{Id}-partial order, then

$$\delta(\{k \in \mathbb{N} : \varpi_{k+1} \preceq_{i_{Id}} \varpi_k\}) = 0.$$

(ii) (ϖ_k) is statistically monotonic decreasing w.r.t. i_{Id}-partial order, then

$$\delta(\{k \in \mathbb{N} : \varpi_k \preceq_{i_{Id}} \varpi_{k+1}\}) = 0.$$

Theorem 2.9. *A statistically convergent sequence of bi-complex numbers which is s.m.i monotone w.r.t. either i_2-partial order or i_{Id}-partial order if and only if it is statistically bounded*

Proof. Necessity: We can easily prove the necessary condition.
Sufficiency:
Case-I: Consider the sequence of bi-complex numbers (ϖ_k) which is statistical monotonic increasing w.r.t. i_2-partial order, then there exists a subset $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\} \subset \mathbb{N}$ such that $\delta(S) = 1$ and $\varpi_{k_n} \preceq_{i_2} \varpi_{k_{n+1}}$, for all $n \in \mathbb{N}$, then $\|\varpi_{k_n}\|_{\mathbb{C}_2} \leq \|\varpi_{k_{n+1}}\|_{\mathbb{C}_2}$. Let (ϖ_k) be a statistically bounded sequences of bi-complex numbers. Then by lemma1.4 there exists a subsets $A = \{a_1 < a_2 < ...\} \subset \mathbb{N}$ such that

$\delta(A) = 1$ and (ϖ_{a_n}) is bounded. Let $B = K \cap A$, then $\delta(B) = 1$. Hence, (ϖ_{b_n}) is bounded and increasing w.r.t. i_2-partial order and hence, (ϖ_{b_n}) is convergent and therefore, (ϖ_k) it is statistically convergent.
Case-II: Consider the sequence of bi-complex numbers (ϖ_k) which is statistical monotonic increasing w.r.t. i_{Id}-partial order. then there exists a subset $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and $\varpi_{k_n} \preceq_{i_{Id}} \varpi_{k_{n+1}}$, for all $n \in \mathbb{N}$, then $\|\varpi_{k_n}\|_{\mathbb{C}_2} \leq \|\varpi_{k_{n+1}}\|_{\mathbb{C}_2}$. We can prove it following simmilar method. □

The decomposition theorem for statistically monotonic sequences of bi-complex numbers is as follows;

Theorem 2.10. *Let (ϖ_k) be $\varpi_n \uparrow_{i_2}$(or $\varpi_n \uparrow_{i_{Id}}$). Then we can write*

$$\varpi_k = \eta_k + \zeta_k, \forall k \in \mathbb{N},$$

where (η_k) is monotonic increasing w.r.t. i_2-parial order(i_{Id}-partial order) and $\delta(\{k : \zeta_k \neq 0_2\}) = 0$. If (η_k) is bounded then (ϖ_k) is statistically convergent and $st - \lim_{k\to\infty} \varpi_k = \lim_{k\to\infty} \eta_k$.

Proof. Let (ϖ_k) be a statistically convergent, then it is statistically bounded, then $\delta(B) = 0$, where $B = \{k : \|\varpi_k\|_{\mathbb{C}_2} \geq M\}$.
Define the sequences $\eta = (\eta_k)$ and $\zeta = (\zeta_k)$ as follows:

$$\eta_k = \begin{cases} \varpi_k, & if\ k \in B^c; \\ e_1 e_2, & otherwise. \end{cases}$$

$$\zeta_k = \begin{cases} e_1 e_2, & if\ k \in B^c; \\ \varpi_k, & otherwise. \end{cases}$$

