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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

# ON NEIGHBOURHOOD SEMI-STAR ROTHBERGER (MENGER AND HUREWIEZ) SPACES

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Abstract. In this paper, we introduce and study new types of star-selection principles, namely semi-neighbourhood semi-star-Rothberger (Menger, Hurewicz, and Lindelöf) and neighbourhood semi-star-Rothberger (Menger, Hurewicz, and Lindelöf) spaces. We establish several properties of these selection principles and investigate their relationships with other selection properties in topological spaces. Furthermore, we present a collection of fundamental theorems and propositions that characterize these spaces.

## 1. Introduction

In 1963, N. Levine (see [11]) introduced the concept of semi-open sets. A subset $A$ of a topological space $X$ is termed semi-open if there exists an open set $U$ such that $U \subset A \subset \overline{U}$, where $\overline{U}$ $'or'$ $Cl(U)$ denotes the closure of $U$ in $X$. S. Crossley and S. Hildebrand (see [5]). Defined a set to be semi-closed if its complement is semi-open. Equivalently, $A$ is semi-open[resp., semi-closed] if and only if $A \subset \overline{Int(A)}$[resp., $Int(\overline{A}) \subset A$].
While every open set is semi-open, a semi-open set may not necessarily be open. The union of any number of semi-open sets is semi-open, but the intersection of two semi-open sets may not be semi-open. However, the intersection of an open set and a semi-open set is always semi-open. The semi-closure $sCl(A)$ of $A \subset X$ is defined as the intersection of all semi-closed sets containing $A$. A set $A$ is semi-open if and only if $sInt(A) = A$, and $A$ is semi-closed if and only if $sCl(A) = A$. It's worth noting that for any subset $A$ of $X$,

$$Int(A) \subset sInt(A) \subset A \subset sCl(A) \subset Cl(A).$$

A subset $A$ of a topological space $X$ is termed semi-regular if it is both semi-open and semi-closed, or equivalently, $A = sCl(sInt(A))$ 'or' $A = sInt(sCl(A))$.
In [8], Kočinac introduced star selection hypothesis similar to the previous ones. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space $X$. Then: (**A**) The symbol $\mathsf{S_{fin}}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(V, \mathcal{U}_n) \ : \ V \in \mathcal{V}_n)\}$ is an element of $\mathcal{B}$.

(**B**) The symbol $\mathsf{SS}^*_{\mathrm{comp}}(\mathcal{A}, \mathcal{B})$ (resp., $SS^*_{fin}(A, B)$) denotes the selection hypothesis that for each sequence $(\mathcal{U}_n \ : n \in \mathbb{N})$ of elements of $A$ there exists a sequence $(K_n : n \in \mathbb{N})$ of compact (resp., finite) subsets of $X$ such that $\{St(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in B$.

Let $\mathcal{O}$ denote the collection of all open covers of a space $X$.

**Definition 1.1.** (see [8]) A space $X$ is said to be *star-Menger* [resp., *star-Rothberger*] if it satisfies the selection hypothesis $\mathrm{S_{fin}}^*(\mathcal{O}, \mathcal{O})$ [resp., $\mathsf{S_1}^*(\mathcal{O}, \mathcal{O})$]

**Definition 1.2.** (see [8]) A space $X$ is said to be *star-Hurewicz* if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$, one can choose finite $\mathcal{V}_n \subset \mathcal{U}_n$ so that for every $x \in X$ we have $x \in \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many $n$.

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The following three generalizations of star selection properties have been introduced (in a general form and under different names) in (see [9]) and studied in details in (see [3]).

**Definition 1.3.** (see [3]) A space $X$ is said to be *neighbourhood star-Menger* (NSM) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$, one can choose finite $(F_n \subset X, n \in \mathbb{N})$, so that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_n \{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} = X$.

**Definition 1.4.** (see [3]) A space $X$ is said to be *neighbourhood star-Rothberger*(NSR) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$, one can choose $(x_n \in X : n \in \mathbb{N})$, so that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}) = X$.

