Methods of Functional Analysis and Topology
Vol. 31 (2025), no. 1, pp. 39–46
doi.org/10.31392/MFAT-npu26_1.2025.04

METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

$\mathcal{N}_F^{\alpha}$-FRACTIONAL SEMI-GROUPS OF OPERATORS

BAHLOUL RACHID, RACHAD HOUSSAME, AND THABET ABDELJAWAD

ABSTRACT. Based on the new definition of the $\mathcal{N}_F^{\alpha}$-derivative function introduced by Juan E. Nápoles Valdés et al. (2020) in [1], we give a new definition and some results of the $\mathcal{N}_F^{\alpha}$-fractional semi-groups of operators.

1. Introduction

Let X be a Banach space, and $L(X)$ be the space of all bounded linear operators on X. A family $\{T(t), t \geq 0\} \subset L(X)$ is called a semigroup of operators if:

(1) $T(0) = I$
(2) $T(s+t) = T(s)T(t)$
(3) $T(t)x \xrightarrow[t \to 0^+]{} x$.

The name semigroup comes from the fact that the family $\{T(t)\}_{t \geq 0}$ forms a semigroup in the algebraic sense.
Since E. Hille [2], K. Yoshida [[3], [4]], Pazy [5], Nathan [6] and Toukmati [7]established the characterization of generators of C_0-semigroups in the 1940s, semigroups of linear operators and its neighboring areas have developed into a beautiful abstract theory. Moreover, the fact that mathematically this abstract theory has many direct and important applications in partial differential equations enhances its importance as a necessary discipline in both functional analysis and differential equations.

The conformable derivative operator was introduced in the literature by Khalil et al (2014) [8] to allow integrating and differentiating with respect to arbitrary order without having memory in the structure and hence falling in a similar category to local fractional calculus and fractal calculus. Since then, many classical problems have been generalized to the conformable case [[9], [1]].

The rest of this paper is structured as follows : In section 2, we introduce the basic definitions and properties of $\mathcal{N}_F^{\alpha}$-derivative, define by Juan E. Napoles Valdés et al [1]. In section 3, we give some results of N_F^{α}-derivative and of $\mathcal{N}_F^{\alpha}$-Fractional Semigroups where we show that:

(1) $\mathcal{N}_F^{\alpha}$-Rolle's Theorem and $\mathcal{N}_F^{\alpha}$-Mean Value Theorem of $\mathcal{N}_F^{\alpha}$-fractional derivative.

(2) Let $(T(t))_{t \geq 0}$ is a $\mathcal{N}_F^{\alpha}-$semigroup, with generator A, then we have :
   (a) $\mathcal{N}_F^{\alpha} T(t)x = AT(t)x = T(t)Ax$

   (b) $\mathcal{N}_F^{\alpha}(\mathcal{I}_F^{\alpha} T(t)x) = T(t)F(t,\alpha)x - F(0,\alpha)x$.

The conclusion is presented in Section 4.

2020 *Mathematics Subject Classification.* 45N05, 44A10, 43A15, 44A35, 43A25, 43A50, 45D05.
Keywords. Conformable derivative, $\mathcal{N}_F^{\alpha}$-derivative, $\mathcal{N}_F^{\alpha}$-Semigroup.

2. Basic definitions and tools

In this section, we introduce the definition of N_F^α-derivative and their important properties.

Definition 2.1. [1] Let $\alpha \in (0,1]$ and $f : [0,+\infty[\to \mathbb{R}$. Then the N_F^α-derivative of order α is defined by :

$$\mathcal{N}_F^\alpha(f)(t) := \lim_{h\to 0} \frac{f(t+\frac{h}{F(t,\alpha)}) - f(t)}{h}.$$

with $F(t,\alpha) \neq 0$, for all $t \in [0,+\infty[$.

