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MULTI-DIMENSIONAL MATRIX CHARACTERIZATION OF $(\mathcal{L}_1, \mathcal{L}_1)$ AND MERCERIAN-TYPE THEOREM VIA MATRIX PRODUCT

SAMI M. HAMID AND RICHARD F. PATTERSON

Abstract. This paper examines four-dimensional matrices in $(\mathcal{L}_1, \mathcal{L}_1)$ under standard matrix product. Using established characterizations of $(\mathcal{L}_1, \mathcal{L}_1; P)$, we demonstrate that $(\mathcal{L}_1, \mathcal{L}_1)$ forms a Banach algebra under standard matrix operations. We prove that $(\mathcal{L}_1, \mathcal{L}_1; P)$ is a closed, convex semigroup with identity under matrix product. Finally, we present a Mercerian-type theorem for four-dimensional matrices via matrix product.

1. Introduction

The study of four-dimensional matrix transformations has developed steadily over many decades. The initial investigations by Hamilton [6] and Robison [11] examined the fundamental properties of matrix regularity in four dimensions. Building on this foundation, Móricz and Rhoades [7] advanced the theory with their work on strongly regular matrices for double sequences. Zeltser et al. [16] then studied the characteristics of almost conservative and almost regular four-dimensional matrices. Başar and Yeşilkayagil [1] later contributed a detailed work covering various matrix types and results on Mercerian and Steinhaus-type theorems. Recent developments in double sequence theory can be found in [2], [3], [4], [12], [13], [14] and [15]. Important theoretical advances were made by Mursaleen and Mohiuddine [8], who developed new techniques to study double sequence convergence.

The space of infinite matrices mapping $\mathcal{L}_1$ into itself, denoted as $(\mathcal{L}_1, \mathcal{L}_1)$, has been well studied in the two-dimensional case, with Natarajan [9] establishing key results on its algebraic structure under both standard and convolution products. This paper extends this work to the four-dimensional setting under the standard matrix product in an analogous space, which we also denote as $(\mathcal{L}_1, \mathcal{L}_1)$ following the established notation in the literature. In our recent work [5], we studied the space of four-dimensional matrices under the convolution operation and established key results about its algebraic structure. The present paper examines this space under the standard matrix product.

The paper is organized as follows: We begin by presenting preliminary definitions and known results to provide the necessary background. Building on our previous characterization of $(\mathcal{L}_1, \mathcal{L}_1; P)$, we focus on the algebraic properties of $(\mathcal{L}_1, \mathcal{L}_1)$ and $(\mathcal{L}_1, \mathcal{L}_1; P)$ under the standard matrix product. We then present a Mercerian-type theorem for four-dimensional matrices in $(\mathcal{L}_1, \mathcal{L}_1)$ utilizing the standard matrix product.

2. Preliminaries and Definitions

Definition 2.1. By Ω, we denote the space of all real or complex-valued double sequences $x = (x_{k,l})_{k,l\geq 0}$, which forms a vector space with coordinatewise addition and scalar multiplication.

Next, we define the space of absolutely summable double sequences, which plays a central role in our study.

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Definition 2.2. The space of absolutely summable double sequences, denoted by $\mathcal{L}_1$, is defined as

$$\mathcal{L}_1 = \left\{ x = (x_{k,l})_{k,l \geq 0} \in \Omega : \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}| < \infty \right\}.$$

The space $\mathcal{L}_1$ is a Banach space with the norm $\|x\|_{\mathcal{L}_1} = \sum_{k,l=0,0}^{\infty,\infty} |x_{k,l}|$.

In this paper, we focus on four-dimensional infinite matrices and their transformations of double sequences. We now state the key concepts related to these matrices.

Definition 2.3. Let $A = (a_{m,n,k,l})_{m,n,k,l \geq 0}$ be a four-dimensional infinite matrix. The A-transform of a double sequence $x = (x_{k,l})_{k,l \geq 0}$ is the double sequence $Ax = \{(Ax)_{m,n}\}$, where

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}, \quad m, n \geq 0$$

assuming that the double series on the right exists.

We now define the class of matrices that map $\mathcal{L}_1$ into itself.

Definition 2.4. We write $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1)$ if for every $x = (x_{k,l}) \in \mathcal{L}_1$, the double sequence $Ax = \{(Ax)_{m,n}\}$ belongs to $\mathcal{L}_1$.

A particularly important subclass of $(\mathcal{L}_1, \mathcal{L}_1)$, denoted as $(\mathcal{L}_1, \mathcal{L}_1; P)$, which we define as follows.