**Definition 1.5.** (see [3]) A space $X$ is said to be *neighbourhood star-Hurewicz* (NSH) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of $X$, one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, each $x \in X$ belongs to $\mathrm{St}(O_n, \mathcal{U}_n)$ for all but finitely many $n$.

## 2. New Selection Principles

In this section we introduce and study some new types of star-selection principles denoted as sNsSM, sNsSR and sNsSH spaces.
A semi-open cover $\mathcal{U}$ of a space $X$ is defined as follows;

- $s\mathcal{O}$ the family of semi-open covers of $X$;
- an *sω-cover* if $X$ does not belong to $\mathcal{U}$ and every finite subset of $X$ is contained in a member of $\mathcal{U}$;
- an *sγ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$;
- *s-groupable* if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many $n$;
- *weakly s-groupable* if it is a countable union of finite, pairwise disjoint sets $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite set $F \subset X$ we have $F \subset \bigcup \mathcal{U}_n$ for some $n$.

**Definition 2.1.** A space $X$ is said to be:
1. *semi-neighbourhood semi star-Menger (resp., neighbourhood semi star-Menger)* sNsSM (resp., NsSM) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of $X$, one can choose finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every semi open $O_n \supset F_n$, $n \in \mathbb{N}$ (resp., for every open $O_n \supset F_n$) , we have $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n) = X$

2. *semi-neighbourhood semi star-Rothberger (resp., neighborhood semi star-Rothberger)* sNsSR(resp., NsSR) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of $X$, one can choose the sequence $(x_n : n \in \mathbb{N})$ of elements of $X$ so that for every semi open $O_n \ni x_n$, $n \in \mathbb{N}$ (resp., for every open $O_n \ni x_n$) , we have $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n) = X$

3. *semi-neighbourhood semi star-Hurewicz (resp., neighbourhood semi star-Hurewicz* sNsSH(resp., NsSH) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi open covers of $X$, one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every semi open $O_n \supset F_n$, $n \in \mathbb{N}$ (resp., for every open $O_n \supset F_n$), each $x \in X$ belongs to $\mathrm{St}(O_n, \mathcal{U}_n)$.

**Remark 2.2.** Every sNsSM space is NsSM, and every NsSM space is NSM. Similarly, for Rothberger-type and Hurewicz-type properties.

Moreover, we have the following relationships among the classes of spaces defined above.

$$\begin{array}{ccccc} sNsSR & \Longrightarrow & NsSR & \Longrightarrow & NSR \\ \Downarrow & & \Downarrow & & \Downarrow \\ sNsSM & \Longrightarrow & NsSM & \Longrightarrow & NSM \\ \Uparrow & & \Uparrow & & \Uparrow \\ sNsSH & \Longrightarrow & NsSH & \Longrightarrow & NSH \end{array}$$

**Diagram** 01

**Theorem 2.3.** *A topological space $X$ is classified as semi-neighborhood semi-star-Menger if for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover. Then, there exist $F_n : n \in \mathbb{N}$ of finite subset of $X$ such that for all $x \in X$ we have $sCl(\mathrm{St}(\{x\},\mathcal{U}_n)) \cap F_n \neq \emptyset$.*

*Proof.*

$\Rightarrow$) Let $X$ be a semi neighborhood semi star-Menger and let $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open cover. Then exists a fin $F_n \subset X$ such that for all semi-open $O_n$ containing $F_n$ : $\{\mathrm{St}(O_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$. This implies that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n,\mathcal{U}_n) = X$. Let $x \in X$, $\exists k \in \mathbb{N}$ fulfilling that $x \in \mathrm{St}(O_k,\mathcal{U}_k)$. Let $f_k \in F_k \subset O_k$. Since $\mathrm{St}(\{x\},\mathcal{U}_k) \cap O_k \neq \emptyset$ , $f_k \in sCl(\mathrm{St}(\{x\},\mathcal{U}_k))$. Hence, $sCl(\mathrm{St}(\{x\},\mathcal{U}_n)) \cap F_n \neq \emptyset$.