From the above construction, we have

$$\varpi_k = \eta_k + \zeta_k, \forall k \in \mathbb{N}.$$

□

3. Statistical monotonicity and statistical order convergence in BC Riesz space

In this section we invesitigate statistical monotonicity, statist ical order convergence and statistical relatuvely uniform convergence in BC Riesz space.
A bi-complex valued vector space [12] E with a partial order $\preceq_{i_2}$(or, i_{Id}) is called an ordered vector space if E is partially ordered such that
(i) $\varpi \preceq_{i_2}$(or, i_{Id})η implies $\varpi + \varpi \preceq_{i_2}(or, i_{Id})\eta + \varpi$, for every $\varpi \in E$,
(ii) $\varpi \preceq_{i_2}$(or, i_{Id})0_2 implies $\alpha\varpi \preceq_{i_2}$(or, i_{Id})0_2, for every $0 \leq \alpha \in C_0$, where $0_2 = 0 + 0i_1 + 0i_2 + 0i_1i_2$. If E additionally represents a lattice with respect to of partial ordering, then it is referred to as a BC Riesz space or a vector lattice. We denote BC Riesz spaces with a partial order i_2 and i_{Id} by $\mathbb{BR}_{i_2}$ space and $\mathbb{BR}_{i_{Id}}$ space respectively. Let E be a $\mathbb{BR}_{i_2}(\mathbb{BR}_{i_{Id}})$ space, then E is called Archimedean if $\varpi, \eta \in E$ and $n\varpi \preceq_{i_2} \eta (n\varpi \preceq_{i_{Id}} \eta)$ for $n \in \mathbb{N}$ imply $\varpi \preceq_{i_2} 0_2$ $(\varpi \preceq_{i_{Id}} 0_2)$.
Let E be a $\mathbb{BR}_{i_2}$ space. If the sequence of bi-complex numbers (ϖ_k) is increasing (decreasing) w.r.t. i_2-partial order we write $\varpi_k \Uparrow_{i_2}$ $(\varpi_k \Downarrow_{i_2})$. If $\sup_k \varpi_k = \varpi(\inf_k \varpi_k = \varpi)$ exists, then we write $\varpi_k \Uparrow_{i_2} \varpi(\varpi_k \Uparrow_{i_2} \varpi)$. Similarly if E be a $\mathbb{BR}_{i_{Id}}$ space then we write $\varpi_k \Uparrow_{i_{Id}} \varpi(\varpi_k \Downarrow_{i_{Id}} \varpi)$.

Definition 3.1. Let E be a $\mathbb{BR}_{i_2}$ space. A sequence (ϖ_k) of bi-complex numbers is said to be statistically monotonic increasing w.r.t. i_2-partial order(in short s.m.i.i_2) if there exists a subset $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and (ϖ_{k_n}) is monotonic w.r.t. i_2-partial order.

Similarly one can define statistical monotonic decreasing w.r.t. i_2-partial order(in short s.m.d.i_2). We denote, s.m.i.i_2 and s.m.d.i_2 in $\mathbb{BR}_{i_2}$ space by $\varpi_n \Uparrow^{st}_{i_2}$ and $\varpi_n \Downarrow^{st}_{i_2}$ respectively.

Definition 3.2. Let E be a $\mathbb{BR}_{i_{Id}}$ space. A sequence (ϖ_k) of bi-complex numbers is said to be statistically monotonic w.r.t. i_{Id}-partial order(in short s.m.i.i_{Id}) if there exists a subset $S = \{k_1 < k_2 < k_3 < ... < k_n < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$ and (ϖ_{k_n}) is monotonic w.r.t. i_{Id}-partial order. Similarly one can define statistical monotonic decreasing w.r.t. i_{Id}-partial order(in short s.m.d.i_{Id}). We denote, s.m.i.i_{Id} and s.m.d.i_{Id} in $\mathbb{BR}_{i_{Id}}$ by $\varpi_n \Uparrow^{st}_{i_{Id}}$ and $\varpi_n \Downarrow^{st}_{i_{Id}}$ respectively.

Definition 3.3. Let E be a $\mathbb{BR}_{i_2}$ space and (ϖ_k) be a sequence in E. If there exists a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$, and (ϖ_{n_k}) is monotone increasing w.r.t i_2- partial order and $\sup_{k\in\mathbb{N}} \varpi_{n_k} = \varpi$, for some $\varpi \in E$, then we write, $\varpi_n \Uparrow^{st}_{i_2} \varpi$ and if E be a $\mathbb{BR}_{i_{Id}}$ space then we write, $\varpi_n \Uparrow^{st}_{i_{Id}} \varpi$.

Definition 3.4. Let E be a $\mathbb{BR}_{i_2}$ space and (ϖ_k) be a sequence in E. If there exists a subset $K = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(K) = 1$, and (ϖ_{n_k}) is monotone decreasing w.r.t i_2- partial order and $\inf_{k\in\mathbb{N}} \varpi_{n_k} = \varpi$, for some $\varpi \in E$, then we write, $\varpi_n \Downarrow^{st}_{i_2} \varpi$ and if E be a $\mathbb{BR}_{i_{Id}}$ space then we write, $\varpi_n \Downarrow^{st}_{i_{Id}} \varpi$.