Definition 2.5. The class $(\mathcal{L}_1, \mathcal{L}_1; P)$ is defined as the set of all four-dimensional matrices $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1)$ that satisfy the additional property

$$\sum_{m,n=0,0}^{\infty,\infty} (Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} x_{k,l}$$

for all $x = (x_{k,l}) \in \mathcal{L}_1$.

3. Characterization of $(\mathcal{L}_1, \mathcal{L}_1; P)$

We begin this section by stating a fundamental theorem by Patterson in [10], which characterizes the class of four-dimensional matrices that map $\mathcal{L}_1$ into itself. The characterization of the class $(\mathcal{L}_1, \mathcal{L}_1; P)$, also established in [5, Theorem 3.2], is then presented as it forms the foundation for our subsequent analysis.

Theorem 3.1. ([10, Theorem 6]). *A four-dimensional matrix $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1)$ if and only if there exists a positive constant M_A such that for each k and l,*

$$\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < M_A.$$

Theorem 3.2. ([5, Theorem 3.2]). *Let $A = (a_{m,n,k,l})$ be a four-dimensional matrix. Then $A \in (\mathcal{L}_1, \mathcal{L}_1; P)$ if and only if*

(i) *there exists $M > 0$ such that* $\sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| \leq M$, *and*

(ii) *for all $k, l \geq 0$,* $\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$.

4. Algebraic Properties of $(\mathcal{L}_1, \mathcal{L}_1)$ and $(\mathcal{L}_1, \mathcal{L}_1; P)$

In this section, we define a norm on $(\mathcal{L}_1, \mathcal{L}_1)$ and then proceed to show that $(\mathcal{L}_1, \mathcal{L}_1)$ equipped with this norm and the standard four-dimensional matrix product, forms a Banach algebra. Finally, we examine the properties of $(\mathcal{L}_1, \mathcal{L}_1; P)$ under the standard four-dimensional matrix product as a subset of $(\mathcal{L}_1, \mathcal{L}_1)$.

Theorem 4.1. *Let $A = (a_{m,n,k,l})$ be a four-dimensional matrix in $(\mathcal{L}_1, \mathcal{L}_1)$. Define the function ϕ by*

$$\begin{aligned} \phi : (\mathcal{L}_1, \mathcal{L}_1) &\longrightarrow \mathbb{R} \\ A &\longmapsto \phi(A) = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|. \end{aligned}$$

Then ϕ is a norm on the space of four-dimensional matrices in $(\mathcal{L}_1, \mathcal{L}_1)$.

Proof. Since the positive definiteness, absolute homogeneity, and the triangle inequality properties of a norm are easily obtained from the ϕ's definition containing the supremum and absolute value properties, we omit the details. □

Definition 4.2 (Norm for Four-Dimensional Matrices in $(\mathcal{L}_1, \mathcal{L}_1)$)**.** For any four-dimensional matrix A in $(\mathcal{L}_1, \mathcal{L}_1)$, we define $\|A\| = \phi(A)$, where ϕ is the function proven to be a norm in Theorem 4.1. This $\|\cdot\|$ is adopted as the standard norm for the space of four-dimensional matrices in $(\mathcal{L}_1, \mathcal{L}_1)$.

Next, we define the four-dimensional matrix product operation, the four-dimensional identity matrix, and the concept of matrix inverse in this context.

Definition 4.3 (Product of Four-Dimensional Infinite Matrices [1])**.** The product $AB = C = (c_{i,j,m,n})$ of the four-dimensional infinite matrices $A = (a_{i,j,k,l})$ and $B = (b_{k,l,m,n})$ is defined by

$$c_{i,j,m,n} := \sum_{k,l=0,0}^{\infty,\infty} a_{i,j,k,l} b_{k,l,m,n} \quad \text{for all } i,j,m,n \geq 0$$

provided the double series on the right-hand side converges for each $i,j,m,n \geq 0$.

Definition 4.4 (Four-Dimensional Identity Matrix [1])**.** The four-dimensional identity matrix $I = (\delta_{i,j,k,l})$ is defined as:

$$\delta_{i,j,k,l} = \begin{cases} 1, & \text{if } (i,j) = (k,l); \\ 0, & \text{otherwise,} \end{cases}$$

for all $i,j,k,l \geq 0$.