$\Leftarrow$) Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open sets. Then exists $F_n$ of fin subsets of $X$ fulfilling that for every $x \in X$ , $\exists n \in \mathbb{N}$ fulfilling that $sCl(\mathrm{St}(\{x\},\mathcal{U}_n)) \cap F_n \neq \emptyset$. This implies that for every semi-open set $O_n \supset F_n$ we have $\mathrm{St}(\{x\},\mathcal{U}_n) \cap F_n \neq \emptyset$. This implies that $x \in \mathrm{St}(O_n,\mathcal{U}_n)$. Hence, $\{\mathrm{St}(O_n,\mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$.

□

**Proposition 2.4.**

(1) *A space $X$ is neighborhood semi star-Rothberger iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist $x_n : n \in \mathbb{N}$ of point of $X$ such that for all $x \in X$ we have $x_n \in \overline{\mathrm{St}(\{x\},\mathcal{U}_n)}$.*

(2) *A space $X$ is semi neighborhood semi star-Rothberger iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist $x_n : n \in \mathbb{N}$ of point of $X$ such that for all $x \in X$ we have $x_n \in sCl(\mathrm{St}(\{x\},\mathcal{U}_n))$.*

(3) *A space $X$ is neighborhood semi star-Menger iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist $F_n : n \in \mathbb{N}$ of finite subset of $X$ such that for all $x \in X$ we have $\overline{\mathrm{St}(\{x\},\mathcal{U}_n)} \cap F_n \neq \emptyset$.*

(4) *A space $X$ is semi neighborhood semi star-Menger iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist $F_n : n \in \mathbb{N}$ of finite subset of $X$ such that for all $x \in X$ we have $sCl(\mathrm{St}(\{x\},\mathcal{U}_n)) \cap F_n \neq \emptyset$.*

(5) *A space $X$ is neighborhood semi star-Hurewicz iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist $F_n : n \in \mathbb{N}$ of finite subset of $X$ such that for all $x \in X$ we have $\overline{\mathrm{St}(\{x\},\mathcal{U}_n)} \cap F_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.*

(6) *A space $X$ is semi neighborhood semi star-Hurewicz iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist $F_n : n \in \mathbb{N}$ of finite subset of $X$ such that for all $x \in X$ we have $sCl(\mathrm{St}(\{x\},\mathcal{U}_n)) \cap F_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$.*

**Proposition 2.5.** *If all finite powers of a space $X$ are $sNsSM$ (resp., $NsSM$) then $X$ satisfies $sNsSM(s\mathcal{O}, s\Omega)$ (resp., $NsSM(s\mathcal{O}, s\Omega)$). Such that the symbol $s\Omega$ denotes the family of semi-$\omega$-covers of a space.*

*Proof.* We prove the $sNsSM$ case, and the other case can be proved similarly.
Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of $X$ and let $\mathbb{N} = \{N_1 \cup N_2 \cup ....\}$