Theorem 2.2. [1] *Let $\alpha \in (0,1]$ and f, g be $\mathcal{N}_F^\alpha$-differentiable at a point $t > 0$. Then:*

(1) $\mathcal{N}_F^\alpha(af + bg)(t) = a\mathcal{N}_F^\alpha(f)(t) + b\mathcal{N}_F^\alpha(g)(t)$.
(2) $\mathcal{N}_F^\alpha(\lambda) = 0, \lambda \in \mathbb{R}$.
(3) $\mathcal{N}_F^\alpha(fg)(t) = \mathcal{N}_F^\alpha(f)(t)g(t) + f(t)\mathcal{N}_F^\alpha(g)(t)$.
(4) $\mathcal{N}_F^\alpha(\frac{f}{g})(t) = \dfrac{g(t)\mathcal{N}_F^\alpha(f)(t) - f(t)\mathcal{N}_F^\alpha(g)(t)}{g^2(t)}$.
(5) *If, in addition, f is differentiable then* $\mathcal{N}_F^\alpha(f)(t) = \dfrac{f'(t)}{F(t,\alpha)}$.

Example 2.3. Let $f(t) = e^t$ and $F(t,\alpha) = \dfrac{1}{ch(\alpha t)}, t \in [0,+\infty[$. Then

$$\mathcal{N}_F^\alpha(f)(t) := ch(\alpha t)e^t = \frac{e^{(1+\alpha)t} + e^{(1-\alpha)t}}{2}.$$

Definition 2.4. Let $0 < \alpha \le 1$ and $f : [0,+\infty[\to \mathbb{R}$.

(1) The function f is called N_F^α-differentiable on $[0,+\infty[$ if f is continuous, $N_F^\alpha f(t)$ exist for all $t \in]0,+\infty[$ and $N_F^\alpha f(0) = \lim_{t\to 0^+} N_F^\alpha f(t)$ exists.
(2) The function f is called continuously N_F^α-differentiable on $[0,+\infty)$ if f is N_F^α-differentiable on $[0,+\infty)$ and $N_F^\alpha f(t)$ is continuous on $[0,+\infty[$.

Definition 2.5. Let $0 < \alpha \le 1, n \in \mathbb{N}$ and $f : [0,+\infty[\to \mathbb{R}$.

(1) The function f is called n times N_F^α-differentiable on $[0,+\infty[$ if f is continuous, $\forall j \in \{0,...n\}$ $N_F^{(j\alpha)} f(t) = N_F^\alpha(N_F^\alpha...(N_F^\alpha(f)))(t), j$ times, exist for all $t \in]0,+\infty[$ and $N_F^\alpha f(0) = \lim_{t\to 0^+} N_F^\alpha f(t)$ exists.
(2) The function f is called n times continuously N_F^α-differentiable on $[0,+\infty)$ if f is n times N_F^α-differentiable on $[0,+\infty)$ and $\forall j \in \{0,...n\}$ $N_F^\alpha f(t)$ is continuous on $[0,+\infty[$.
(3) The function f is called infinitely continuously N_F^α-differentiable, if f is n times continuously N_F^α-differentiable for all $n \in \mathbb{N}$.

Note that for $n = 0$, f is n time N_F^α-differentiable if there is continuous and $N_F^0 f(t) = f(t)$.

Definition 2.6. ($\mathcal{N}_F^\alpha$-integral)
Let $0 < \alpha \le 1$ and f be a function defined in $[0,t]$. The function N_F^α-fractional integral of order α of f is defined by

$$\mathcal{I}_F^\alpha(f)(t) = \int_0^t F(s,\alpha)f(s)ds, \quad t \in [0,+\infty[$$