Lemma 4.5. *The four-dimensional unit matrix $I = (\delta_{m,n,k,l})$ is an element of $(\mathcal{L}_1, \mathcal{L}_1; P)$ and serves as its identity element under the four-dimensional standard matrix product.*

Proof. To show that $I \in (\mathcal{L}_1, \mathcal{L}_1)$, we need to prove that $\|I\| < \infty$. Now

$$\|I\| = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty,\infty} |\delta_{m,n,k,l}| < \infty.$$

For I to be in $(\mathcal{L}_1, \mathcal{L}_1; P)$, we need to show that $\sum_{m,n=0,0}^{\infty,\infty} \delta_{m,n,k,l} = 1$ for all $k,l \geq 0$.

Now

$$\sum_{m,n=0,0}^{\infty,\infty} \delta_{m,n,k,l} = \delta_{k,l,k,l} + \sum_{(m,n) \neq (k,l)} \delta_{m,n,k,l} = 1.$$

This holds for all $k,l \geq 0$. Therefore, $I \in (\mathcal{L}_1, \mathcal{L}_1, P)$.

To prove that I is the identity element of $(\mathcal{L}_1, \mathcal{L}_1, P)$, we need to show that for any $A = (a_{m,n,k,l}) \in (\mathcal{L}_1, \mathcal{L}_1, P)$, both $IA = A$ and $AI = A$. The (m, n, k, l)-th element of IA is given by

$$\begin{aligned}(IA)_{m,n,k,l} &= \sum_{i,j=0,0}^{\infty,\infty} \delta_{m,n,i,j} a_{i,j,k,l} \\ &= \delta_{m,n,m,n} a_{m,n,k,l} + \sum_{(i,j)\neq(m,n)} \delta_{m,n,i,j} a_{i,j,k,l} = a_{m,n,k,l}.\end{aligned}$$

Similarly, computing $(AI)_{m,n,k,l}$ yields $(AI)_{m,n,k,l} = a_{m,n,k,l}$. Therefore, $IA = AI = A$ for all $A \in (\mathcal{L}_1, \mathcal{L}_1, P)$, proving that I is the identity element of $(\mathcal{L}_1, \mathcal{L}_1, P)$ under the standard four-dimensional matrix product. ☐

Definition 4.6 (Inverse of a Four-Dimensional Matrix [1])**.** If the four-dimensional matrices $A = (a_{i,j,k,l})$ and $B = (b_{i,j,k,l})$ are commutative and their product is equal to the identity, i.e., $AB = BA = I$, then the matrix B is called the inverse of the matrix A and is written $B = A^{-1}$.

We now prove some important properties of the matrix product operation in $(\mathcal{L}_1, \mathcal{L}_1)$.

Lemma 4.7. *If $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l})$ are in $(\mathcal{L}_1, \mathcal{L}_1)$, then their product AB is also in $(\mathcal{L}_1, \mathcal{L}_1)$. Moreover, the norm in $(\mathcal{L}_1, \mathcal{L}_1)$ is submultiplicative, i.e., $\|AB\| \leq \|A\|\|B\|$.*

Proof. Let $A = (a_{m,n,k,l})$ and $B = (b_{m,n,k,l})$ be in $(\mathcal{L}_1, \mathcal{L}_1)$, then by definition,

$$\sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \infty \quad \text{and} \quad \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |b_{m,n,k,l}| < \infty.$$

Let $C = AB = (c_{m,n,k,l})$, then $c_{m,n,k,l} = \sum_{i,j=0,0}^{\infty,\infty} a_{m,n,i,j} b_{i,j,k,l}$ and

$$\begin{aligned}\sum_{m,n=0,0}^{\infty,\infty} |c_{m,n,k,l}| &= \sum_{m,n=0,0}^{\infty,\infty} \left| \sum_{i,j=0,0}^{\infty,\infty} a_{m,n,i,j} b_{i,j,k,l} \right| \leq \sum_{m,n=0,0}^{\infty,\infty} \sum_{i,j=0,0}^{\infty,\infty} |a_{m,n,i,j}||b_{i,j,k,l}| \\ &= \sum_{i,j=0,0}^{\infty,\infty} \left(\sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,i,j}| \right) |b_{i,j,k,l}| \\ &\leq \left(\sup_{i,j\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,i,j}| \right) \left(\sum_{i,j=0,0}^{\infty,\infty} |b_{i,j,k,l}| \right) \leq \|A\|\|B\| < \infty.\end{aligned}$$

Since this bound is independent of k and l, we have

$$\|C\| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |c_{m,n,k,l}| \leq \|A\|\|B\| < \infty,$$

and therefore, $AB = C \in (\mathcal{L}_1, \mathcal{L}_1)$ and that the norm is submultiplicative. ☐

The following lemma regarding the completeness of $(\mathcal{L}_1, \mathcal{L}_1)$ was established in [5, Lemma 4.8] and is fundamental for our subsequent results.