be a partition of $\mathbb{N}$ into infinite (pairwise disjoint) sets. For every $k \in \mathbb{N}$ and every $m \in N_k$ let $\mathcal{W}_m = (\mathcal{U}_m)^k$. Then $(\mathcal{W}_m : m \in N_k)$ is a sequence of semi-open covers of $X^k$. Applying to this sequence the fact that $X^k$ is $sNsSM$ we find a sequence $(A_m : m \in N_k)$ of finite subsets of $X^k$ such that for every semi open sequence $(O_m(A_m) : m \in N_k)$ of neighborhoods of $A_m,\ m \in N_k,\ in\ X^k$. (Since $X^k$ is semi-neighbourhood semi star-Menger) the family $\{\mathrm{St}(O_m, \mathcal{W}_m) : m \in N_k\}$ is semi-open cover of $X^k$(that is, $X^k \subseteq \bigcup \mathrm{St}(O_m, \mathcal{W}_m)$). For every $m \in N_k$, let $S_m$ be a finite subset of $X$ such that $S^k_m \supset A_m$ consider the sequence of all $S_m,\ m \in N_k,\ k \in \mathbb{N}$, chosen in this way and denote it $(S_n : n \in \mathbb{N})$. Let $(G_n(S_n) : n \in \mathbb{N})$ be a sequence of semi neighborhoods of $S_n,\ n \in \mathbb{N}$. We claim that $\{\mathrm{St}(G_n(S_n)), \mathcal{U}_n) : n \in \mathbb{N}\}$ is an s$\omega$-cover of $X$. Let $F = \{x_1 \cdot \cdots \cdot x_p\}$ be a finite subset of $X$. Then $(x_1 \cdot \cdots \cdot x_p) \in X^p$. There exists $n \in N_p$ such that $\{(G_n(S_n))^p\ n \in \mathbb{N}\}$ is a sequence of semi neighborhoods of $A_n$. So that there exists $n \in \mathbb{N}$ such that, $(x_1 \cdot \cdots \cdot x_p) \in \mathrm{St}(O_n, \mathcal{W}_n) \subset \mathrm{St}((G_n(S_n))^p, \mathcal{W}_n)$, so that we have $F \subset \mathrm{St}(G_n(S_n), \mathcal{U}_n)$. Then $X$ satisfies $sNsSM(s\mathcal{O}, s\Omega)$. □

In similarity we have,

**Theorem 2.6.** *If each finite power of a space $X$ is semi-neighborhood semi star-Hurewicz (resp., neighborhood semi star-Hurewicz) then $X$ satisfies $sNsSH(s\mathcal{O}, s\Omega)$ (resp., $X$ satisfies $NsSH(s\mathcal{O}, s\Omega)$) .*

**Theorem 2.7.** *Let $X$ be an extremally disconnected space, $X$ is $sNsSH$ space (resp., $NsSH$ space) if and only if $X$ satisfies $sNsSH(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$ (resp., $NsSH(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$ ). Such that, $s\mathcal{O}^{gp}$ is family of s-groupable covers of $X$.*

*Proof.* We prove the $sNsSH$ case, and the other case can be proved similarly. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of $X$ by semi-open sets. Since $X$ is semi neighborhood semi star-Hurewicz space, one can choose finite $F_n \subset X,\ n \in \mathbb{N}$, so that for every semi-open $O_n \supset F_n,\ n \in \mathbb{N}$, and each $x \in X$ belongs to $\mathrm{St}(O_n, \mathcal{U}_n)$ for all but finitely many $n$. This implies that $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\gamma$-cover of $X$. Since each countable $s\gamma$-cover is $s$-groupable, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}^{\mathrm{gp}}$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of $X$ by semi-open sets. Let

$$\mathcal{H}_n = \bigwedge\nolimits_{i \leq n} \mathcal{U}_i.$$

Then $(\mathcal{H}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of $X$ since $X$ is extremally disconnected. Use now $sNsSH(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$ property of $X$. For each $\mathcal{H}_n$ and for each $n \in \mathbb{N}$ select semi-open set $O_n \supset F_n$, such that the set $\{\mathrm{St}(O_n, \mathcal{H}_n) : n \in \mathbb{N}\}$ is an $s$-groupable cover of $X$. Let $n_1 < n_2 < ... < n_k < \ldots$ be a sequence of natural numbers which witnesses this fact, that is, for each $x \in X$, $x$ belongs to $\bigcup\{\mathrm{St}(O_i, \mathcal{H}_i) : n_k < i \leq n_{k+1}\}$ for all but finitely many $k$. Put

$$\mathcal{W}_n = \bigcup_{i<n} O_i, \text{ for } n < n_1;$$
$$\mathcal{W}_n = \bigcup_{n_k<i\leq n_{k+1}} O_i, \text{ for } n_k \leq n < n_{k+1}.$$

Then we shows that $\bigcup \mathcal{W}$ is semi neighborhood of $F_n$ and $X$ satisfies $sNsSH$ property because each $x \in X$ belongs to all but finitely many $\mathrm{St}(\bigcup \mathcal{W}_n, \mathcal{U}_n)$. □

**Definition 2.8.** (see [5]) A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is called irresolute if $f^{-1}(O)$ is semi-open in $X$ for every $O$ semi-open in $Y$.