Lemma 2.7. [1] *Let $0 < \alpha \le 1$ and assume that $f : [0,+\infty) \to \mathbb{R}$ is continuous. Then, for all $t > 0$ we have:*

$$\mathcal{N}_F^\alpha(\mathcal{I}_F^\alpha(f)(t)) = f(t)$$

Lemma 2.8. [1] *Let $0 < \alpha \le 1$ and assume that $f : [0,+\infty) \to \mathbb{R}$ is N_F^α-differentiable. Then, for all $t > 0$ we have:*

$$\mathcal{I}_F^\alpha(\mathcal{N}_F^\alpha f(t)) = f(t) - f(0)$$

Consider a continuous function $F(t,\alpha)$ such that $F(t,\alpha) > 0$ for all $t > 0$ and $G_\alpha(t)$ its primitive function verifies $G_\alpha(0) = 0$ and $\lim_{t\to+\infty} G_\alpha(t) = +\infty$, where $0 < \alpha \leq 1$.

Example 2.9. Let $0 < \alpha \leq 1$ and $F(t,\alpha) = \alpha sh(t), t \in [0, +\infty[$, then
1. $F(t,\alpha) > 0$, for all $t > 0$.
2. $G_\alpha(t) = \frac{1}{\alpha} argsh(t)$, $G_\alpha(0) = 0$ and $\lim_{t\to+\infty} G_\alpha(t) = +\infty$.

3. Some results of $\mathcal{N}_F^\alpha$-derivative and $\mathcal{N}_F^\alpha$-Fractional Semigroups

Theorem 3.1. ***($\mathcal{N}_F^\alpha$-Rolle's theorem)***
Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies:
 (1) *f is continuous on $[a,b]$.*
 (2) *f is N_F^α-differentiable for some $\alpha \in (0,1)$.*
 (3) *$f(a) = f(b)$.*

Then, there exists $c \in (a,b)$ such that $\mathcal{N}_F^\alpha f(c) = 0$.

Proof. Since f is continuous on $[a,b]$ and $f(a) = f(b)$, there exists $c \in (a,b)$ which is a point of local extrema. Assume that for example c is a point of local minimum. So,

$$\mathcal{N}_F^\alpha f(c) = \lim_{\epsilon\to 0^+} \frac{f(c+\epsilon\frac{1}{F(c,\alpha)}) - f(c)}{\epsilon} = \lim_{\epsilon\to 0^-} \frac{f(c+\epsilon\frac{1}{F(c,\alpha)}) - f(c)}{\epsilon}$$

But, the first limit is non-negative, and the second limit is non-positive. Hence,

$$\mathcal{N}_F^\alpha f(c) = \mathcal{N}_F^{\alpha,+} f(c) = \mathcal{N}_F^{\alpha,-} f(c) = 0.$$

□

Theorem 3.2. ***($\mathcal{N}_F^\alpha$-Mean Value Theorem)***
Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies:
 (1) *f is continuous on $[a,b]$.*
 (2) *f is $\mathcal{N}_F^\alpha$-differentiable for some $\alpha \in (0,1)$.*

Then, there exists $c \in (a,b)$ such that:

$$\mathcal{N}_F^\alpha f(c) = \frac{f(b) - f(a)}{G_\alpha(b) - G_\alpha(a)}$$

Proof. Consider the function:

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{G_\alpha(b) - G_\alpha(a)} (G_\alpha(x) - G_\alpha(a))$$

we have

$$g(a) = 0, \quad g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{G_\alpha(b) - G_\alpha(a)} (G_\alpha(b) - G_\alpha(a)) = 0$$

The function g satisfies the conditions of $\mathcal{N}_F^\alpha$-Rolle's Theorem. Hence, there exists $c \in (a,b)$ such that:

$$\mathcal{N}_F^\alpha g(c) = 0$$

Using the fact that $\mathcal{N}_F^\alpha(G_\alpha(x)) = 1$, we have:

$$\mathcal{N}_F^\alpha f(c) - \frac{f(b) - f(a)}{G_\alpha(b) - G_\alpha(a)} = 0$$

Therefore:

$$\mathcal{N}_F^\alpha f(c) = \frac{f(b) - f(a)}{G_\alpha(b) - G_\alpha(a)}$$

□

Definition 3.3. Let $0 < \alpha \leq 1$ and a Banach space X. A family $\{T(t), t \geq 0\} \subset L(X)$ is called a fractional $\mathcal{N}_F^\alpha$-semigroup of operators if:

(1) $T(0) = I$, where I is the identity operator.
(2) $T(G_\alpha^{-1}(t+s)) = T(G_\alpha^{-1}(t))T(G_\alpha^{-1}(s))$, for all $t, s \in [0, \infty)$.