Lemma 4.8 (Completeness of $(\mathcal{L}_1, \mathcal{L}_1)$)**.** *The space $(\mathcal{L}_1, \mathcal{L}_1)$ of four-dimensional matrices $A = (a_{m,n,k,l})$ with the norm $\|A\| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ is complete.*

With these properties established, we can now prove our main result in this section about the algebraic structure of $(\mathcal{L}_1, \mathcal{L}_1)$.

Theorem 4.9. *The class $(\mathcal{L}_1, \mathcal{L}_1)$ of four-dimensional matrices $A = (a_{m,n,k,l})$ is a Banach algebra under the norm $\|A\| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |a_{m,n,k,l}|$, with the usual matrix addition, scalar multiplication, and the four-dimensional matrix product.*

Proof. We begin by observing that $(\mathcal{L}_1, \mathcal{L}_1)$ forms a vector space over the complex field. The closure under addition and scalar multiplication follows directly from the properties of absolute convergence of the series defining the norm. The vector space axioms are readily verified. In Definition 4.2, the norm on $(\mathcal{L}_1, \mathcal{L}_1)$ has been defined and shown to satisfy all norm axioms in Theorem 4.1. Therefore, $(\mathcal{L}_1, \mathcal{L}_1)$ is a normed linear space under this norm. The completeness of $(\mathcal{L}_1, \mathcal{L}_1)$ under this norm has been established in Lemma 4.8, thus confirming that $(\mathcal{L}_1, \mathcal{L}_1)$ is indeed a Banach space. To show that $(\mathcal{L}_1, \mathcal{L}_1)$ is an algebra, we need to demonstrate closure under the standard four-dimensional matrix product and the continuity of this operation. The closure of $(\mathcal{L}_1, \mathcal{L}_1)$ under matrix product and the submultiplicativity of the norm have been established in Lemma 4.7. This lemma also demonstrates that the product operation is continuous, as required for a Banach algebra. The associativity of matrix product follows from the associativity of series multiplication for absolutely convergent series, and the distributivity over addition is a consequence of the linearity of limits of absolutely convergent series.

Thus, we have shown that $(\mathcal{L}_1, \mathcal{L}_1)$ is a vector space, equipped with a complete norm that is submultiplicative with respect to the defined product operation. The algebra operations are continuous under this norm, as demonstrated by the submultiplicativity property. Therefore, $(\mathcal{L}_1, \mathcal{L}_1)$ satisfies all the requirements of a Banach algebra. □

We now examine the properties of the subclass $(\mathcal{L}_1, \mathcal{L}_1; P)$.

Theorem 4.10. *The class $(\mathcal{L}_1, \mathcal{L}_1; P)$, as a subset of $(\mathcal{L}_1, \mathcal{L}_1)$, is a closed, convex semigroup with identity, the product being the usual four-dimensional matrix product.*

Proof. The convexity and closedness of $(\mathcal{L}_1, \mathcal{L}_1; P)$ were established in [5, Theorem 4.10]. As shown in Lemma 4.5, the identity element for the four-dimensional matrix product was shown to be in $(\mathcal{L}_1, \mathcal{L}_1; P)$. To complete the proof of the theorem, it suffices to check closure under matrix product. Let $A = (a_{m,n,i,j})$ and $B = (b_{i,j,k,l})$ be matrices in $(\mathcal{L}_1, \mathcal{L}_1; P)$. We need to show that their product AB is also in $(\mathcal{L}_1, \mathcal{L}_1; P)$. The (m, n, k, l)-th element of AB is given by

$$(AB)_{m,n,k,l} = \sum_{i,j=0,0}^{\infty,\infty} a_{m,n,i,j} b_{i,j,k,l}.$$

Now, we need to show that $\sum_{m,n=0,0}^{\infty,\infty} (AB)_{m,n,k,l} = 1$ for all $k, l \geq 0$. Now for a fixed $k, l \geq 0$, we have

$$\begin{aligned}\sum_{m,n=0,0}^{\infty,\infty} (AB)_{m,n,k,l} &= \sum_{m,n=0,0}^{\infty,\infty} \sum_{i,j=0,0}^{\infty,\infty} a_{m,n,i,j} b_{i,j,k,l} \\ &= \sum_{i,j=0,0}^{\infty,\infty} \left(\sum_{m,n=0,0}^{\infty,\infty} a_{m,n,i,j} \right) b_{i,j,k,l} = 1.\end{aligned}$$

The interchange of summation order is justified by the absolute convergence of the series involved, which follows from the fact that $A, B \in (\mathcal{L}_1, \mathcal{L}_1)$. This shows that

$AB \in (\mathcal{L}_1, \mathcal{L}_1; P)$. Therefore, $(\mathcal{L}_1, \mathcal{L}_1; P)$ is a closed, convex semigroup with identity under the standard four-dimensional matrix product. □

5. Mercerian-type Theorem for Four-Dimensional Matrices

We first establish a connection between double sequences in $\mathcal{L}_1$ and four-dimensional matrices in $(\mathcal{L}_1, \mathcal{L}_1)$. We begin by defining a correspondence and then proving its properties in the following lemma.

Definition 5.1. Define a correspondence Φ between $\mathcal{L}_1$ and a subset of $(\mathcal{L}_1, \mathcal{L}_1)$ as follows: For any $(z_{m,n}) \in \mathcal{L}_1$, let $\Phi((z_{m,n})) = Z = (z_{m,n,k,l})$ where

$$z_{m,n,k,l} = \begin{cases} z_{m,n}, & \text{if } k = m \text{ and } l = n; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.2. *The correspondence Φ given by Definition 5.1 is a bijection between $\mathcal{L}_1$ and a subset of $(\mathcal{L}_1, \mathcal{L}_1)$. Moreover, this correspondence is norm-preserving, i.e.,*

$$\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}| = \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}|$$

for all $(z_{m,n}) \in \mathcal{L}_1$.

Proof. The well-definedness and injectivity properties of Φ are immediate consequences of its definition and fundamental properties of summation and supremum. As for the surjectivity of Φ, given such a Z, define $(z_{m,n})$ by $z_{m,n} = z_{m,n,m,n}$ for all $m, n \geq 0$. We show that $(z_{m,n}) \in \mathcal{L}_1$ and that $\Phi((z_{m,n})) = Z$. First, $(z_{m,n}) \in \mathcal{L}_1$ because

$$\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}| = \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,m,n}| \leq \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}| < \infty.$$

Now, we show that $\Phi((z_{m,n})) = Z$. Let $\Phi((z_{m,n})) = Y = (y_{m,n,k,l})$. By definition of Φ,

$$y_{m,n,k,l} = \begin{cases} z_{m,n}, & \text{if } k = m \text{ and } l = n; \\ 0, & \text{otherwise.} \end{cases}$$

But this is exactly how Z is defined, so $Y = Z$.

For the norm-preserving property of Φ, let $(z_{m,n}) \in \mathcal{L}_1$. For any $k, l \geq 0$,

$$\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}| = |z_{k,l}|.$$

Taking the supremum over all k and l,

$$\sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}| = \sup_{k,l\geq 0} |z_{k,l}| \leq \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}|, \tag{1}$$

where the last inequality holds because for any $k, l \geq 0$, we have $|z_{k,l}| \leq \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}|$, and since $|z_{k,l}|$ is one term in the sum $\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}|$. Since this is true for all $k, l \geq 0$, it must also be true for the supremum $\sup_{k,l\geq 0} |z_{k,l}| \leq \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}|$. On the other hand, we also granted

$$\sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}| = \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,m,n}| \leq \sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}|, \tag{2}$$

where the equality follows from the definition of $z_{m,n,k,l}$, and the inequality holds because the left-hand side is one of the sums considered in the supremum on the right. Combining inequalities (1) and (2) yields the desired result $\sup_{k,l\geq 0} \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n,k,l}| = \sum_{m,n=0,0}^{\infty,\infty} |z_{m,n}|$. Therefore, the correspondence Φ is a norm-preserving bijection between $\mathcal{L}_1$ and the subset of $(\mathcal{L}_1, \mathcal{L}_1)$ defined by the form of Z in the definition of Φ. □

With this bijective correspondence established, we can now state and prove our main theorem.

Theorem 5.3 (Mercerian-type Theorem under Standard Product)**.** *Let $(y_{m,n})$ and $(x_{m,n})$ be double sequences of complex numbers related by*

$$y_{m,n} = x_{m,n} + \lambda \sum_{k=0}^{m} \sum_{l=0}^{n} c^{m-k} d^{n-l} x_{k,l},$$

where λ, c, and d are complex numbers satisfying $|c| < 1$, $|d| < 1$, and $(y_{m,n}) \in \mathcal{L}_1$. Then $(x_{m,n}) \in \mathcal{L}_1$ provided $|\lambda| < (1 - |c|)(1 - |d|)$.