Theorem 3.5. *Let E be a $\mathbb{BR}_{i_2}$ space. If $\varpi_n \Downarrow^{st}_{i_2} \varpi$ then for every subsequence (ϖ_{k_n}) such that $\delta\{k_1 < k_2 < ...\} = 1$ and (ϖ_{k_n}) is decreasing then $\inf_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$. Similarly if $\varpi_n \Uparrow^{st}_{i_2} \varpi$ then for every subsequence (ϖ_{k_n}) such that $\delta\{k_1 < k_2 < ...\} = 1$ and (ϖ_{k_n}) is increasing then $\sup_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$.*

Proof. Consider $\varpi_n \Uparrow^{st}_{i_2} \varpi$, then there exists a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$, (ϖ_{k_n}) is decreasing w.r.t. i_2-partial order and $\inf_{k\in\mathbb{N}} \varpi_{k_n} = \varpi$. Let $P = \{p_1 < p_2 < ...\}$ of $\mathbb{N}$ such that $S \neq P, \delta(P) = 1$ and (ϖ_{k_n}) is decreasing w.r.t. i_2-partial order. We show that $\inf_{k\in\mathbb{N}} \varpi_{p_k} = \varpi$. We have $\varpi \preceq_{i_2} \varpi_{k_i}$, for every $k \in \mathbb{N}$. As $\delta(S \cap P) = 1$, so the subsets S and P are almost equal. Then, we have found an $k_n \in K$ and $p_r \in P$ such that $k_n = p_r$ and $\varpi_{k_n} = \varpi_{p_r}$. Hence, $\varpi = \varpi_{k_n} = \varpi_{p_r} \preceq_{i_2} \varpi_{p_{r-1}} \preceq_{i_2} ... \preceq_{i_2} \varpi_{p_1}$. Hence, $\varpi \in E$ is a lower bound of (ϖ_{k_n}). Let us suppose that $\eta \in E$ is another lower bound of (ϖ_{k_n}). Let $\eta \preceq_{i_2} \varpi_{p_n}$, for every $k \in \mathbb{N}$. Then we have found an $\varpi_{k_{n_1}}$ such that $\eta \preceq_{i_2} \varpi_{k_{n_1}} = \varpi_{m_k}$ for some $k \in \mathbb{N}$. Continuing in this way, we may generate a subsequence $\varpi_{k_{n_1}}, \varpi_{k_{n_2}}, ...$ of (ϖ_{n_k}) such that η is a lower bound of $(\varpi_{k_{n_p}})_{p\in\mathbb{N}}$. Since, $\varpi_n \Downarrow^{st}_{i_2} \varpi$, so ϖ is the infimum of every subsequence of (ϖ_{k_n}). Hence, $\eta \preceq_{i_2} \varpi$. Since, η is arbitrary, it follows that $\inf_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$.

In similar way we can established the part. □

Lemma 3.6. *Let E be a $\mathbb{BR}_{i_{Id}}$ space. If $\varpi_n \Downarrow^{st}_{i_{Id}} \varpi$ then for every subsequence (ϖ_{k_n}) such that $\delta\{k_1 < k_2 < ...\} = 1$ and (ϖ_{k_n}) is decreasing then $\inf_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$. Similarly if $\varpi_n \Uparrow^{st}_{i_{Id}} \varpi$ then for every subsequence (ϖ_{k_n}) such that $\delta\{k_1 < k_2 < ...\} = 1$ and (ϖ_{k_n}) is increasing then $\sup_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$.*

Theorem 3.7.
(i) If $\varpi_n \Uparrow^{st}_{i_2} \varpi$ and $\gamma \succeq_{i_2} 0_2$, then $\gamma\varpi_n \Uparrow^{st}_{i_2} \gamma\varpi$. Similarly, for a s.m.d.i_2 sequence.
(ii) If $\varpi_{k_n} \Downarrow^{st}_{i_2} 0_2$ and $\eta_{k_n} \Downarrow^{st}_{i_2} 0_2$, then $(\varpi_{k_n} + \eta_{k_n}) \Downarrow^{st}_{i_2} 0_2$.
(iii) If there exists a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1, 0_2 \preceq_{i_2} \eta_k \preceq_{i_2} \varpi_k$ for every $k \in K, \eta_k \Downarrow^{st}_{i_2} 0_2$ and (η_k) is s.m.d.i_2, then $\varpi_k \Downarrow^{st}_{i_2} 0_2$. If (η_k) is s.m.i.i_2, then $\eta_k = 0_2$ for a.a.k.