Clearly, if $G_\alpha^{-1} = I$, the $\mathcal{N}_F^\alpha$-semigroups are just the usual semigroups.

Example 3.4. Let A be a bounded linear operator. Define:

$$G_\alpha(t) = \alpha sh(t)$$

and

$$T(t) = e^{\alpha A sh(t)}$$

we have : $G_\alpha^{-1}(t) = \frac{1}{\alpha} argsh(t)$ and
$T(\frac{1}{\alpha} argsh(t+s)) = e^{A(t+s)} = e^{At}e^{As} = T(\frac{1}{\alpha} argsh(t))T(\frac{1}{\alpha} argsh(s))$.

Definition 3.5. An $\mathcal{N}_F^\alpha$-semigroup $\{T(t)\}_{t\geq 0}$ is called $C_0\mathcal{N}_F^\alpha-$semigroup if, for each fixed $x \in X$, $T(t)x \longrightarrow x \quad$ as $t \longrightarrow 0^+$.

The $\mathcal{N}_F^\alpha$-derivative of $T(t)$ at $t = 0$ is called the $\mathcal{N}_F^\alpha-$ infinitesimal generator of the fractional $\mathcal{N}_F^\alpha$-semigroup $T(t)$, with domain equals

$$\left\{ x \in X : \lim_{t\to 0^+} \mathcal{N}_F^\alpha T(t)x \text{ exists } \right\}.$$

We will write A for such generator.

Proposition 3.6.
1. Let $(T_F^\alpha(t))_{t\geq 0}$ be a C_0 $\mathcal{N}_F^\alpha$-semigroup. For any $t \geq 0$, we set

$$S(t) = T_F^\alpha\left(G_\alpha^{-1}(t)\right),$$

then $(S(t))_{t\geq 0}$ C_0-semigroup.
2. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup. For any $t \geq 0$, we set

$$T_F^\alpha(t) = S\left(G_\alpha(t)\right),$$

then $(T_F^\alpha(t))_{t\geq 0}$ is a C_0 $\mathcal{N}_F^\alpha$-semigroup.
3. Let $(T_F^\alpha(t))_{t\geq 0}$ be a C_0 $\mathcal{N}_F^\alpha$-semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that for all $t \geq 0$,

$$\|T_F^\alpha(t)\| \leq Me^{\omega G_\alpha(t)}.$$

Proof. 1. Show that $S(t) = T_F^\alpha\left(G_\alpha^{-1}(t)\right)$ is a C_0-semigroup.
We have

$$S(0) = T_F^\alpha\left(G_\alpha^{-1}(0)\right) = T_F^\alpha(0) = I.$$

For all $t, s \geq 0$,

$$S(t+s) = T_F^\alpha\left(G_\alpha^{-1}(t+s)\right).$$

Using the $\mathcal{N}_F^\alpha$-semigroup property,

$$T_F^\alpha\left(G_\alpha^{-1}(t+s)\right) = T_F^\alpha\left(G_\alpha^{-1}(t)\right)T_F^\alpha\left(G_\alpha^{-1}(s)\right) = S(t)S(s).$$