Proof. We represent the double sequences $(x_{m,n})$ and $(y_{m,n})$ as four-dimensional matrices

$$X_{m,n,k,l} = \begin{cases} x_{k,l}, & \text{if } m = k \text{ and } n = l; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$Y_{m,n,k,l} = \begin{cases} y_{m,n}, & \text{if } k = m \text{ and } l = n; \\ 0, & \text{otherwise.} \end{cases}$$

Define the four-dimensional matrix A as

$$A_{m,n,k,l} = \begin{cases} c^{m-k} d^{n-l}, & \text{if } k \leq m \text{ and } l \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

The (m, n, k, l)-th element of AX is given by

$$(AX)_{m,n,k,l} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{m,n,i,j} X_{i,j,k,l}.$$

By the definitions of A and X, this sum is non-zero only when $i \leq m$, $j \leq n$, $i = k$, and $j = l$, so we can rewrite it as

$$(AX)_{m,n,k,l} = \sum_{i=0}^{m} \sum_{j=0}^{n} A_{m,n,i,j} X_{i,j,i,j} = \sum_{i=0}^{m} \sum_{j=0}^{n} c^{m-i} d^{n-j} x_{i,j}.$$

The (m, n, k, l)-th element of $X + \lambda AX$ is

$$(X + \lambda AX)_{m,n,k,l} = X_{m,n,k,l} + \lambda (AX)_{m,n,k,l},$$

which is non-zero only when $m = k$ and $n = l$. In this case, we have

$$(X + \lambda AX)_{m,n,m,n} = X_{m,n,m,n} + \lambda (AX)_{m,n,m,n}.$$

Substituting the definitions of X and AX, we get

$$(X + \lambda AX)_{m,n,m,n} = x_{m,n} + \lambda \sum_{i=0}^{m} \sum_{j=0}^{n} c^{m-i} d^{n-j} x_{i,j}.$$

Thus, we have $Y_{m,n,k,l} = (X + \lambda AX)_{m,n,k,l}$ for all m, n, k, l. This equality demonstrates that our original equation

$$y_{m,n} = x_{m,n} + \lambda \sum_{k=0}^{m} \sum_{l=0}^{n} c^{m-k} d^{n-l} x_{k,l}$$

is equivalent to the matrix equation $Y = X + \lambda AX$.

Now

$$\|A\| = \sup_{k,l \geq 0} \sum_{m,n=0,0}^{\infty} |A_{m,n,k,l}| = \sup_{k,l \geq 0} \sum_{m=k}^{\infty} \sum_{n=l}^{\infty} |c|^{m-k} |d|^{n-l}$$
$$= \sup_{k,l \geq 0} \left(\sum_{m=k}^{\infty} |c|^{m-k} \right) \left(\sum_{n=l}^{\infty} |d|^{n-l} \right) = \left(\frac{1}{1 - |c|} \right) \left(\frac{1}{1 - |d|} \right).$$

We have previously established in Theorem 4.9 that $(\mathcal{L}_1, \mathcal{L}_1)$ is a Banach algebra under four-dimensional matrix product. Therefore, if $|\lambda| < \|A\|^{-1} = (1 - |c|)(1 - |d|)$, then $I + \lambda A$ has an inverse in $(\mathcal{L}_1, \mathcal{L}_1)$. When $|\lambda| < (1 - |c|)(1 - |d|)$, we can solve for X

$$X = (I + \lambda A)^{-1} Y.$$

Since $Y \in (\mathcal{L}_1, \mathcal{L}_1)$ (as $(y_{m,n}) \in \mathcal{L}_1$) and $(I + \lambda A)^{-1} \in (\mathcal{L}_1, \mathcal{L}_1)$, we conclude that $X \in (\mathcal{L}_1, \mathcal{L}_1)$. By Lemma 5.2, this implies $(x_{m,n}) \in \mathcal{L}_1$, completing the proof. $\square$

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Sami M. Hamid: shamid@unf.edu
Department of Mathematics & Statistics, University of North Florida, Jacksonville, FL 32224, USA

Richard F. Patterson: rpatters@unf.edu
Department of Mathematics & Statistics, University of North Florida, Jacksonville, FL 32224, USA

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