**Theorem 2.9.** *Let $X$ be a $sNsSM$ topological space and let $Y$ be a topological space. If $f : X \to Y$ is an irresolute. Then $Y$ is a $sNsSM$.*

*Proof.* Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi open covers of $Y$. For each $n \in \mathbb{N}$, the set $\mathcal{U}_n' = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is an semi open cover of $X$. Since $X$ is semi neighbourhood semi star-Menger, there are finite $(F_n \subset X)$, $n \in \mathbb{N}$, so that for every semi-open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\{\mathrm{St}(O_n, \mathcal{U}_n') : n \in \mathbb{N}\}$ is a semi-cover of $X$. The sets $f(F_n)$, $n \in \mathbb{N}$, are finite in $Y$. Let $G_n \supset f(F_n)$ for each $n$ be semi-open set in $Y$. Then $f^{-1}(G_n) = H_n$ is an semi-open subset of $X$ for each $n \in \mathbb{N}$ and $H_n \supset F_n$. Thus $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(H_n, \mathcal{U}_n')$. We prove that $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n, \mathcal{U}_n)$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then there is $k \in \mathbb{N}$ such that $x \in \mathrm{St}(H_k, \mathcal{U}_k')$. Then $y = f(x) \in f(\mathrm{St}(H_k, \mathcal{U}_k')$. Because $f(\mathrm{St}(H_k, \mathcal{U}_k')) \subset f(\mathrm{St}(f^{-1}(G_k), \mathcal{U}_k')) \subset \mathrm{St}(G_k, \mathcal{U}_k)$ we have $y \in \mathrm{St}(G_k, \mathcal{U}_k)$. Therefore $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_k, \mathcal{U}_k)$, that is, $Y$ is semi-neighbourhood semi star Menger. □

Similarly, we can prove the following.

**Theorem 2.10.**

(1) *Let $X$ be a $sNsSR$ topological space and let $Y$ be a topological space. If $f : X \to Y$ is an irresolute. Then $Y$ is a $sNsSR$.*
(2) *Let $X$ be a $sNsSH$ topological space and let $Y$ be a topological space. If $f : X \to Y$ is an irresolute. Then $Y$ is a $sNsSH$.*

**Definition 2.11.** A space X is :

(1) meta semi-compact if every semi-open cover $\mathcal{U}$ of $X$ has a point-finite semi-open refinement $\mathcal{V}$, that means, (every point of $X$ belongs to at most finitely many members of $\mathcal{V}$),
(2) meta semi-Lindelöf if every semi-open cover $\mathcal{U}$ of $X$ has a point-countable semi-open refinement $\mathcal{V}$.

**Theorem 2.12.** *If a space $X$ is semi neighborhood semi star-Menger meta semi-compact space, then $X$ is semi-Menger.*

*Proof.* Let $X$ be a $sNsSM$ meta semi-compact space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of $X$. Let $\mathcal{V}_n$ be a point-finite semi-open refinement of $\mathcal{U}_n$. Since $X$ is semi-neighborhood semi star-Menger, one can choose a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $X$ such that for every semi-open $(O_n \supset F_n) : \bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{V}_n) = X$. As $\mathcal{V}_n$ is a point-finite refinement and each $F_n \subset O_n$ is finite belongs to finitely many members of $\mathcal{V}_n$ say $V_{n_1}, V_{n_2}, V_{n_3}, V_{n_4}...., V_{n_k}$. Let $\mathcal{V}_n' = \{V_{n_1}, V_{n_2}, V_{n_3}, V_{n_4}...., V_{n_k}\}$. Then $St(O_n, \mathcal{V}_n) = \bigcup\{\mathcal{V}_n' : n \in \mathbb{N}\}$.This mean $\bigcup_{n\in\mathbb{N}}(\cup\mathcal{V}_n') = X$. For every $V \in \mathcal{V}_n'$, take $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. Then we have, $W_n = \{U_V : V \in \mathcal{V}_n'\}$ is a finite subfamily of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \cup W_n = X$. Then $X$ is semi-Menger $(sM)$ space. □