Since $T_F^\alpha(t)$ is strongly continuous and $G_\alpha^{-1}(t)$ is continuous, $S(t)$ is strongly continuous. Therefore, $S(t)$ is a C_0-semigroup.
2. Show that $T_F^\alpha(t) = S\left(G_\alpha(t)\right)$ is a C_0 $\mathcal{N}_F^\alpha$-semigroup.
We have

$$T_F^\alpha(0) = S\left(G_\alpha(0)\right) = S(0) = I.$$

For all $t, s \geq 0$,

$$T_F^\alpha\left(G_\alpha^{-1}(t+s)\right) = S\left(G_\alpha\left(G_\alpha^{-1}(t+s)\right)\right) = S(t+s).$$

Also,

$$T_F^\alpha\left(G_\alpha^{-1}(t)\right)T_F^\alpha\left(G_\alpha^{-1}(s)\right)=S(t)S(s)=S(t+s).$$

Hence,

$$T_F^\alpha\left(G_\alpha^{-1}(t+s)\right)=T_F^\alpha\left(G_\alpha^{-1}(t)\right)T_F^\alpha\left(G_\alpha^{-1}(s)\right).$$

Since $S(t)$ is strongly continuous and $G_\alpha(t)$ is continuous, $T_F^\alpha(t)$ is strongly continuous. Therefore, $T_F^\alpha(t)$ is a C_0 $\mathcal{N}_F^\alpha$-semigroup.
3. Since $S(t)$ is a C_0-semigroup, there exist $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0.$$

Therefore,

$$\|T_F^\alpha(t)\|=\|S\left(G_\alpha(t)\right)\| \leq Me^{\omega G_\alpha(t)}.$$

□

Corollary 3.7. *Let $(T_F^\alpha(t))_{t\geq 0}$ be a $C_0\mathcal{N}_F^\alpha$-semigroup. Then for any $x \in X$, the map $t \mapsto T_F^\alpha(t)x$ is continuous; that is, $(T_F^\alpha(t))_{t\geq 0}$ is strongly continuous.*

Theorem 3.8. *Let $\{T(t)\}_{t\geq 0} \subseteq L(X)$ be a $C_0\mathcal{N}_F^\alpha-$semigroup with infinitesimal generator A and $0<\alpha\leq 1$. If $T(t)$ is continuously $\mathcal{N}_F^\alpha$-differentiable and $x \in D(A)$, then*

$$\mathcal{N}_F^\alpha T(t)x = AT(t)x = T(t)Ax$$

Proof. Let us begin with

$$\begin{aligned}\mathcal{N}_F^\alpha T(t)x &= \lim_{\varepsilon\to 0}\frac{T\left(t+\frac{\varepsilon}{F(t,\alpha)}\right)x-T(t)x}{\varepsilon}\\ &=\lim_{\varepsilon\to 0}\frac{T\left(G_\alpha^{-1}(G_\alpha(t)+G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)-G_\alpha(t))\right)x-T(t)x}{\varepsilon}\\ &=\lim_{\varepsilon\to 0}\frac{T\left(G_\alpha^{-1}\left(G_\alpha(t)+(G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)-G_\alpha(t))\right)\right)x-T(t)x}{\varepsilon}\end{aligned}$$

Since $T(t)$ is an $\mathcal{N}_F^\alpha-$semigroup of operators, then $T(G_\alpha^{-1}(t+s))=T(G_\alpha^{-1}(t))T(G_\alpha^{-1}(s))$, for all $t,s\in[0,\infty)$. Hence

$$\begin{aligned}\mathcal{N}_F^\alpha T(t)x &= \lim_{\varepsilon\to 0}\frac{T(G_\alpha^{-1}(G_\alpha(t))T(G_\alpha^{-1}(G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)-G_\alpha(t)))x-T(t)x}{\varepsilon}\\ &=\lim_{\varepsilon\to 0}\frac{T(t)\left[T\left(G_\alpha^{-1}((G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)-G_\alpha(t))\right)x-T(0)x\right]}{\varepsilon}\end{aligned}$$

Now, using the Mean Value Theorem for $\mathcal{N}_F^\alpha$-fractional derivative, we get
$\frac{T(t)\left[T\left(G_\alpha^{-1}((G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)-G_\alpha(t))\right)x-T(0)x\right]}{\varepsilon}=T(t)\mathcal{N}_F^\alpha T(c)x\frac{\left[(G_\alpha^{-1}((G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t))-G_\alpha(t))\right]}{\varepsilon}$
for some c

$$0<c<G_\alpha^{-1}(G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)-G_\alpha(t))$$

If $\varepsilon\to 0$, then $c\to 0$, and $\lim_{\varepsilon\to 0}\mathcal{N}_F^\alpha T(c)=N_F^\alpha T(0)=A$.
Consequently,

$$\mathcal{N}_F^\alpha T(t)x=T(t)Ax\lim_{\varepsilon\to 0}\frac{\left[\left((G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)\right)-G_\alpha(t))\right]}{\varepsilon}.$$

Then

$$\lim_{\varepsilon\to 0}\frac{\left[\left((G_\alpha(\frac{\varepsilon}{F(t,\alpha)}+t)\right)-G_\alpha(t))\right]}{\varepsilon}=1. \quad \text{Hence} \quad \mathcal{N}_F^\alpha T(t)x=T(t)Ax.$$

Similarly, one can show that $T(t)x \in D(A)$ and $\mathcal{N}_F^{\alpha}T(t)x = AT(t)x$. This end the proof. $\square$

Consider the Banach space $X = C([0,+\infty))$ of bounded uniformly continuous functions from $[0,+\infty)$ to $\mathbb{R}$, equipped with the norm

$$\|f\|_{\infty} = \sup_{x\in[0,+\infty)} |f(x)|,$$

Let $(T(t))_{t\geq 0}$ be the family of operators defined on X by

$$(T(t)f)(s) = f\left(s + G_{\alpha}(t)\right). \quad \forall f \in X, \forall s \in [0,+\infty).$$

Claim: T is an $\mathcal{N}_F^{\alpha}$-semigroup. Indeed:

$$\begin{aligned}\left(T(G_{\alpha}^{-1}(t+r))f\right)(s) &= f(s + G_{\alpha}(G_{\alpha}^{-1}(t+r)) \\ &= f\left(s+t+r\right) \\ &= \left(T\left(G_{\alpha}^{-1}(t)\right)\left(T\left(G_{\alpha}^{-1}(r)f\right)\right)\right)(s)\end{aligned}$$

It is almost immediate that $T(0) = I$ and $T(t)f \in X$ whenever $f \in X$ and that

$$\|T(t)f\|_{\infty} \leq \|f\|_{\infty}, \quad t \geq 0,$$

so that $T(t) \in L(X)$. Since the operator $T(t)$ is a translation operator corresponding to moving the graph of $f(G_{\alpha}^{-1})(t)$ units to the left and chopping off the part to the left of the origin, it is known from the literature that $T(t)f$ is continuous right at 0. Therefore, $T(t)$ is an $\mathcal{N}_F^{\alpha}$-semigroup.

Theorem 3.9. *The infinitesimal generator of the above semigroup is*

$$\begin{aligned}Af(s) &= f'(s) \\ D(A) &= \{f \in X : f' \text{ exists in } X\}.\end{aligned}$$

Proof. Let $0 < \alpha \leq 1$ ant $t > 0$.

$$\begin{aligned}\mathcal{N}_F^{\alpha}T(t)f(s) &= \frac{1}{F(t,\alpha)}T'(t)f(s) \\ &= \frac{1}{F(t,\alpha)}\lim_{\varepsilon\to 0}\frac{T(t+\varepsilon)f(s) - T(t)f(s)}{\varepsilon} \\ &= \frac{1}{F(t,\alpha)}\lim_{\varepsilon\to 0}\frac{f\left(s + (G_{\alpha}(t+\varepsilon)\right) - f\left(s + (G_{\alpha}(t)\right)}{\varepsilon} \\ &= \frac{1}{F(t,\alpha)}\lim_{\varepsilon\to 0}\frac{f\left(s + (G_{\alpha}(t+\varepsilon)\right) - f\left(s + (G_{\alpha}(t)\right)}{\varepsilon}\cdot\frac{f(s+t+\varepsilon) - f(s+t)}{f(s+t+\varepsilon) - f(s+t)} \\ &= \frac{1}{F(t,\alpha)}\lim_{\alpha\to 0}\frac{f\left(s + (G_{\alpha}(t+\varepsilon)\right) - f\left(s + (G_{\alpha}(t)\right)}{f(s+t+\varepsilon) - f(s+t)}\frac{f(s+t+\varepsilon) - f(s+t)}{\varepsilon}.\end{aligned}$$

Now

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{f(s+t+\varepsilon) - f(s+t)}{\varepsilon} = f'(s+t) \\ &\lim_{\varepsilon\to 0}\frac{f\left(s + (G_{\alpha}(t+\varepsilon)\right) - f\left(s + (G_{\alpha}(t)\right)}{f(s+t+\varepsilon) - f(s+t)} = \frac{0}{0}\end{aligned}$$

By applying L'Hôpital's rule, the limit becomes:

$$\lim_{\varepsilon\to 0}\frac{f'\left(s + G_{\alpha}(t+\varepsilon)\right)F(t+\varepsilon;\alpha)}{f'(s+t+\varepsilon)}.$$

Taking the limit as $\varepsilon \to 0$, we obtain:

$$\frac{f'\left(s + G_{\alpha}(t)\right)F(t;\alpha)}{f'(s+t)}.$$

Thus the product gives

$$\mathcal{N}_F^\alpha T(t)f(s) = f'\left(s + (G_\alpha(t)\right).$$

Now take the limit as $t \to 0$ to get

$$\mathcal{N}_F^\alpha T(0)f(s) = f'(s)$$

Hence $\quad Af = f'$. This completes the proof. □

Lemma 3.10. *Let $(T(t))_{t\geq 0}$ be a $C_0\mathcal{N}_F^\alpha-$semigroup, with generator A. Then*

$$\lim_{h\mapsto 0} \frac{1}{h}\int_t^{t+h} T(u)F(u,\alpha)du = T(t)F(t,\alpha) \quad \forall t \geq 0$$

Proof. For $t > 0$, by continuity of f and $t \mapsto T(t)$ on $[0,+\infty[$, we see that the function $g(t) = T(t)F(t,\alpha)$ is continuous on $[0,+\infty\,[$. Define $F(x) = \int_t^{t+x} g(u)du$. Then

$$F'(0) = \lim_{h\mapsto 0}\frac{F(h)-F(0)}{h} = \lim_{h\mapsto 0}\frac{1}{h}\int_t^{t+h} T(u)F(u,\alpha)du.$$

On the other hand, we have $F'(x) = g(t+x)$, so that $F'(0) = g(t) = T(t)F(t,\alpha)$. Finally,

$$\lim_{h\mapsto 0} \frac{1}{h}\int_t^{t+h} T(u)F(u,\alpha)du = T(t)F(t,\alpha) \quad \forall t \geq 0.$$

□

Theorem 3.11. *Let $(T(t))_{t\geq 0}$ be a $C_0\mathcal{N}_F^\alpha-$semigroup, with generator A. Then:*

$$A\left(\int_0^t T(s)F(s,\alpha)x\,\mathrm{d}s\right) = T(t)F(t,\alpha)x - F(0,\alpha)x.$$

Proof. For all $x \in X$ and $h > 0$. We have

$$\begin{aligned}
&N_F^\alpha\left(\int_0^t T(s)F(s,\alpha)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\frac{1}{h}\left(\int_0^t T\left(s+\frac{h}{F(t,\alpha)}\right)F\left(s+\frac{h}{F(t,\alpha)},\alpha\right)x\,\mathrm{d}s - \int_0^t T(s)F(s,\alpha)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\frac{1}{h}\left(\int_0^t T\left(s+\frac{h}{F(t,\alpha)}\right)F\left(s+\frac{h}{F(t,\alpha)},\alpha\right)x\,\mathrm{d}s - \int_0^t T(s)F(s,\alpha)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\frac{1}{h}\left(\int_{\frac{h}{F(t,\alpha)}}^{t+\frac{h}{F(t,\alpha)}} T\left(s\right)F\left(s-\frac{h}{F(t,\alpha)},\alpha\right)x\,\mathrm{d}s - \int_0^t T(s)F(s,\alpha)x\,\mathrm{d}s\right)\\
&= \lim_{h\to 0^+}\left(\frac{\int_t^{t+\frac{h}{F(t,\alpha)}} T(s)F\left(s-\frac{h}{F(t,\alpha)},\alpha\right)x\,\mathrm{d}s}{h} - \frac{\int_0^{\frac{h}{F(t,\alpha)}} T(s)F(s,\alpha)x\,\mathrm{d}s}{h}\right)\\
&\underset{h\to 0^+}{\longrightarrow} T(t)F(t,\alpha)x - F(0,\alpha)x.
\end{aligned}$$

Thus, we obtain :

$$A\left(\int_0^t T(s)F(s,\alpha)x\,\mathrm{d}s\right) = T(t)F(t,\alpha)x - F(0,\alpha)x.$$

□

4. Conculusion and future work

In this paper, we replace the classical derivative with the $\mathcal{N}_F^{\alpha}$-derivative and obtain new results related to this definition applied to $\mathcal{N}_F^{\alpha}$-Semigroups. and obtain good results that can be used in the future to solve $\mathcal{N}_F^{\alpha}$-differential equations, some models as $\mathcal{N}_F^{\alpha}$-Fractional Epidemic Model, population dynamics.

Acknowledgements

The authors are indebted to the Referee for giving valuable comments.

References

[1] J. E. Napoles, M. N. Quevedo, *On the Oscillatory Nature of Some Generalized Emden-Fowler Equation* Punjab University Journal of Mathematics Vol. 52(6)(2020) pp. 97-106.
[2] Hille, E. *Functional Analysis and semigroups.* A.M.S., New York, 1948.
[3] Yosida, K. *On the differentiability and the representation of one-parameter semi-groups of linear operators.* J. Math. Soc. Japan, 1, (1948), 15-21.
[4] Yosida, K. *Lectures on Semi-group Theory and its application to Cauchy problem in Partial Differential Equations.* Tata Institute of Fundamental Research, Bombay, 1957.
[5] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.
[6] Nathan, D.S. *One-parameter semigroups of transformations in abstract vector spaces.* Duke Math. J., 1(1935), 518-526.
[7] Ahmed Toukmati, *Spectral inclusions of exponentialy bounded C-semigroups.* Methods of Functional Analysis and Topology Vol. 28 (2022), no. 2, pp. 169–175. doi.org/10.31392/MFAT-npu26_2.2022.09.
[8] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. 264 (2014) 65–70. dx.doi.org/10.1016/j.cam.2014.01.002.
[9] V.Stojiljkovic, *A New Conformable Fractional Derivative and Applications.* Sel.Mat. 2022, 9, 370 - 380. doi.10.17268/sel.mat.2022.02.12.

Bahloul Rachid: bahloulrachid363@gmail.com
Department of Mathematics, polidisciplinary faculty, Sultan Moulay Slimane University, Beni Mellal, Morocco.

Rachad Houssame: houssamer405@gmail.com
Department of Mathematics, polidisciplinary faculty, Sultan Moulay Slimane University, Beni Mellal, Morocco.

Thabet Abdeljawad: tabdeljawad@psu.edu.sa
Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia.

Received 04/12/2024; Revised 21/01/2025