Proof. (i) Let, $\varpi_n \Uparrow_{i_2}^{st} \varpi$, then there exists a subset $K = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(K) = 1$, and (ϖ_{n_k}) is monotone increasing w.r.t i_2- partial order and $\sup\limits_{k\in\mathbb{N}} \varpi_{n_k} = \varpi$, for some $\varpi \in E$. Since, $\gamma \succeq_{i_2} 0$ so, $\sup\limits_{k\in\mathbb{N}} \gamma\varpi_{n_k} = \gamma\varpi$.

(ii) Let $\varpi_{k_n} \Downarrow_{i_2}^{st} 0_2$ and $\eta_{k_n} \Downarrow_{i_2}^{st} 0_2$, then there exists a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1$, $\inf\limits_{n\in\mathbb{N}} \varpi_{k_n} = 0_2$ and (ϖ_{k_n}) is decreasing w.r.t. i_2. Also there exists a set $P = \{p_1 < p_2 < ...\} \subset \mathbb{N}$ such that $\delta(P) = 1$, $\inf\limits_{n\in\mathbb{N}} \eta_{k_n} = 0_2$ and (η_{k_n}) is decreasing w.r.t. i_2. Now, $\delta(K \cap M) = 1$. Let $S \cap P = \{p_1 < p_2 < ...\}$. Hence, the sequence $(\varpi_{p_k} + \eta_{p_k})$ is decreasing w.r.t. i_2 and for every $k \in \mathbb{N}, (\varpi_{p_k} + \eta_{p_k}) \succeq_{i_2} 0_2$. Let ϖ be a lower bound of this sequence such that $\varpi \neq 0_2$. Then, $\varpi_{p_k} + \eta_{p_k} \preceq_{i_2} \varpi_l + \eta_l$ for $p_k \leq l \in K \cap M$. Thus $\varpi \preceq_{i_2} \inf\limits_{k\in\mathbb{N}} (\varpi_l + \eta_{p_k}) = \varpi_l$. Which implies $\varpi \preceq_{i_2} \inf\limits_{l\in K\cap M} \varpi_l = 0_2$. Thus, $\inf\limits_{k\in\mathbb{N}} (\varpi_{p_k} + \eta_{p_k}) = 0_2$, which shows that $(\varpi_{k_n} + \eta_{k_n}) \Downarrow_{i_2}^{st} 0_2$.

(iii) Consider two sets $L = \{l_1 < l_2 < ...\}$ and $M = \{m_1 < m_2 < ...\}$ of $\mathbb{N}$ such that $\delta(L) = \delta(M) = 1$, $\inf\limits_{n\in\mathbb{N}} \varpi_{l_n} = 0_2$, Also, (η_{l_n}) and (ϖ_{m_n}) are decreasing w.r.t i_2. Now consider the set $W = K \cap L \cap M$, then $\delta(W) = 1$. Then, $0_2 \preceq_{i_2} \varpi_w \preceq_{i_2} \eta_w, \eta_w \Downarrow_{i_2}^{st} 0_2$, for all $w \in W$. and (ϖ_w) is decreasing w.r.t. i_2. Hence, $\eta_w \Downarrow_{i_2}^{st} 0_2$. Now let (ϖ_k) is s.m.i.i_2, then there exists a set $S = \{s_1 < s_2 < ...\} \subset \mathbb{N}$ with $\delta(S) = 1$ such that ϖ_{s_n} is increasing w.r.t. i_2. Then we have $\delta(K \cap S \cap L) = 1$ and $0_2 \preceq_{i_2} \varpi_j \preceq_{i_2} \varpi_n \preceq_{i_2} \eta_n, \eta_n \Downarrow_{i_2}^{st} 0_2$, for every $j, n \in K \cap S \cap L$ with $j \leq n$. Thus, $\varpi_n = 0_2$, for every $n \in K \cap S \cap L$, which shows $\varpi_k = 0_2$, for a.a.k. □

Theorem 3.8. *Let $\varpi_k = u_{1k} + i_2 v_{1k}$, for all $k \in \mathbb{N}, \varpi = u_1 + i_2 v_1$.*
(i) If $\varpi_k \Uparrow_{i_2}^{st} \varpi$, then $u_{1k} \Uparrow_{i_1}^{st} u_1$ and $v_{1k} \Uparrow_{i_1}^{st} v_1$.
(ii) If $\varpi_k \Downarrow_{i_2}^{st} \varpi$, then $u_{1k} \Downarrow_{i_1}^{st} u_1$ and $v_{1k} \Downarrow_{i_1}^{st} v_1$.

Proof. (i) Let us assume that $\varpi_k \Uparrow_{i_2}^{st} \varpi$ and let $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ such that $\delta(S) = 1, \sup\limits_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$ and (ϖ_{k_n}) is increasing w.r.t.i_2.
Now, $\sup\limits_{n\in\mathbb{N}} \varpi_{k_n} = \varpi$
$\implies \sup\limits_{n\in\mathbb{N}} u_{1k_n} + i_2 \sup\limits_{n\in\mathbb{N}} v_{2k_n} = u_1 + i_2 v_1$.
Which implies that $\sup\limits_{n\in\mathbb{N}} u_{1k_n} = u_1$ and $\sup\limits_{n\in\mathbb{N}} v_{1k_n} = v_1$. Since, (ϖ_{k_n}) is increasing w.r.t.i_2, then from Definition1.2, the sequences (u_{1k_n}) and (v_{1k_n}) are increasing w.r.t.i_1.
Hence, the theorem is prove. □

Theorem 3.9. *Let $\varpi_k = \mu_{1k} e_1 + \mu_{2k} e_2$, for all $k \in \mathbb{N}, z = \nu_1 e_1 + \nu_2 e_2$.*
(i) If $\varpi_n \Uparrow_{i_{Id}}^{st} \varpi$, then $\mu_{1n} \Uparrow_{i_1}^{st} \nu_1$ and $\mu_{2n} \Uparrow_{i_1}^{st} \nu_2$.
(ii) If $\varpi_n \Downarrow_{i_{Id}}^{st} \varpi$, then $\mu_{1n} \Downarrow_{i_1}^{st} \nu_1$ and $\mu_{2n} \Downarrow_{i_1}^{st} \nu_2$.

Proof. The proof is straightforward. □

3.1. **Statistical order convergence and statistical relative uniform convergence.** Let E be $\mathbb{BR}_{i_2}$(or $\mathbb{BR}_{i_{Id}}$) space and $m \in E$. A sequence of bi-complex numbers (ϖ_k) in E is said to converge m-uniformly to $\varpi \in E$ whenever, for every $\varepsilon > 0$, there exists a natural number n_0 such that $\|\varpi_k - \varpi\|_{\mathbb{C}_2} \leq \varepsilon \|m\|_{\mathbb{C}_2}$, for all $K \geq n_0$. We write $\varpi_k \to \varpi(m)$. A sequence of bi-complex numbers (ϖ_k) in E is said to relatively uniformly converges to $\varpi \in E$ if $\varpi_k \to \varpi(m)$. We write $\varpi_k \xrightarrow{ru} \varpi$. .

Definition 3.10. Let E be $\mathbb{BR}_{i_2}$(or $\mathbb{BR}_{i_{Id}}$) space and $m \in E$. A sequence (ϖ_k) in E is said to be statistically m-uniformly convergent to $\varpi \in E$ if

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k : k \leq n, \left(\|\varpi_k - \varpi\|_{\mathbb{C}_2} - \varepsilon \|m\|_{\mathbb{C}_2} \right) > 0 \right\} \right| = 0, \text{ for each } \varepsilon > 0.$$

We write $\varpi_k \xrightarrow{st} \varpi(m)$. A sequence (ϖ_k) in E is said to be statistically relatively uniformly convergent to $\varpi \in E$ if $\varpi_k \xrightarrow{st} \varpi(m)$, for some $m \in E$. We write $\varpi_k \xrightarrow{stru} \varpi$.

Lemma 3.11. *Let E be an Archemedian $\mathbb{BR}_{i_2}$(or $\mathbb{BR}_{i_{Id}}$) space and $m \in E$. Then the following statements are equivalent;*
(i) $\varpi_k \xrightarrow{st} \varpi(m)$;
(ii)There is a subset $S = \{k_1 < k_2, ...\}$ of $\mathbb{N}$ with $\delta(S) = 1$ such that $\varpi_k \xrightarrow{ru} \varpi(m)$;
(iii) There is a sequence (η_k) in E such that $\varpi_k = \eta_k$ a.a.k and $\eta_k \xrightarrow{ru} \varpi(m)$.

Let E be a $\mathbb{BR}_{i_2}$(or $\mathbb{BR}_{i_{Id}}$) space. A sequence of bi-complex numbers (ϖ_k) in E is said to be order convergent to $\varpi \in E$, if there exists a sequence $\eta_k \Downarrow_{i_2} 0(\eta_k \Downarrow_{i_{Id}} 0)$ in E such that $\|\varpi_k - \varpi\|_{\mathbb{C}_2} \le \|\eta_k\|_{\mathbb{C}_2}$

Definition 3.12. Let E be $\mathbb{BR}_{i_2}$ space. A sequence (ϖ_k) in E is statistical i_2-order convergent to $\varpi \in E$ that there exists a sequence (ϖ_k) such that $\varpi_k \Uparrow^{st}_{i_2} 0_2$ and a subset $S = \{k_1 < k_2 < ...\} \subset \mathbb{N}$ with $\delta(S) = 1$ such that $\|\varpi_k - \varpi\|_{\mathbb{C}_2} \le \|\varpi_k\|_{\mathbb{C}_2}$, for every $k \in K$ and we write $\varpi_k \xrightarrow{st-ord} [i_2]\varpi$

Remark 3.13. $\varpi_k \xrightarrow{st-ord} [i_2]\varpi$ if and only if there is a sequence (ϖ_k) in E with $\varpi_k \Downarrow_{i_2} 0_2$ such that $\|\varpi_k - \varpi\|_{\mathbb{C}_2} \le \|\varpi_k\|_{\mathbb{C}_2}$ a.a.k.

Remark 3.14. $\varpi_k \xrightarrow{st-ord} [i_{Id}]\varpi$ if and only if there is a sequence (ϖ_k) in E with $\varpi_k \Downarrow_{i_{Id}} 0_2$ then $\|\varpi_k - \varpi\|_{\mathbb{C}_2} \le \|\varpi_k\|_{\mathbb{C}_2}$ a.a.k.

Definition 3.15. Let E be $\mathbb{BR}_{i_{Id}}$ space. A sequence (ϖ_k) in E is statistically i_{Id}-order convergent to $\varpi \in E$ provided there exists a sequence (ϖ_k) such that $\varpi_k \Downarrow^{st}_{i_{Id}} 0_2$ and a subset $S = \{k_1 < k_2 < ...\} \subset \mathbb{N}$ with $\delta(S) = 1$ such that $\|\varpi_k - \varpi\|_{\mathbb{C}_2} \le \|\varpi_k\|_{\mathbb{C}_2}$, for every $k \in S$ and we write $\varpi_k \xrightarrow{st-ord} [i_{Id}]\varpi$.

Theorem 3.16. *If $\varpi_k \xrightarrow{st-ord} [i_2]\varpi$ if and only if $u_{1k} \xrightarrow{st-ord} [i_1]u_1$ and $v_{1k} \xrightarrow{st-ord} [i_1]v_1$.*

Proof. Let $\varpi_k \xrightarrow{st-ord} [i_2]\varpi$, then there exists a sequence (ϖ_k) such that $\varpi_k \Downarrow^{st}_{i_2} 0_2$ and a subset $S = \{k_1 < k_2 < ...\}$ with $\delta(S) = 1$ such that

$$\|\varpi_k - \varpi\|_{\mathbb{C}_2} \le \|\varpi_k\|_{\mathbb{C}_2}. \tag{3.1}$$

Let $\varpi_k = u_{1k} + i_2 v_{1k}, k \in \mathbb{N}$ and $\varpi_k = z'_{1k} + i_2 z'_{2k}, k \in \mathbb{N}$. Then from Theorem3.8, we have $z'_{1k} \Downarrow^{st}_{i_1} 0_1$ and $z'_{2k} \Downarrow^{st}_{i_1} 0_1$. Taking $v_{1k} = 0_1, \forall k \in \mathbb{N}$ and from (3.1) we have

$$|u_{1k} - u_1| \le \|\varpi_k - \varpi\|_{\mathbb{C}_2} \le |z'_{1k}|.$$

Hence, $u_{1k} \xrightarrow{st-ord} [i_1]u_1$. Similarly we can proved that $v_{1k} \xrightarrow{st-ord} [i_1]v_1$. Converse part can be prove in general way so we skip the converse part. □

Theorem 3.17. *If $\varpi_k \xrightarrow{st-ord} [i_{Id}]\varpi$ if and only if $\mu_{1k} \xrightarrow{st-ord} [i_1]\nu_1$ and $\mu_{2k} \xrightarrow{st-ord} [i_1]\nu_2$.*

Proof. The proof is straightforward. □

Theorem 3.18. *Let (ϖ_k) be a sequence in E and $\varpi, m \in E$. Then $\varpi_k \xrightarrow{st} \varpi(m)$ if and only if $\varpi_k \xrightarrow{st-ord} [i_2]\varpi$ and (ϖ_k) is a statistically m-uniformly Cauchy sequence.*

Proof. We can prove the necessity part easily so we have to prove only sufficient part. Let us assume that $\varpi_k \xrightarrow{st-ord} [i_2]\varpi$ and (ϖ_k) is a statistically m-uniformly Cauchy sequence. We get a subset $S = \{k_1 < k_2 < ...\}$ of $\mathbb{N}$ with $\delta(S) = 1$ such that the sequence

(ϖ_{k_n}) convergence in order to ϖ and is m-uniform Cauchy. Let $\varepsilon > 0$, then there exits a natural number n_0 such that

$$\|\varpi_{k_n} - \varpi_{k_p}\|_{\mathbb{C}_2} \leq \varepsilon \|m\|_{\mathbb{C}_2}, \text{ for all } n, p \geq n_0.$$

Fix any $p \geq n_0$. Letting $n \to \infty$, we get

$$\|\varpi - \varpi_{k_p}\|_{\mathbb{C}_2} \leq \varepsilon \|m\|_{\mathbb{C}_2}.$$

Then (ϖ_{k_p}) is m-uniformly convergent to ϖ. Hence, $\varpi_k \xrightarrow{st} \varpi(m)$. □

Theorem 3.19. *Let E denote an Archeimedan $\mathbb{BR}_{i_2}$ space. The following statements are equivalent:*
(1) Statistical order convergence remains stable.
(2) Statistical order convergence is the same as statistically relatively uniform convergence.

Proof. (1) implies (2). Let us suppose that $\varpi_k \xrightarrow{st-ord} [i_2]0_2$. From (1), there exits a sequence of bi-complex $(\alpha_k)(|\alpha_k|_{i_1} \neq 0)$ with $\alpha_k \Uparrow_{i_2} \infty^*(\infty^* = \infty + \infty i_1 + \infty i_2 + \infty i_1 i_2)$ such that $\|\alpha_k\|_{\mathbb{C}_2}\varpi_k \xrightarrow{st-ord} [i_2]0_2$. Then there exits a sequence (ϖ_k) with $\varpi_k \Downarrow^{st}_{i_2} 0_2$ such that

$$\|\alpha_k\|_{\mathbb{C}_2}\|\varpi_k\|_{\mathbb{C}_2} \leq \|\varpi_k\|_{\mathbb{C}_2} \text{ a.a.k}$$

Which implies that

$$\|\varpi_k\|_{\mathbb{C}_2} \leq \left\| \frac{\varpi_k}{\alpha_k} \right\|_{\mathbb{C}_2} \text{ a.a.k}$$

Thus, $\varpi_k \xrightarrow{st} 0_2(\varpi_k)$. Hence, $\varpi_k \xrightarrow{stru} 0_2$.
(2) implies (1). Let us consider $\varpi_k \xrightarrow{st-ord} 0_2$, from (2) we see $\varpi_k \xrightarrow{stru} 0_2$. Then there exists $m \in E$ and sequence $\varepsilon_n \downarrow o$ such that

$$\|\varpi_k\|_{\mathbb{C}_2} \leq \varepsilon_k \|m\|_{\mathbb{C}_2} \text{ a.a.k}$$
$$\frac{1}{\sqrt{\varepsilon_k}}\|\varpi_k\|_{\mathbb{C}_2} \leq \sqrt{\varepsilon_k}\|m\|_{\mathbb{C}_2} \text{ a.a.k}$$

Thus, we get $\frac{1}{\sqrt{\varepsilon_k}}\varpi_k \xrightarrow{st-ord} 0_2$. □

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