Similarly, we can prove the following

**Theorem 2.13.** *If a space $X$ is semi neighborhood semi star-Hurewicz meta semi-compact space then $X$ is semi-Hurewicz.*

**Theorem 2.14.** *If a space $X$ is semi neighborhood semi star-Rothberger meta semi-compact space then $X$ is semi-Rothberger.*

**Definition 2.15.** Let $T$ be a subset of $X$ then:

(1) $T$ is relatively $sNsSM$(resp., relatively $sNsSH$) in $X$ if for each $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of $X$, one can choose a $(A_n : n \in \mathbb{N})$ of finite subsets of $X$, such that for every semi-open ( $O_n \supset A_n, n \in \mathbb{N}$), we have $\bigcup\{St(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset T$ (resp., $\forall t \in T, t \in St(O_n, \mathcal{U}_n)$ for all but finitely many $n$).
(2) $T$ is relatively $sNsSR$ in $X$ if for each $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of $X$, there are ( $x_n \in X, n \in \mathbb{N}$), such that for all semi-open $(x_n \in O_n, n \in \mathbb{N})$, we have $\bigcup\{St(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset T$.

**Proposition 2.16.** *If $X = \bigcup\{T_k : k \in \mathbb{N}\}$, and every $T_k$ is relatively $sNsSM$ (resp., relatively $sNsSH$, relatively $sNsSR$) in $X$, then $X$ is $sNsSM$ (resp., $sNsSH$, $sNsSR$).*

*Proof.* We shall prove the $sNsSM$. The other cases follow in the same way. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of a semi-open covers of $X$. Then for all $k, n \in \mathbb{N}, \mathcal{U}_n$ covers $T_k$, and since $T_k$ is relatively $sNsSM$ , there are countable sets $G_{k,n} \subset X$ , such that for each semi-open set $O_{k,n} \supset G_{k,n}$, we have $St(O_{k,n}, \mathcal{U}_n) \supset T_k$. Consider the $(G_{k,n})$ and $(F_{k,n})$ of neighborhood of $G'_{k,n}$. It is easy to conclude that $\bigcup_{k\in\mathbb{N}} St(F_{k,n}, \mathcal{U}_n) \supset \bigcup_{k\in\mathbb{N}} T_k = X$, which means that $X$ is $sNsSM$. □

## 3. About $sNsSL$ and $NsSL$ spaces

In this section, we give some facts about semi neighborhood semi star-Lindelöf and neighborhood semi star-Lindelöf.

**Definition 3.1.** A space $X$ is said to be semi neighborhood semi star-Lindelöf (sNsSL) (resp., neighborhood semi star-Lindelöf (NsSL), if for every semi open cover $\mathcal{U}$ of $X$, one can choose a countable subset $A \subset X$ such that for every semi neighbourhood $O$(resp., neighbourhood $O$) of $A$, we have $\mathrm{St}(O, \mathcal{U}) = X$.

**Remark 3.2.** Of course, every sNsSL space is NsSL and NsSL space is NSL.

In fact, we have the following relations among classes of spaces defined above.

$$\begin{array}{ccccc} sNsSM & \Longrightarrow & NsSM & \Longrightarrow & NSM \\ \Downarrow & & \Downarrow & & \Downarrow \\ sNsSL & \Longrightarrow & NsSL & \Longrightarrow & NSL \end{array}$$

**Diagram** 02

**Proposition 3.3.**

(1) *A space $X$ is semi neighborhood semi star-Lindelöf iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist a countable subset $F_n : n \in \mathbb{N}$ of $X$ such that for all $x \in X$ we have $sCl(\mathrm{St}(\{x\}, \mathcal{U}_n)) \cap F_n \neq \emptyset$.*

(2) *A space $X$ is neighborhood semi star-Lindelöf iff for every $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of semi-open cover there exist a countable subset $F_n : n \in \mathbb{N}$ of $X$ such that for all $x \in X$ we have $\overline{\mathrm{St}(\{x\}, \mathcal{U}_n)} \cap F_n \neq \emptyset$.*

*Proof.* We shall prove the $sNsSL$ case.

$\Rightarrow$) Let $X$ be a semi neighborhood semi star-Lindelöf and let $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open cover. There exists a countable $F_n \subset X$ such that for all semi-open $O$ containing $F_n : \{\mathrm{St}(O, \mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$. This implies that $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O, \mathcal{U}_n) = X$. Let $x \in X$, $\exists k \in \mathbb{N}$ fulfilling that $x \in \mathrm{St}(O, \mathcal{U}_k)$. Let $f_k \in F_k \subset O$. Since $\mathrm{St}(\{x\}, \mathcal{U}_k) \cap O \neq \emptyset$ , $f_k \in sCl(\mathrm{St}(\{x\}, \mathcal{U}_k))$. Hence, $sCl(\mathrm{St}(\{x\}, \mathcal{U}_n)) \cap F_n \neq \emptyset$.

$\Leftarrow$) Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open sets. Then exists a countable $F_n$ of subsets of $X$ fulfilling that for every $x \in X \exists n \in \mathbb{N}$ fulfilling that $sCl(\mathrm{St}(\{x\}, \mathcal{U}_n)) \cap F_n \neq \emptyset$. This implies that for every semi-open set $O \supset F_n$ we have $\mathrm{St}(\{x\}, \mathcal{U}_n) \cap F_n \neq \emptyset$. This implies that $x \in \mathrm{St}(O, \mathcal{U}_n)$. Hence, $\{\mathrm{St}(O, \mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}$.

□

**Theorem 3.4.** *An semi-open $F_\sigma$-subset of a semi-neighbourhood semi star-Lindelöf space (resp., $NsSL$) is semi-neighbourhood semi star-Lindelöf (resp., $NsSL$).*
*( $F_\sigma$-set is a countable union of closed sets ).*

*Proof.* Let $(X, \tau)$ be a $sNsSL$ space and let $Y = \bigcup\{H_n : n \in \mathbb{N}\}$ be a semi-open $F_\sigma$-subset of $X$, where the set $H_n$ is closed in $X$ for each $n \in \mathbb{N}$. We show that $Y$ is semi-neighborhood semi star-Lindelöf. Let $\mathcal{U}$ be a semi-open cover of $(Y, \tau_Y)$. We have to find a countable subset $F$ of $Y$ such that for each $\tau_Y$semi-open $O \supseteq F$, $Y \subseteq \mathrm{St}(O,\mathcal{U})$.

For each $n \in \mathbb{N}$, consider the semi-open cover $\mathcal{U}_n = \mathcal{U} \cup \{X \setminus (H_n)\}$ of $X$. Since $X$ is semi-neighborhood semi star-Lindelöf, there exists a countable subset $F_n$ of $X$ such that for each semi-open $O^{'} \supseteq F_n$, we have $X = \mathrm{St}(O^{'},\mathcal{U})$. For each $n \in \mathbb{N}$, let $M_n = F_n \cap Y$. Then $M_n$ is a countable subset of $Y$ such that for each semi-open $O \supseteq M_n$, $H_n \subseteq \mathrm{St}(O,\mathcal{U})$. If we put $F = \bigcup\{M_n : n \in \omega\}$, then $F$ is a countable subset of $Y$ such that for each semi-open $O \supseteq F$, $\mathrm{St}(O,\mathcal{U}) \supseteq Y$, which shows that $Y$ is semi-neighborhood semi star-Lindelöf. □

A *cozero-set* in a space $X$ is a set of the form $f^{\leftarrow}(\mathbb{R} \setminus \{0\})$ for some real-valued continuous function $f$ on $X$ (see [6]).

Since a cozero-set is a semi-open $F_\sigma$-set, we have the following corollary of theorem (3.4).

**Corollary 3.5.** *A cozero-set of a semi-neighborhood semi star-Lindelöf(resp., $NsSL$) space is semi-neighborhood semi star-Lindelöf (resp., $NsSL$).*

**Theorem 3.6.** *Let $X$ be a semi-neighbourhood semi star-Lindelöf topological space and let $Y$ be a topological space. If $f : X \to Y$ is an irresolute, then $Y$ is a semi-neighbourhood semi star-Lindelöf space.*

*Proof.* Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a semi-open cover of $Y$. Then for each $n \in \mathbb{N}$, the set $\mathcal{U}_n^{'} = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ is a semi open cover of $X$. Since $X$ is semi-neighbourhood semi star-Lindelöf, there are countable $(F_n \subset X)$, $n \in \mathbb{N}$, so that for every semi-open $O \supset F_n$, $n \in \mathbb{N}$, we have $\{\mathrm{St}(O,\mathcal{U}_n^{'}) : n \in \mathbb{N}\}$ is a semi cover of $X$. The sets $f(F_n)$, $n \in \mathbb{N}$, are countable in $Y$. Let $G \supset f(F_n)$ for each $n$ be semi-open set in $Y$. Then $f^{-1}(G) = H$ is a semi-open supset of $X$ for each $n \in \mathbb{N}$ and $H \supset F_n$. Thus $X = \mathrm{St}(H,\mathcal{U}_n')$. We prove that $Y = \mathrm{St}(G,\mathcal{U}_n)$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then there is $k \in \mathbb{N}$ such that $x \in \mathrm{St}(H,\mathcal{U}_k^{'})$. Then $y = f(x) \in f(\mathrm{St}(H,\mathcal{U}_k^{'})$. Because $f(\mathrm{St}(H,\mathcal{U}_k^{'})) \subset f(\mathrm{St}(f^{-1}(G),\mathcal{U}_k^{'})) \subset \mathrm{St}(G,\mathcal{U}_k)$ we have $y \in \mathrm{St}(G,\mathcal{U}_k)$. Therefore $Y = \mathrm{St}(G,\mathcal{U}_k)$, which shows that $Y$ is semi-neighbourhood semi star-Lindelöf. □

Recall that a space $X$ is semi-paraLindelöf if every semi-open cover $\mathcal{U}$ of $X$ has a locally countable semi-open refinement.

**Theorem 3.7.** *Every semi-paraLindelöf semi neighborhood semi star-Lindelöf space is semi Lindelöf.*

*Proof.* Let $X$ be a semi-paraLindelöf semi neighborhood semi star-Lindelöf space and $\mathcal{U}$ be a semi-open cover of $X$. Then there exists a locally countable semi-open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists a semi-open neighborhood $V_x$ of $x$ such that $V_x \subseteq V$ for some $V \in \mathcal{V}$ and $\{V \in \mathcal{V} : V_x \cap V \neq \emptyset\}$ is countable. Let $\mathcal{V}^{'} = \{V_x : x \in X\}$. Then $\mathcal{V}^{'}$ is a semi-open refinement of $\mathcal{V}$. Since $X$ is semi neighborhood semi star-Lindelöf, there exists a countable subset $A$ of $X$ such that for every semi-open $O \supseteq A$, $X \subseteq \mathrm{St}(O, \mathcal{V})$. We take $O = \bigcup\{V_x \in \mathcal{V}^{'} : x \in A\}$. Then $O$ is semi-open subset of $X$ and $A \subseteq O$. Thus $St(O, \mathcal{V}) \supseteq X$. