**METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY**

# CUMULANT EXPANSIONS OF GROUPS OF OPERATORS FOR PARTICLE SYSTEMS WITH TOPOLOGICAL NEAREST-NEIGHBOR INTERACTION

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ABSTRACT. This article discusses the structure of expansions that represent non-perturbative solutions of the Cauchy problem for the evolution equation hierarchies for the state and observables of many-particle systems with topological nearest-neighbor interaction. The generating operators for these expansions are derived using a proposed cluster expansion method applied to the groups of operators in the Liouville equations for both states and observables, respectively. The article also introduced the concept of a cumulant representation for distribution functions that describe the state of many particles with topological interactions and constructs a non-perturbative solution to the Cauchy problem for the hierarchy of nonlinear evolution equations for the cumulants of distribution functions. Furthermore, a relationship is established between the constructed solution and the series expansion structures for reduced distribution and correlation functions.

## 1. INTRODUCTION

A few years ago, to describe complex systems, particularly systems of mathematical biology, "topological" interaction models were introduced [4, 5, 8, 3]. Namely, the notion of interaction was introduced, which is described as a constituent of the system reacting to the presence of another not according to the distance between them, analogous to many-particle systems, but according to the rank of proximity or their order of preference. Interactions between living beings in nature are weighted as a function of their rank, regardless of relative distance, that is, the probability of an individual interacting with its nearest neighbor is the same whether that individual is close or far away. This new type of interaction was called "topological" in contrast to the usual "metric" interaction, i.e., a function of the relative distance between entities.

A typical example of a topological model of the interaction in many-particle systems is a one-dimensional system of particles that interact with their nearest neighbors. In particular, such systems include the well-known Toda lattice, which models a crystal in solid-state physics, the Ising model, which describes ferromagnetism phenomena, or a one-dimensional hard-rod system, which is characterized by non-symmetric functions of phase space variables; in other words, functions that are not invariant under particle renumbering [11],[26].

The basic approach to constructing solutions for the evolution equations for many-particle systems has been developed since the work of M. Bogolyubov, H. Grad, and O. Lanford [7, 18]. This approach primarily relies on perturbation theory methods. For this reason, to describe the evolution of the state of infinitely many colliding particles, the solution to the BBGKY hierarchy of equations (Bogolyubov–Born–Green–Kirkwood–Yvon) and its scaling asymptotic behavior are represented by the iteration series of these equations. In recent works [16],[19],[20], a non-perturbative representation of the solution to the BBGKY hierarchy for states, as well as the dual hierarchy of equations for

observables of many colliding particles was constructed. This approach classifies various non-perturbative representations of a solution, including the perturbative solutions as a partial case.

As is known [7],[16], the evolution of the state of finitely many classical particles is traditionally described within the framework of probability distribution functions that are symmetric over to the phase variables characterizing the particles and are governed by the Liouville equations. In the paper [17], an alternative approach to describing the evolution of the state of many colliding particles was developed, which consists of using functions defined as cumulants of probability distribution functions. Cumulants of probability distribution functions are interpreted as correlations of particle states, therefore the term correlation functions [24, 27, 1] is also used for them. The evolution of correlation functions is described by the Liouville hierarchy of nonlinear evolution equations, constructed for a system of many hard spheres in the paper [17].

This article formulates the concept of a cumulant representation for probability distribution functions, which describes the evolution of state correlations of many-particle systems with topological interaction. It is established that the structure of cumulant expansions for such systems depends on the symmetry of the probability distribution functions of many-particle systems. Then based on the dynamics of correlations, expansions in series for reduced distribution functions and their cumulants, known as reduced correlation functions, are determined. In particular, this allowed us to explain the structure of the generating operators of the nonperturbative solution of the Cauchy problem for the BBGKY hierarchy for particle systems with topological interaction. Additionally, the expansion structure of a solution to the Cauchy problem for the dual hierarchy of evolution equations for reduced observables of topologically interacting particles is established.

## 2. PRELIMINARIES: TOPOLOGICAL NEAREST-NEIGHBOR INTERACTION

In one-dimensional space, we consider a system of many identical particles that interact with their nearest neighbors through a short-range interaction potential $\Phi$, namely, $\Phi(q) = 0$, if $|q| > R > 0$, with a hard core of length $\sigma$, i.e. $\Phi(\sigma) = +\infty$. Let the interaction potential $\Phi$ satisfy the following conditions: $\Phi \in C^2[\sigma, R]$. For the configurations of many particles with such "topological" nearest-neighbor interaction, the following inequalities hold: $\sigma + q_i \leq q_{i+1}$, where $q_i \in \mathbb{R}$ is the coordinate of the center of the $i$-th particle. Since a system of a non-fixed number of ordered particles is considered, it natural to number the particles using positive and negative integers from the set $\mathbb{Z} \setminus \{0\}$ as follows [11],[26]. The phase coordinates of a single-particle subsystem are numbered as canonical quantities characterizing the first particle. The phase coordinates of particles in many-particle subsystems located to the right of the first particle are numbered with positive integers, and those located to the left of the first particle are numbered with negative integers.

Let $L^1_{n_1+n_2}$ be the Banach space of double sequences $f = \{f_{n_1+n_2}(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})\}_{n_1+n_2 \geq 0}$ of integrable functions $f_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2})$ defined on the phase space $\mathbb{R}^{n_1+n_2} \times (\mathbb{R}^{n_1+n_2} \setminus \mathbb{W}_{n_1+n_2})$, equal zero on the set of forbidden configurations $\mathbb{W}_{n_1+n_2} \equiv \{(q_{-n_1}, \ldots, q_{-1}, q_1, \ldots, q_{n_2}) \in \mathbb{R}^{n_1+n_2} \mid \sigma + q_i < q_{i+1} \text{ at least for one pair } (i, i+1) \in ((-n_1, -n_1+1), \ldots, (-1, 1), \ldots, (n_2-1, n_2))\}$, and non-symmetric with respect to the permutation of arguments $x_i \equiv (q_i, p_i) \in \mathbb{R} \times \mathbb{R}$ with the norm

$$\|f_{n_1+n_2}\|_{L^1_{n_1+n_2}} = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} dx_{-n_1} \ldots dx_{n_2} \, |f_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2})|.$$

Under the conditions formulated above for the interaction potential $\Phi$, the functions $X_i(-t)$, $i \in (-n_1, \ldots, -1, 1, \ldots, n_2)$ are global-in-time solutions of the Cauchy problem to the Hamilton equations of the system $n_1 + n_2$ of particles with initial data $X_i(0) = x_i$, $i \in (-n_1, \ldots, -1, 1, \ldots, n_2)$ [7]. Then in the space of integrable functions $L^1_{n_1+n_2}$, the

group of operators of the Liouville equation is defined as follows [7],[2]:

$$S_{n_1+n_2}^*(t)f_{n_1+n_2} \equiv (S_{n_1+n_2}^*(t, -n_1, \ldots, n_2) f_{n_1+n_2})(x_{-n_1}, \ldots, x_{n_2}) \doteq \quad (2.1)$$

$$\begin{cases} f_{n_1+n_2}(X_{-n_1}(-t, x_{-n_1}, \ldots, x_{n_2}), \ldots, X_{n_2}(-t, x_{-n_1}, \ldots, x_{n_2})), \\ \qquad\qquad\qquad (x_{-n_1}, \ldots, x_{n_2}) \in (\mathbb{R}^{n_1+n_2} \times (\mathbb{R}^{n_1+n_2} \setminus \mathbb{W}_{n_1+n_2})), \\ 0, \qquad\qquad\qquad (q_{-n_1}, \ldots, q_{n_2}) \in \mathbb{W}_{n_1+n_2}. \end{cases}$$

In the space $L_{n_1+n_2}^1$, the one-parameter mapping (2.1) is a strongly continuous group of isometric operators. The infinitesimal generator of the group of operators (2.1) has the following structure

$$\mathcal{L}_{n_1+n_2}^* = \sum_{j \in (-n_1, \ldots, -1, 1, \ldots, n_2)} \mathcal{L}^*(j) + \sum_{(j,j+1) \in ((-n_1, -n_1+1), \ldots, (n_2-1, n_2))} \mathcal{L}_{\mathrm{int}}^*(j, j+1), \quad (2.2)$$

where on the subspace of continuously differentiable functions with compact supports the Liouville operators are determined by the following formulas, respectively [26]:

$$(\mathcal{L}^*(j)f_{n_1+n_2})(x_{-n_1}, \ldots, x_{n_2}) \doteq -p_j \frac{\partial}{\partial q_j} f_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}),$$

$$(\mathcal{L}_{\mathrm{int}}^*(j, j+1)f_{n_1+n_2})(x_{-n_1}, \ldots, x_{n_2}) \doteq \frac{\partial}{\partial q_j} \Phi(q_j - q_{j+1})(\frac{\partial}{\partial p_j} -$$

$$- \frac{\partial}{\partial p_{j+1}})f_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}).$$

Let $D_{n_1+n_2}^0 \in L_{n_1+n_2}^1(\mathbb{R}^{n_1+n_2} \times (\mathbb{R}^{n_1+n_2} \setminus \mathbb{W}_{n_1+n_2}))$ be a probability distribution function. Then all possible states of the system of $n_1 + n_2$ particles at an arbitrary instant $t \in \mathbb{R}$ are described by the probability distribution function

$$D_{n_1+n_2}(t) = S_{n_1+n_2}^*(t)D_{n_1+n_2}^0, \quad (2.3)$$

which is a unique solution to the Cauchy problem for the Liouville equation. This is a strong solution for initial data from the subspace of finite sequences of continuously differentiable functions with compact supports. For arbitrary initial data from the space $L_{n_1+n_2}^1$, $n_1 + n_2 \geq 1$, it is a weak solution [7].

## 3. Cluster and cumulant expansions of distribution functions

For a system of a non-fixed number of particles [26], which will be considered further, the state is described by a double sequence

$$D = \{D_{n_1+n_2}(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})\}_{n_1+n_2 \geq 0}$$

of non-symmetric to the permutation of their arguments $x_i \equiv (q_i, p_i) \in \mathbb{R} \times \mathbb{R}$ probability distribution functions, which are defined on the phase space $\mathbb{R}^{n_1+n_2} \times (\mathbb{R}^{n_1+n_2} \setminus \mathbb{W}_{n_1+n_2})$.

Let us introduce a sequence of correlation functions $g = \{g_s(x_{-s_1}, \ldots, x_{s_2})\}_{s=s_1+s_2 \geq 0}$ of particle states as cumulants (semi-invariants) of probability distribution functions using cluster expansions:

$$D_{s_1+s_2}(x_{-s_1}, \ldots, x_{s_2}) = \sum_{\mathrm{P} : (x_{-s_1}, \ldots, x_{s_2}) = \bigcup_i X_i} \prod_{X_i \subset \mathrm{P}} g_{|X_i|}(X_i), \quad s_1 + s_2 \geq 1, \quad (3.4)$$

where the symbol $\sum_\mathrm{P}$ denotes the sum over all ordered partitions P of the partially ordered set $(x_{-s_1}, \ldots, x_{-1}, x_1, \ldots, x_{s_2})$ into $|\mathrm{P}|$ nonempty partially ordered subsets $X_i \in (x_{-s_1}, \ldots, x_{-1}, x_1, \ldots, x_{s_2})$ that do not intersect. The origin of the presented approach

of describing states in terms of correlation functions will be justified in Section 5. We give examples of cluster expansions (3.4):

$$D_{0+1}(x_1) = g_{0+1}(x_1),$$
$$D_{1+1}(x_{-1}, x_1) = g_{1+1}(x_{-1}, x_1) + g_{1+0}(x_{-1})g_{0+1}(x_1),$$
$$D_{1+2}(x_{-1}, x_1, x_2) = g_{1+2}(x_{-1}, x_1, x_2) + g_{1+1}(x_{-1}, x_1)g_{0+1}(x_2)+$$
$$g_{1+0}(x_{-1})g_{0+2}(x_1, x_2) + g_{1+0}(x_{-1})g_{0+1}(x_1)g_{0+1}(x_2),$$
$$\vdots$$

It should be noted that the concept of cluster expansions was initially introduced for Gibbs distribution functions in the work [25]. This was to describe the equilibrium state using Ursell functions for many-particle systems characterized by distribution functions symmetric concerning permutations of arguments. It is important to highlight that the structure of cluster expansions (3.4) differs significantly from the structure of analogous cluster expansions for systems described by symmetric functions [17].

On the set $\mathbb{R}^{s_1+s_2} \setminus \mathbb{W}_{s_1+s_2}$ of allowed configurations, the solutions of the recursive relations (3.4) are represented by the following cumulant expansions:

$$g_{s_1+s_2}(x_{-s_1}, \ldots, x_{s_2}) = \sum_{\substack{\mathrm{P}\,:\,(x_{-s_1}, \ldots, x_{s_2}) = \bigcup_i X_i \\ s_1 + s_2 \geq 1,}} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} D_{|X_i|}(X_i), \qquad (3.5)$$

where the used denotations are similar to that of the recursive equations (3.4). We give examples of cumulant expansions (3.5):

$$g_{0+1}(x_1) = D_{0+1}(x_1),$$
$$g_{1+1}(x_{-1}, x_1) = D_{1+1}(x_{-1}, x_1) - D_{1+0}(x_{-1})D_{0+1}(x_1),$$
$$g_{1+2}(x_{-1}, x_1, x_2) = D_{1+2}(x_{-1}, x_1, x_2) - D_{1+1}(x_{-1}, x_1)D_{0+2}(x_2)-$$
$$D_{1+0}(x_{-1})D_{0+2}(x_1, x_2) + D_{1+0}(x_{-1})D_{0+1}(x_1)D_{0+1}(x_2),$$
$$\vdots$$

Thus, the structure of the expansions (3.5) for correlation functions is such that they have the meaning of cumulants (semi-invariants) of the corresponding order of probability distribution functions [16],[9],[10] (see also [6],[28]). We note that correlation functions provide an equivalent method of describing the states of many-particle systems analogous to distribution functions. The interpretation of correlation functions is related to the fact that the state of statistically independent particles on allowed configurations is described by the product of one-particle correlation functions, i.e., the probability distribution functions of each particle.

To compare the structure of the expansions (3.4) and (3.5) with similar expansions for symmetric functions, we will represent them on sequences of functions. In the set of double sequences $f = \{f_{n_1+n_2}(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})\}_{n_1+n_1=n\geq 0}$ of measurable functions $f_{n_1+n_2}$, where $f_0$ is an arbitrary number, we introduce the following tensor $\star$-product

$$(f_1 \star f_2)_{|X|}(X) \doteq \sum_{Y \subset X} (f_1)_{|Y|}(Y)\,(f_2)_{|X \setminus Y|}(X \setminus Y),$$

where the symbol $\sum_{Y \subset X}$ denotes the sum over all possible partially ordered subsets $Y$ of the partially ordered set $X \equiv (x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})$.

On the set of double sequences $f$ of measurable functions, the mapping $(I - \circ)^{-\mathbb{I}_\star}$ is defined as the $\star$-resolvent, namely,

$$(I - f)^{-\mathbb{I}_\star} = I + \sum_{n=1}^{\infty} f^{\star n}, \tag{3.6}$$

where $I = (1, 0, 0, \ldots)$ is a unit sequence and is represented component by component form by the following expansions:

$$\left((I - f)^{-\mathbb{I}_\star}\right)_{s_1+s_1}(x_{-s_1}, \ldots, x_{s_2}) =$$

$$\delta_{s_1+s_2,0} + \sum_{\mathrm{P}\,:\,(x_{-s_1},\ldots,x_{s_2}) = \bigcup_i X_i} \prod_{X_i \subset \mathrm{P}} f_{|X_i|}(X_i), \quad s_1 + s_2 \geq 1.$$

The inverse mapping $I - (I + \circ)^{-\mathbb{I}_\star}$ to the mapping $(I - \circ)^{-\mathbb{I}_\star}$ is defined in terms of the $\star$-product by the following series

$$I - (I + f)^{-\mathbb{I}_\star} = \sum_{n=1}^{\infty} (-1)^{n-1} f^{\star n}, \tag{3.7}$$

which is represented in component-wise form by the following expansions:

$$\left(I - (I + f)^{-\mathbb{I}_\star}\right)_{s_1+s_1}(x_{-s_1}, \ldots, x_{s_2}) =$$

$$\sum_{\mathrm{P}\,:\,(x_{-s_1},\ldots,x_{s_2}) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} f_{|X_i|}(X_i), \quad s_1 + s_2 \geq 1.$$

Thus, cluster expansions for the sequence of probability distribution functions (3.4) of a system of particles interacting with their nearest neighbors have a structure determined by mapping (3.6), namely,

$$I + D = (I - g)^{-\mathbb{I}_\star}.$$

The structure of cumulant expansions for the sequence of correlation functions (3.5) is defined by the inverse mapping (3.7)

$$g = I - (I + D)^{-\mathbb{I}_\star}.$$

The origin of the structure of expansions (3.4) and (3.8) will be argued in Section 5. We note that in the case of many particles whose state is described by symmetric distribution functions, the structure of cluster and cumulant expansions is determined by exponential and logarithmic mappings defined by the suitable tensor $*$-product [17], respectively.

We will also introduce some generalizations of the concept of cumulant expansions. Namely, we will define cumulant expansions for a cluster of particles $(-s_1, \ldots, s_2)$ and particles $(-(n_1 + s_1), \ldots, -(s_1 + 1)), (s_2 + 1, \ldots, s_2 + n_2)$, i.e., correlation functions of a cluster of particles and particles

$$g_{1+n_1+n_2}(x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, x_{s_2+1}, \ldots, x_{s_2+n_2}) = \tag{3.8}$$

$$\sum_{\substack{\mathrm{P}\,:\,(x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, \\ x_{s_2+1}, \ldots, x_{s_2+n_2}) = \bigcup_i X_i}} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} D_{|\theta(X_i)|}(\theta(X_i)),$$

$$n_1 + n_2 \geq 0.$$

In the expansion for the $(1 + n_1 + n_2)$-th order cumulant (semi-invariants) of probability distribution functions, the following notation is used: the symbol $\sum_{\mathrm{P}}$ denotes the sum over each ordered partition of P of the partially ordered set of indices $(-(n_1 + s_1), \ldots, -(s_1 + 1), \{-s_1, \ldots, s_2\}, s_2 + 1, \ldots, s_2 + n_2)$ into $|\mathrm{P}|$ nonempty partially ordered subsets $X_i$ that do not intersect each other, the connected set $\{-s_1, \ldots, s_2\}$ belongs entirely to

one of the subsets $X_i$, and the declusterization mapping $\theta(\cdot)$ is defined by the equality:
$\theta(\{X_i\}) = (X_i)$.

We emphasize that cumulants (3.8) are solutions to the recursion relations known as the cluster expansions of the groups of operators (2.1) of a cluster of particles and particles, namely,

$$D_{n_1+s_1+s_2+n_2}(x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, x_{-s_1}, \ldots, x_{s_2}, x_{s_2+1}, \ldots, x_{s_2+n_2})) = \qquad (3.9)$$
$$\sum_{\substack{\mathrm{P}:\, (x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, \\ x_{s_2+1}, \ldots, x_{s_2+n_2})) \bigcup_i X_i}} \prod_{X_i \subset \mathrm{P}} g_{|X_i|}(X_i), \quad n_1 + n_2 \geq 1.$$

The correlation functions of a particle cluster and particles are related to the correlation functions of particles by the following relations:

$$g_{1+n_1+n_2}(x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, x_{s_2+1}, \ldots, x_{s_2+n_2}) = \qquad (3.10)$$
$$\sum_{\substack{\mathrm{P}:\, (x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, \\ x_{s_2}\}, x_{s_2+1}, \ldots, x_{s_2+n_2}) = \bigcup_i X_i}} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} \sum_{\mathrm{P'}:\, \theta(X_i) = \bigcup_{j_i} Z_{j_i}} \prod_{Z_{j_i} \subset \mathrm{P'}} g_{|Z_{j_i}|}(Z_{j_i}).$$

Let us provide instances of these relations:

$$g_{1+0+0}(\{x_1\}) = g_{0+1}(x_1),$$
$$g_{1+0+0}(\{x_{-1}, x_1\}) = g_{1+1}(x_{-1}, x_1) + g_{1+0}(x_{-1})g_{0+1}(x_1),$$
$$g_{1+0+1}(\{x_{-1}, x_1\}, x_2) = g_{1+2}(x_{-1}, x_1, x_2) + g_{1+1}(x_{-1}, x_1)g_{0+1}(x_2) +$$
$$g_{1+0}(x_{-1})g_{0+2}(x_1, x_2) + g_{1+0}(x_{-1})g_{0+1}(x_1)g_{0+1}(x_2),$$
$$\vdots$$

In the particular case $n_1 + n_2 = 0$, i.e., a cluster consisting of $s_1 + s_2$ particles, relations (3.10) take the form

$$g_{1+0+0}(\{x_{-s_1}, \ldots, x_{s_2}\}) = \sum_{\mathrm{P}:\, \theta(\{x_{-s_1}, \ldots, x_{s_2}\}) = \bigcup_i X_i} \prod_{X_i \subset \mathrm{P}} g_{|X_i|}(X_i).$$

Note that the structure of cluster and cumulant expansions for systems of particles with topological interaction of nearest neighbors is determined by the symmetry of the functions to the renumbering of their phase variables and is not related to the dimensionality of the space, in which the particles move.

## 4. A HIERARCHY OF EVOLUTION EQUATIONS
### FOR CUMULANTS OF DISTRIBUTION FUNCTIONS

As noted above, the sequence of correlation functions (3.5) describes the state of systems of a non-fixed number of particles in an alternative way to the sequence of probability distribution functions.

The evolution of the state of a system of particles with topological interaction is described in terms of correlation functions which are governed by the hierarchy of the nonlinear Liouville equations:

$$\frac{\partial}{\partial t} g_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \mathcal{L}^*_{s_1+s_2} g_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) + \qquad (4.11)$$
$$\sum_{\mathrm{P}:\, (x_{-s_1}, \ldots, x_{s_2}) = X_1 \bigcup X_2} \sum_{j=\max \widehat{X}_1} \mathcal{L}^*_{\mathrm{int}}(j, j+1) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2),$$
$$g_{s_1+s_2}(t)\big|_{t=0} = g^0_{s_1+s_2}, \quad s_1 + s_2 \geq 1, \qquad (4.12)$$

where the Liouville operator $\mathcal{L}^*_{s_1+s_2}$ is defined by expression (2.2), $\sum_{\text{P}: (x_{-s_1},\ldots,x_{s_2})=X_1 \bigcup X_2}$ is the sum over all possible ordered partitions P of the partially ordered set $(x_{-s_1},\ldots,x_{s_2})$ into two subsets $X_1$ and $X_2$ that do not intersect, the symbol $\widehat{X}_i$ denotes the set of indices of the corresponding set $X_i$ of phase coordinates of particles, and $\max\widehat{X}_1$ is the maximum value of the set of indices $\widehat{X}_1$.

We emphasize that the Liouville hierarchy (4.11) is a hierarchy of recurrent evolution equations. We give examples of evolution equations (4.11):

$$\frac{\partial}{\partial t}g_{0+1}(t,x_1) = \mathcal{L}^*(1)g_1(t,x_1),$$

$$\frac{\partial}{\partial t}g_{1+1}(t,x_{-1},x_1) = \Big(\sum_{j\in(-1,1)}\mathcal{L}^*(j) + \mathcal{L}^*_{\text{int}}(-1,1)\Big)g_{1+1}(t,x_{-1},x_1)+$$

$$\mathcal{L}^*_{\text{int}}(-1,1)g_{1+0}(t,x_{-1})g_{0+1}(t,x_1),$$

$$\frac{\partial}{\partial t}g_{1+2}(t,x_{-1},x_1,x_2) =$$

$$\Big(\sum_{j\in(-1,1,2)}\mathcal{L}^*(j) + \sum_{(j,j+1)\in((-1,1),(1,2))}\mathcal{L}^*_{\text{int}}(j,j+1)\Big)g_{1+2}(t,x_{-1},x_1,x_2)+$$

$$\mathcal{L}^*_{\text{int}}(-1,1)g_{1+0}(t,x_{-1})g_{0+2}(t,x_1,x_2) + \mathcal{L}^*_{\text{int}}(1,2)g_{1+1}(t,x_{-1},x_1)g_{0+1}(t,x_2),$$

$$\vdots$$

To construct a solution to the Cauchy problem of the Liouville hierarchy (4.11), we introduce the concept of cumulants of the groups of operators of the Liouville equations, i.e. a connected part of the groups of operators of the Liouville equations. The cumulant of the $(s_1 + s_2)$-th order of the groups of operators (2.1) is defined by the following expansion

$$\mathfrak{A}_{s_1+s_2}(t,-s_1,\ldots,s_2) \doteq \sum_{\text{P}\,:\,(-s_1,\ldots,s_2)\,=\,\bigcup_i X_i} (-1)^{|\text{P}|-1}\prod_{X_i\subset\text{P}} S^*_{|X_i|}(t,X_i), \quad (4.13)$$

where the symbol $\sum_{\text{P}}$ denotes the sum over all ordered partitions P of the partially ordered set of indexes $(-s_1,\ldots,-1,1,\ldots,s_2)$ into $|\text{P}|$ nonempty partially ordered subsets $X_i \in (-s_1,\ldots,-1,1,\ldots,s_2)$ that do not intersect. Here are some examples of cumulant expansions (4.13):

$$\mathfrak{A}_{0+1}(t,1) = S^*_{0+1}(t,1),$$

$$\mathfrak{A}_{1+1}(t,-1,1) = S^*_{1+1}(t,-1,1) - S^*_{0+1}(t,-1)S^*_{0+1}(t,1),$$

$$\mathfrak{A}_{1+2}(t,-1,1,2) = S^*_{1+2}(t,-1,1,2) - S^*_{1+1}(t,-1,1)S^*_{0+1}(t,2)-$$

$$S^*_{1+0}(t,-1)S^*_{0+2}(t,1,2) + S^*_{1+0}(t,-1)S^*_{0+1}(t,1)S^*_{0+1}(t,2),$$

$$\vdots$$

We remark that cumulants (4.13) are solutions of the recursive equations known as cluster expansions of the groups of operators (2.1) of the Liouville equations

$$S^*_{s_1+s_2}(t,-s_1,\ldots,s_2) = \sum_{\text{P}\,:\,(-s_1,\ldots,s_2)\,=\,\bigcup_i X_i}\prod_{X_i\subset\text{P}}\mathfrak{A}_{|X_i|}(t,X_i), \quad s_1+s_2\geq 1. \quad (4.14)$$

The structure of these cluster expansions is determined by the structure of the generator (2.2) of the group of operators (2.1) for topologically interacting particles.

The evolution of initial cumulants of distribution functions (3.8) is described by the following correlation functions:

$$g_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \tag{4.15}$$
$$\sum_{\mathrm{P}:\, (x_{-s_1}, \ldots, x_{s_2}) = \bigcup_j X_j} \prod_{X_j \subset \mathrm{P}} \mathfrak{A}_{|\widehat{X}_j|}(t, \widehat{X}_j) g_{|\mathrm{P}|}^0(\{X_1\}, \ldots, \{X_{|\mathrm{P}|}\}), \quad s_1 + s_2 \geq 1,$$

where the symbol $\sum_{\mathrm{P}}$ denotes the sum over all ordered partitions P of the partially ordered set of indexes $(x_{-s_1}, \ldots, x_{s_2})$ into $|\mathrm{P}|$ nonempty partially ordered subsets $X_i \in (x_{-s_1}, \ldots, x_{s_2})$ that do not intersect, the symbol $\widehat{X}_i$ denotes the set of indices of the corresponding set $X_i$ of phase coordinates of particles, and the generating operators of these expansions are products of cumulants (4.13) of the groups of operators (2.1). We adduce some examples of correlation functions (4.15)

$$g_{0+1}(t, x_1) = \mathfrak{A}_1(t, 1) g_{0+1}^0(x_1),$$
$$g_{1+1}(t, x_{-1}, x_1) = \mathfrak{A}_2(t, -1, 1) g_{1+0}^0(\{x_{-1}, x_1\}) + \mathfrak{A}_1(t, -1)\mathfrak{A}_1(t, 1) g_{1+1}^0(x_{-1}, x_1),$$
$$g_3(t, x_{-1}, x_1, x_2) = \mathfrak{A}_3(t, -1, 1, 2) g_{1+0}^0(\{x_{-1}, x_1, x_2\}) +$$
$$+ \mathfrak{A}_2(t, -1, 1)\mathfrak{A}_1(t, 2) g_{1+1}^0(\{x_{-1}, x_1\}, x_2) +$$
$$+ \mathfrak{A}_1(t, -1)\mathfrak{A}_2(t, 1, 2) g_{1+1}^0(x_{-1}, \{x_1, x_2\}) +$$
$$+ \mathfrak{A}_1(t, -1)\mathfrak{A}_1(t, 1)\mathfrak{A}_1(t, 2)) g_{1+2}^0(x_{-1}, x_1, x_2),$$
$$\vdots$$

Indeed, by applying cluster expansions (4.14) of the groups of operators (2.1) to operator groups in the definition of cumulants (3.5) of distribution functions (2.3) and considering the definition of initial correlation functions (3.5), we derive expansions (4.15).

For arbitrary initial states from the space of double sequences of integrable functions the solution of the Cauchy problem of the Liouville hierarchy (4.11),(4.12) is also represented by the following expansions which are equivalent to expansions (4.15):

$$g_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \tag{4.16}$$
$$\sum_{\mathrm{P}:\, (x_{-s_1}, \ldots, x_{s_2}) = \bigcup_i X_i} \mathfrak{A}_{|\mathrm{P}|}(t, \{\widehat{X}_1\}, \ldots, \{\widehat{X}_{|\mathrm{P}|}\}) \prod_{X_i \subset \mathrm{P}} g_{|X_i|}^0(X_i), \quad s_1 + s_2 \geq 1.$$

The generating operator $\mathfrak{A}_{|\mathrm{P}|}(t)$ of the expansion (4.16) is the $|\mathrm{P}|$-th order cumulant (4.13) of the groups of operators (2.1) which is represented by the following expansion

$$\mathfrak{A}_{|\mathrm{P}|}(t, \{\widehat{X}_1\}, \ldots, \{\widehat{X}_{|\mathrm{P}|}\}) \doteq \tag{4.17}$$
$$\sum_{\mathrm{P}':\, (\{\widehat{X}_1\}, \ldots, \{\widehat{X}_{|\mathrm{P}|}\}) = \bigcup_k \widehat{Z}_k} (-1)^{|\mathrm{P}'|-1} \prod_{\widehat{Z}_k \subset \mathrm{P}'} S_{|\theta(\widehat{Z}_k)|}^*(t, \theta(\widehat{Z}_k)),$$

where the connected set of ordered indices $\widehat{X}_i$, i.e., one element of the set, is denoted by the symbol $\{\widehat{X}_i\}$, the declusterization mapping $\theta(\cdot)$ is defined by the equality: $\theta(\{\widehat{X}_i\}) = (\widehat{X}_i)$ and the symbol $\sum_{\mathrm{P}':\, (\{\widehat{X}_1\}, \ldots, \{\widehat{X}_{|\mathrm{P}|}\}) = \bigcup_k \widehat{Z}_k}$ denotes the sum over all possible ordered partitions $\mathrm{P}'$ of the ordered set of indices $(\{\widehat{X}_1\}, \ldots, \{\widehat{X}_{|\mathrm{P}|}\})$ on $|\mathrm{P}'|$ nonempty partially ordered subsets $\widehat{Z}_k \subset (\{\widehat{X}_1\}, \ldots, \{\widehat{X}_{|\mathrm{P}|}\})$. We adduce some examples of correlation

functions (4.16)

$$g_{0+1}(t, x_1) = \mathfrak{A}_1(t, 1)g_{0+1}^0(x_1),$$

$$g_{1+1}(t, x_{-1}, x_1) = \mathfrak{A}_1(t, \{-1, 1\})g_{1+1}^0(x_{-1}, x_1) + \mathfrak{A}_2(t, -1, 1)g_{1+0}^0(x_{-1})g_{0+1}^0(x_1),$$

$$g_{1+2}(t, x_{-1}, x_1, x_2) = \mathfrak{A}_1(t, \{-1, 1, 2\})g_{1+2}^0(x_{-1}, x_1, x_2) +$$

$$+\mathfrak{A}_2(t, -1, \{1, 2\})g_{1+0}^0(x_{-1})g_{0+2}^0(x_1, x_2) + \mathfrak{A}_2(t, \{-1, 1\}, 2)g_{0+1}^0(x_2)g_{1+1}^0(x_{-1}, x_1) +$$

$$+\mathfrak{A}_3(t, -1, 1, 2)g_{1+0}^0(x_{-1})g_{0+1}^0(x_1)g_{0+1}^0(x_2),$$

$$\vdots$$

The validity of the expansions (4.16) for the solution of the Liouville hierarchy (4.11) is proved by point-wise differentiation for the time variable.

The following existence theorem for the Liouville hierarchy (4.11) holds.

**Theorem 4.1.** *For $t \in \mathbb{R}$ the solution of the Cauchy problem of the Liouville hierarchy (4.11),(4.12) is represented by expansions (4.16). For infinitely differentiable initial correlation functions with compact supports from the space $L_{n_1+n_2}^1$, $n_1+n_2 \geq 1$, expansions (4.16) represent the strong (classical) solution and for arbitrary initial correlation functions from the space $L_{n_1+n_2}^1$, $n_1 + n_2 \geq 1$ it is the weak (generalized) solution.*

We emphasize that the characteristic properties of the constructed solution (4.16) are generated by the properties of the cumulants (4.17) of the groups of operators (2.1) of the Liouville equations.

The following criterion holds.

**Criterion 1.** *A solution of the Cauchy problem for the Liouville hierarchy (4.11),(4.12) for particle systems with topological interaction is represented by expansions (4.16) if and only if the generating operators of expansions (4.16) are solutions of cluster expansions (4.14) of the groups of operators (2.1).*

The necessary condition means that cluster expansions (4.14) are valid for groups of operators (2.1). These recurrence relations are derived from the definition (3.5) of correlation functions, provided that they are represented as expansions (4.16) for the solution of the Cauchy problem to the Liouville hierarchy (4.11),(4.12). The sufficient condition means that the infinitesimal generator of one-parameter mapping (4.16) coincides with the generator of the Liouville hierarchy (4.11), which is a consequence of Theorem 1.

## 5. THE BBGKY HIERARCHY FOR TOPOLOGICALLY INTERACTING PARTICLES

An alternative approach equivalent to the one presented above is used to describe the evolution of states of both finite and infinite numbers of particles. This approach is based on the description of the evolution of states by reduced distribution functions.

Further, we will formulate an approach to describing the evolution of the state of a system of many particles with topological interaction in terms of reduced distribution functions defined within the framework of the dynamics of correlations (4.11). Recall [26] that reduced distribution functions are defined through the sequence $D(0) = \{D_{n_1+n_2}^0(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})\}_{n_1+n_2 \geq 0}$ of probability distribution functions at the

initial instant and the groups of operators (2.1):

$$F_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) \doteq (I, D(0))^{-1} \sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-(n_1+s_1)} \cdots$$

(5.18)

$$dx_{-(s_1+1)} dx_{s_2+1} \ldots dx_{s_2+n_2} (S^*(t)D(0))_{n_1+s_1+s_2+n_2}(x_{-(n_1+s_1)}, \ldots, x_{s_2+n_2}),$$
$$s_1 + s_2 \geq 1,$$

where the normalizing factor

$$(I, D(0)) \doteq$$

$$\sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-n_1} \ldots dx_{n_2} D^0_{n_1+n_2}(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})$$

known as the grand canonical partition function. The justifications for introducing this definition of reduced distribution functions for topologically nearest-neighbor interacting particles are provided in [26]. Notice that the sequence of reduced distribution functions (5.18) satisfies the BBGKY hierarchy (5.23) for topologically nearest-neighbor interacting particles [11].

The result of dividing the series in expression (5.18) by the series of the normalizing factor naturally allows one to redefine the reduced distribution functions as a series expansion over the cumulants of distribution functions (3.5)

$$F_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) \doteq$$

$$\sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-(n_1+s_1)} \cdots dx_{-(s_1+1)} dx_{s_2+1} \ldots dx_{s_2+n_2}$$

$$\sum_{\substack{\mathrm{P}: (x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, \\ x_{s_2+1}, \ldots, x_{s_2+n_2}) = \bigcup_i X_i}} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} (S^*(t)D(0))_{|X_i|}(X_i),$$

$$s_1 + s_2 \geq 1.$$

Thus, the definition of reduced distribution functions, equivalent to the definition of (5.18), is formulated based on correlation functions (3.5) by the following series expansions:

$$F_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) \doteq \qquad (5.19)$$

$$\sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-(n_1+s_1)} \cdots dx_{-(s_1+1)} dx_{s_2+1} \cdots$$

$$dx_{s_2+n_2} g_{1+n_1+n_2}(t, x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, x_{s_2+1}, \ldots, x_{s_2+n_2}),$$
$$s_1 + s_2 \geq 1.$$

On the set of allowed configurations, the correlation functions of the particle cluster and particles $g_{1+n_1+n_2}(t)$, $n_1 + n_2 \geq 1$, are defined by the following expansions:

$$g_{1+n_1+n_2}(t, x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, x_{s_2+1}, \ldots, x_{s_2+n_2}) = \qquad (5.20)$$

$$\sum_{\substack{\mathrm{P}: (x_{-(n_1+s_1)}, \ldots, x_{-(s_1+1)}, \{x_{-s_1}, \ldots, x_{s_2}\}, \\ x_{s_2+1}, \ldots, x_{s_2+n_2}) = \bigcup_i X_i}} \mathfrak{A}_{|\mathrm{P}|}\big(t, \{\theta(\widehat{X}_1)\}, \ldots, \{\theta(\widehat{X}_{|\mathrm{P}|})\}\big) \prod_{X_i \subset \mathrm{P}} g^0_{|X_i|}(X_i).$$

Recall that the generating operators of the expansions (5.20) are represented by the $|\mathrm{P}|th$-th order cumulants (4.17) of the groups of operators (2.1) and the symbol $\sum_{\mathrm{P}}$

denotes the sum over all ordered partitions of the set of indices $(-(n_1 + s_1), \ldots, -(s_1 + 1), \{-s_1, \ldots, -1, 1, \ldots, s_2\}, s_2 + 1, \ldots, s_2 + n_2)$ into intersecting subsets $X_i$.

Thus, we observe that the structure of cumulant expansions for states (3.4) for topologically interacting particles is determined by the method incorporating the normalizing factor in the mean value functional which is determined through reduced distribution functions.

Let's introduce the space $L^1_\alpha = \sum_{n=0}^\infty \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \bigoplus \alpha^{n_1+n_2} L^1_{n_1+n_2}$ of double sequences $f = \{f_{n_1+n_2}\}_{n_1+n_2 \geq 0}$, of integrable functions $f_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2})$, defined on the phase space $\mathbb{R}^{n_1+n_2} \times (\mathbb{R}^{n_1+n_2} \setminus \mathbb{W}_{n_1+n_2})$ with the norm

$$\|f\| = \sum_{n=0}^\infty \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \alpha^{n_1+n_2} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-n_1} \ldots dx_{n_2} |f_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2})|,$$

where the parameter $\alpha > 1$.

The following bounded operators are defined on sequences of functions $f \in L^1_\alpha$:

$$\big(\mathfrak{a}_+ f\big)_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}) = \int_{\mathbb{R} \times \mathbb{R}} dx_{n_2+1}\, f_{n_1+n_2+1}(x_{-n_1}, \ldots, x_{n_2}, x_{n_2+1}), \quad (5.21)$$

$$\big(\mathfrak{a}_- f\big)_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}) = \int_{\mathbb{R} \times \mathbb{R}} dx_{-(n_1+1)}\, f_{n_1+1+n_2}(x_{-(n_1+1)}, x_{-n_1}, \ldots, x_{n_2}),$$

which are analogs of the particle annihilation operator of quantum field theory, as well as the following operators:

$$(I - \mathfrak{a}_\pm)^{-1} = \sum_{n=0}^\infty \mathfrak{a}_\pm^n. \quad (5.22)$$

Since the correlation functions (5.20) are determined by the Liouville hierarchy (4.11), the reduced distribution functions (5.19) satisfy the BBGKY hierarchy for a system of many topologically interacting particles [11]

$$\frac{d}{dt}F(t) = \mathcal{L}^* F(t) + [\mathfrak{a}_-, \mathcal{L}^*]F(t) + [\mathfrak{a}_+, \mathcal{L}^*]F(t), \quad (5.23)$$

where $\mathcal{L}^* = \sum_{n=0}^\infty \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \bigoplus \mathcal{L}^*_{n_1+n_2}$ is the direct sum of Liouville operators (2.2), the brackets $[\circ, \circ]$ denote the commutator of two operators, and, therefore, in component form, these terms of the generator BBGKY are represented by the following expressions:

$$\big([\mathfrak{a}_-, \mathcal{L}^*]F(t)\big)_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}) =$$
$$\int dx_{-(n_1+1)} \frac{\partial}{\partial q_{-n_1}} \Phi(q_{-n_1} - q_{-(n_1+1)}) \frac{\partial}{\partial p_{-n_1}} F_{n_1+1+n_2}(t),$$
$$\big([\mathfrak{a}_+, \mathcal{L}^*]F(t)\big)_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}) =$$
$$\int dx_{n_2+1} \frac{\partial}{\partial q_{n_2}} \Phi(q_{n_2} - q_{n_2+1}) \frac{\partial}{\partial p_{n_2}} F_{n_1+n_2+1}(t).$$

A solution of the Cauchy problem to the BBGKY hierarchy (5.23) of a many-particle system with topological nearest-neighbor interaction is represented by the following series

expansions for the sequence of reduced distribution functions [23, 22, 12]:

$$F_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-(n_1+s_1)} \ldots dx_{-(s_1+1)} \quad (5.24)$$

$$dx_{s_2+1} \ldots dx_{s_2+n_2} \, \mathfrak{A}_{1+n_1+n_2}(t, -(n_1+s_1), \ldots, -(s_1+1), \{-s_1, \ldots, s_2\},$$

$$s_2+1, \ldots, s_2+n_2) \, F^0_{s_1+n_1+s_2+n_2}(x_{-(n_1+s_1)}, \ldots, x_{s_2+n_2}), \quad s_1 + s_2 \geq 1,$$

where the generating operators of these series are cumulants of the groups of operators (2.1) of the Liouville equations

$$\mathfrak{A}_{1+n_1+n_2}(t, -(n_1+s_1), \ldots, -(s_1+1), \{-s_1, \ldots, s_2\}, s_2+1, \ldots, s_2+n_2) = \quad (5.25)$$

$$\sum_{\substack{\mathrm{P} : (-(n_1+s_1), \ldots, -(s_1+1), \{-s_1, \ldots, s_2\}, \\ s_2+1, \ldots, s_2+n_2) = \bigcup_i X_i}} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} S^*_{|\theta(X_i)|}(t, \theta(X_i)).$$

In the expansion for the $(1 + n_1 + n_2)$-th order cumulant of the groups of operators (2.1), the following notation is used: the symbol $\sum_{\mathrm{P}}$ denotes the sum over each ordered partition of P of the partially ordered set of indices $(-(n_1+s_1), \ldots, -(s_1+1), \{-s_1, \ldots, s_2\}, s_2 + 1, \ldots, s_2+n_2)$ into $|\mathrm{P}|$ nonempty partially ordered subsets $X_i$ that do not intersect each other, and the connected set $\{-s_1, \ldots, s_2\}$ belongs entirely to one of the subsets $X_i$.

We emphasize that cumulants (5.25) are solutions to the recursion relations known as the cluster expansions (4.13) of the groups of operators (2.1), namely

$$S^*_{n_1+s_1+s_2+n_2}(t, -(n_1+s_1), \ldots, -(s_1+1), -s_1, \ldots, s_2, s_2+1, \ldots, s_2+n_2)) = \quad (5.26)$$

$$\sum_{\substack{\mathrm{P} : (-(n_1+s_1), \ldots, -(s_1+1), \{-s_1, \ldots, s_2\}, \\ s_2+1, \ldots, s_2+n_2) \bigcup_i X_i}} \prod_{X_i \subset \mathrm{P}} \mathfrak{A}_{|X_i|}(t, X_i), \quad n_1 + n_2 \geq 1.$$

In terms of the introduced operators (5.21) in the space $L^1_\alpha$, the series expansions (5.24) of the sequence of reduced distribution functions have the following structure

$$F(t) = (I - \mathfrak{a}_-)^{-1}(I - \mathfrak{a}_+)^{-1}(I - S^*(t))^{-\mathbb{I}_\star} F(0). \quad (5.27)$$

For the initial reduced distribution functions of the space $L^1_\alpha$, the series (5.24) is defined. Since for the generating operators (5.25) of series expansions (5.24) in the space $L^1_{s_1+n_1+s_2+n_2}$ the following estimate holds:

$$\|\mathfrak{A}_{1+n_1+n_2}(t) \, f_{s_1+n_1+s_2+n_2}\|_{L^1_{s_1+n_1+s_2+n_2}} \leq 2^{n_1+n_2} \, \|f_{s_1+n_1+s_2+n_2}\|_{L^1_{s_1+n_1+s_2+n_2}},$$

then for the sequence of functions (5.24) in the space $L^1_\alpha$, under the condition $\alpha > 2$, we have the following estimate

$$\|F(t)\|_{L^1_\alpha} \leq c_\alpha \|F^0\|_{L^1_\alpha},$$

where $c_\alpha = (1 - \frac{2}{\alpha})^{-1}$. The parameter $\alpha$ can be interpreted as a quantity inversely proportional to the density of a system of many particles, that is, the average number of particles per unit volume.

Thus, for a non-perturbative solution (5.24) of the Cauchy problem for the BBGKY hierarchy of particle systems with topological interaction, the following theorem holds [23],[22].

**Theorem 5.1.** *If $F(0) \in L^1_\alpha$ is a double sequence of non-negative initial distribution functions, then for $t \in \mathbb{R}$ under the condition $\alpha > 2$, there exists a unique solution of the Cauchy problem for the BBGKY hierarchy (5.23) for particle systems with topological interaction, which is represented by series expansions (5.24). This is a strong solution for initial data from the subspace of finite sequences of continuously differentiable functions with compact supports. For arbitrary initial data from the space $L^1_\alpha$, one is a weak solution.*

The following criterion holds.

**Criterion 2.** *A solution of the Cauchy problem for the BBGKY hierarchy* (5.23) *for particle systems with topological interaction is represented by series expansions* (5.24) *if and only if the generating operators of series expansions* (5.24) *are solutions of cluster expansions* (5.26) *of the groups of operators* (2.1).

The necessary condition means that cluster expansions (5.26) are valid for groups of operators (2.1). These recurrence relations are derived from the definition (5.18) of reduced distribution functions, provided that they are represented as series expansions (5.24) for the solution of the Cauchy problem to the BBGKY hierarchy (5.23). The sufficient condition means that the infinitesimal generator of one-parameter mapping (5.27) coincides with the generator of the BBGKY hierarchy (5.23), which is the consequence of Theorem 2.

In the articles [11],[26], a solution of the Cauchy problem for the BBGKY hierarchy (5.23) was constructed in a reduced form, the expression of which in terms of operators (5.21) has the following structure

$$F(t) = (I - \mathfrak{a}_-)^{-1}(I - \mathfrak{a}_+)^{-1}S^*(t)(I - \mathfrak{a}_-)(I - \mathfrak{a}_+)F(0).$$

Component-wise, such a sequence of reduced distribution functions is represented by expansions in a series:

$$F_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-(n_1+s_1)} \ldots dx_{-(s_1+1)} \quad (5.28)$$

$$dx_{s_2+1} \ldots dx_{s_2+n_2} U_{1+n_1+n_2}(t, -(n_1+s_1), \ldots, -(s_1+1), \{-s_1 \ldots, -1, 1, \ldots, s_2\},$$
$$s_2 + 1, \ldots, s_2 + n_2)F^0_{n_1+s_1+s_2+n_2}(x_{-(n_1+s_1)}, \ldots, x_{s_2+n_2}), \quad s_1 + s_2 \geq 1.$$

were the generating operators $\{U_{1+n_1+n_2}(t)\}_{n_1+n_2 \geq 0}$, of series (5.28) are determined by the following expansions:

$$U_{1+n_1+n_2}(t, -(n_1+s_1), \ldots, -(s_1+1), \{-s_1, \ldots, s_2\}, s_2+1, \ldots, s_2+n_2) =$$

$$\sum_{k_1=0}^{\min(1,n_1)} \sum_{k_2=0}^{\min(1,n_2)} (-1)^{k_1+k_2} S^*_{n_1+s_1-k_1+s_2+n_2-k_2}(t, -(n_1+s_1-k_1), \ldots, s_2+n_2-k_2).$$

Such expressions are solutions of reduced cluster expansions of the groups of operators (2.1) in terms of cumulants (semi-invariants).

In the space $L^1_{\alpha}$ of sequences of integrable functions, due to the isometrics of the group of operators (2.1), the representations (5.24) and (5.28) of the solution of the Cauchy problem for the hierarchy of BBGKY equations (5.23) are equivalent. We emphasize that a traditional representation of the solution of the Cauchy problem for the BBGKY hierarchy is the series expansion of the perturbation theory [7],[12]. Under appropriate conditions for the initial data and the interaction potential of particles, the equivalence of the representations of these solutions is proved due to the validity of the analogs of the Duhamel equations for the generating operators of the series (5.28).

The characteristics of fluctuations of the mean value of observables on a macroscopic scale are directly determined by reduced correlation functions describing the state of a system of many particles on a microscopic scale [16].

Then an alternative approach to describing the evolution of topologically interacting particles is based on reduced correlation functions defined by the cumulant expansions

over the reduced distribution functions (5.18):

$$G_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \sum_{\mathrm{P}:(x_{-s_1}, \ldots, x_{s_2}) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} F_{|X_i|}(t, X_i), \quad (5.29)$$
$$s_1 + s_2 \geq 1,$$

or in terms of sequences of functions, these expansions are defined by the inverse mapping (3.7) and take the form

$$G(t) = I - (I + F(t))^{-\mathbb{I}_*},$$

where the used denotations are similar to that of the recursive equations (3.4).

Then the definition of reduced correlation functions is formulated based on the solution (4.16) of the Cauchy problem for the hierarchy of Liouville equations (4.11),(4.12) by the following expansion into a series:

$$G_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) \doteq \sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1, n_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{n_1+n_2}} dx_{-(n_1+s_1)} \cdots \qquad (5.30)$$
$$dx_{-(s_1+1)} dx_{s_2+1} \ldots dx_{s_2+n_2} \, g_{n_1+s_1+s_2+n_2}(t, x_{-(n_1+s_1)}, \ldots, x_{s_2+n_2}),$$
$$s_1 + s_2 \geq 1,$$

where the generating functions $g_{n_1+s_1+s_2+n_2}(t)$ are determined by the expansions (4.16). In terms of sequences of functions, the sequence of reduced correlation functions (5.30) has the following structure

$$G(t) = (I - \mathfrak{a}_-)^{-1}(I - \mathfrak{a}_+)^{-1} g(t).$$

We emphasize that the $(n_1 + n_2)$-th term of expansions (5.30) of reduced correlation functions is determined by the correlation function of $(n_1 + s_1 + s_2 + n_2)$ particles (4.16), in contrast to the expansions of reduced distribution functions (5.19), which are determined by the $(1 + n_1 + n_2)$-th correlation function of a particle cluster and particles (5.20).

Since the correlation functions $g_{n_1+s_1+s_2+n_2}(t)$ are determined by the hierarchy of Liouville equations (4.11), the reduced correlation functions (5.30) satisfy the hierarchy of nonlinear evolution equations.

Note that for many-particle systems whose states are described by symmetric distribution functions, a non-perturbative solution of the Cauchy problem to the BBGKY hierarchy was constructed in the works [17],[19, 20, 21].

## 6. The hierarchy of evolution equations for observables of topologically interacting particles

Because of the mean value functional of observables determines the duality of observables and the state of particle systems, we will consider another approach to describing the evolution of a system of a non-fixed number of ordered particles with the topological interaction of nearest neighbors. This equivalent approach is based on the concept of observables governed by the hierarchy of evolution equations for reduced observables.

In the case of a non-fixed number of many identical ordered particles interacting with their nearest neighbors through a potential defined above, the mean value of the double sequences of reduced observables $B_{n_1+n_2}(t)$, $n_1 + n_2 \geq 0$, is determined by a continuous linear functional represented by the following expansion into the series [26]

$$(B(t), F(0)) \doteq$$

$$\sum_{s=0}^{\infty} \sum_{\substack{s = s_1 + s_2 \\ s_1, s_2 \geq 0}} \int_{(\mathbb{R} \times \mathbb{R})^{s_1+s_2}} dx_{-s_1} \ldots dx_{s_2} B_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) F_{s_1+s_2}^0(x_{-s_1}, \ldots, x_{s_2}).$$

Let $C_{n_1+n_2} \equiv C(\mathbb{R}^{n_1+n_2} \times (\mathbb{R}^{n_1+n_2} \setminus \mathbb{W}_{n_1+n_2}))$ be the space of bounded continuous functions which are equal to zero on the set of forbidden configurations $\mathbb{W}_{n_1+n_2}$. The following operators are defined on sequences of functions $b = \{b_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2})\}_{n_1+n_2 \geq 0}$ from the space $b_{n_1+n_2} \in C_{n_1+n_2}$:

$$\left(\mathfrak{a}_+^+ b\right)_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2}) = b_{n_1+n_2-1}(x_{-n_1}, \ldots, x_{n_2-1}), \qquad (6.31)$$
$$\left(\mathfrak{a}_-^+ b\right)_{n_1+n_2}(x_{-n_2}, \ldots, x_{n_1}) = b_{n_1-1+n_2}(x_{-(n_1-1)}, \ldots, x_{n_2}),$$

which are analogs of the particle creation operator of quantum field theory, as well as the operators and their inverses:

$$\left(I - \mathfrak{a}_\pm^+\right)^{-1} = \sum_{n=0}^{\infty} (\mathfrak{a}_\pm^+)^n, \quad \left(I - \mathfrak{a}_\pm^+\right) = I - \mathfrak{a}_\pm^+.$$

In terms of operators (5.22) the mean value functional takes the form

$$\left(B(t), F(0)\right) = \left((I - \mathfrak{a}_-)^{-1}(I - \mathfrak{a}_+)^{-1} B(t) F(0)\right)_0. \qquad (6.32)$$

Note that operators (6.31) are adjoint to operators (5.21) in the sense of functional (6.32), i.e., $\left(\mathfrak{a}_\pm^+ b, f\right) = (b, \mathfrak{a}_\pm f)$.

On sequences of functions $b = \{b_{n_1+n_2}(x_{-n_1}, \ldots, x_{n_2})\}_{n_1+n_2 \geq 0}$ from the space $C_{n_1+n_2}$, the direct sum of the adjoint groups of operators to the operators (2.1) is defined, $\left(S(t) b, f\right) = \left(b, S^*(t) f\right)$, and, consequently,

$$S_{n_1+n_2}(t) = S_{n_1+n_2}^*(-t). \qquad (6.33)$$

Accordingly, on finite sequences of differential functions, a direct sum of the Liouville generators' adjoint to the operators (2.2) is defined, $\left(\mathcal{L} b, f\right) = \left(b, \mathcal{L}^* f\right)$, and, consequently, $\mathcal{L}_{n_1+n_2} = -\mathcal{L}_{n_1+n_2}^*$.

Recall [26] that a sequence of reduced observables are defined using the sequence of initial observables $A(0) = \{A_{n_1+n_2}^0(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2})\}_{n_1+n_2 \geq 0}$ and the groups of operators (6.33):

$$B(t) = (I - \mathfrak{a}_-^+)(I - \mathfrak{a}_+^+) S(t) A(0), \qquad (6.34)$$

or in a component-wise form reduced observables are represented by the following expansions:

$$B_{s_1+s_2}(t, x_{-s_2}, \ldots, x_{s_1}) = ((I - \mathfrak{a}_-^+)(I - \mathfrak{a}_+^+) S(t) A(0))_{s_1+s_2}(x_{-s_2}, \ldots, x_{s_1}) =$$

$$\sum_{n_1=0}^{\min(1,s_1)} \sum_{n_2=0}^{\min(1,s_2)} (-1)^{n_1+n_2} S_{s_1-n_1+s_2-n_2}(t) A_{s_1-n_1+s_2-n_2}^0(x_{-(s_1-n_1)}, \ldots, x_{s_2-n_2}).$$

The sequence of reduced observables (6.34) satisfies the hierarchy of evolution equations for reduced observables [7]. In terms of operators (6.31) this hierarchy has the following abstract form

$$\frac{\partial}{\partial t} B(t) = \left(I - \mathfrak{a}_-^+\right)\left(I - \mathfrak{a}_+^+\right) \mathcal{L} \left(I - \mathfrak{a}_-^+\right)^{-1} \left(I - \mathfrak{a}_+^+\right)^{-1} B(t),$$

or in the case of a pair interaction potential under consideration

$$\frac{\partial}{\partial t} B(t) = \mathcal{L} B(t) + [\mathcal{L}, \mathfrak{a}_-^+] B(t) + [\mathcal{L}, \mathfrak{a}_+^+] B(t), \qquad (6.35)$$

where the brackets $[\circ, \circ]$ denote the commutator of two operators.

Therefore, in a component-wise form, the Cauchy problem for the hierarchy of evolution equations (6.35) for reduced observables is represented as follows:

$$\frac{\partial}{\partial t} B_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \mathcal{L}_{s_1+s_2}(-s_1, \ldots, s_2) B_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) +$$
$$+ \mathcal{L}_{\text{int}}(-s_1, -s_1+1) B_{s_1-1+s_2}(t, x_{-s_1+1}, \ldots, x_{s_2}) +$$
$$+ \mathcal{L}_{\text{int}}(s_2-1, s_2) B_{s_1+s_2-1}(t, x_{-s_1}, \ldots, x_{s_2-1}),$$
$$B_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = B_{s_1+s_2}^0(x_{-s_1}, \ldots, x_{s_2}), \quad s_1 + s_2 \geq 1.$$

A solution to the Cauchy problem for the hierarchy of reduced observables in component-wise form is represented by the expansions:

$$B_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \sum_{n_1=0}^{s_1} \sum_{n_2=0}^{s_2} \mathfrak{A}_{1+s_1-n_1+s_2-n_2}(t, -s_1, \ldots, -(n_1+1), \quad (6.36)$$
$$\{-n_1, \ldots, -1, 1, \ldots, n_2\}, n_2+1, \ldots, s_2) B_{n_1+n_2}^0(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2}),$$
$$s_1 + s_2 \geq 1.$$

The generating operators of these expansions are corresponding-order cumulants of the group of operators (6.33)

$$\mathfrak{A}_{1+s_1-n_1+s_2-n_2}(t, -s_1, \ldots, -(n_1+1), \{-n_1, \ldots, -1, 1, \ldots, n_2\}, n_2+1, \ldots, s_2) =$$
$$(6.37)$$
$$\sum_{\substack{\mathrm{P} : (-s_1, \ldots, -(n_1+1), \{-n_1, \ldots, -1, 1, \ldots, n_2\}, \\ n_2+1, \ldots, s_2) = \bigcup_i X_i}} (-1)^{|\mathrm{P}|-1} \prod_{X_i \subset \mathrm{P}} S_{|\theta(X_i)|}(t, \theta(X_i)),$$

where the notations adopted in formula (3.8) are used. We observe that cumulants (6.37) are solutions to the recursion relations known as cluster expansions of the groups of operators (6.33):

$$S_{s_1+s_2}(t, -s_1, \ldots, -(n_1+1), -n_1, \ldots, -1, 1, \ldots, n_2, n_2+1, \ldots, s_2) = \quad (6.38)$$
$$\sum_{\substack{\mathrm{P} : (-s_1, \ldots, -(n_1+1), \{-n_1, \ldots, -1, 1, \ldots, n_2\}, \\ n_2+1, \ldots, s_2) = \bigcup_i X_i}} \prod_{X_i \subset \mathrm{P}} \mathfrak{A}_{|X_i|}(t, X_i),$$

In terms of the introduced above operators, the series expansions representing a solution of the Cauchy problem to the hierarchy of evolution equations for reduced observables has the following structure

$$B(t) = \big(I - \mathfrak{a}_-^+\big)\big(I - \mathfrak{a}_+^+\big)\big(I - (I + S(t))^{-\mathbb{I}_\star}\big) B(0). \quad (6.39)$$

The following criterion holds.

**Criterion 3.** *A solution of the Cauchy problem of the hierarchy of reduced observables* (6.35) *is represented by expansions* (6.36) *if and only if the generating operators of expansions* (6.36) *are solutions of cluster expansions* (6.38) *of the groups of operators* (6.33).

The necessary condition means that cluster expansions (6.38) are valid for groups of operators (6.33). These recurrence relations are derived from the definition (6.34) of reduced observables provided that they are represented as expansions (6.36) for the solution of the Cauchy problem of the hierarchy of evolution equations for reduced observables (6.35).

The sufficient condition means that the infinitesimal generator of one-parameter mapping (6.39) coincides with the hierarchy of evolution equations generator for reduced observables (6.35).

**Theorem 6.1.** *A non-perturbative solution of the Cauchy problem of the hierarchy of evolution equations for reduced observables* (6.35) *is represented by expansions* (6.36) *in which the generating operators are cumulants of the corresponding order* (6.37) *of the groups of operators* (6.33). *For initial data* $B(0) \in C_0$ *of finite sequences of infinitely differentiable functions with compact supports sequence* (6.36) *is a unique global-in-time classical solution and for arbitrary initial data* $B(0) \in C_\gamma$ *is a unique global-in-time generalized solution.*

Note that cluster expansions (6.38) of the groups of operators (6.33) underlie the classification of possible non-perturbative solution representations of the Cauchy problem of the hierarchy of evolution equations for reduced observables (6.35). For example, solutions of the recursive relations (6.38) to first-order cumulants can be represented as expansions in terms of cumulants acting on variables on which initial reduced observables depend, and in terms of cumulants not acting on these variables. Then the component-wise sequence of reduced observables (6.36) is represented by such expansions into series:

$$
B_{s_1+s_2}(t, x_{-s_1}, \ldots, x_{s_2}) = \sum_{n_1=0}^{s_1} \sum_{n_2=0}^{s_2} \sum_{k_1=0}^{\min(1,s_1-n_1)} \sum_{k_2=0}^{\min(1,s_2-n_2)} (-1)^{k_1+k_2} \, S_{s_1-k_1+s_2-k_2}(t,
$$
$$
-(s_1-k_1), \ldots, -1, 1, \ldots, s_2-k_2) B_{n_1+n_2}^0(x_{-n_1}, \ldots, x_{-1}, x_1, \ldots, x_{n_2}), \quad s_1+s_2 \geq 1.
$$

In terms of operators (6.31), the series expansions of sequence of reduced observables have the following structure:

$$
B(t) = (I - \mathfrak{a}_-^+)(I - \mathfrak{a}_+^+) S(t)(I - \mathfrak{a}_-^+)^{-1}(I - \mathfrak{a}_+^+)^{-1} B(0).
$$

We note that in the case of many-particle systems whose observables are symmetric functions, the structure of corresponding cluster and cumulant expansions was analyzed in [16],[19].

## 7. CONCLUSION

The structure of expansions into series (5.24) that represent a non-perturbative solution of the Cauchy problem of the BBGKY hierarchy (5.23) for a state of many particles with topological nearest-neighbor interactions as well as expansions (6.36) that represent a non-perturbative solution of the dual hierarchy (6.35) for observables was above justified.

Non-perturbative solutions to these hierarchies are represented in the form of expansions over the particle clusters whose evolution is governed by the corresponding order cumulants (5.25) of the groups of operators (2.1) or cumulants (6.37) of the groups of adjoint operators (6.33) for finitely many particles with the topological nearest-neighbor interaction. Cluster expansions (5.26) of the groups of operators (2.1) and cluster expansions (6.38) of the groups of adjoint operators (6.33) underlie the classification of all possible solution representations (5.24) and (6.36) of the BBGKY hierarchy (5.23) and the dual hierarchy (6.35), respectively.

Thus, the notion of cumulants of the groups of operators (6.37) or (5.25) is the basis for non-perturbative expansions solutions to hierarchies of fundamental evolution equations, which respectively describe the evolution of observables and states of many particles with topological nearest-neighbor interactions.

A new approach is presented to describe the evolution of correlations for finitely many particles with topological nearest-neighbor interaction. This approach is based on a concept of the cumulants of the probability distribution functions, which are governed by the Liouville hierarchy (4.11). It was established that the constructed dynamics of correlations underlie the description of the evolution of the states of infinitely many particles within the framework of reduced distribution functions (5.19) or reduced correlation functions (5.30), namely, cumulants of reduced distribution functions (5.29). Let us emphasize that

the correlations created during the evolution of a system of particles with topological interaction (5.20) by nature differ from the structure of correlations of particle systems whose state is traditionally described by symmetric over to the phase variables distribution functions [17].

In conclusion, we note that the statistical properties of physical systems are simulated by the collective behavior of infinitely many particles [7]. It should also note the importance of a mathematical description of the processes of creation and propagation of correlations, in particular, for numerous applications. In the case of systems of colliding particles, the dynamics of correlations were constructed in [17], and for quantum many-particle systems in [13, 14, 15].

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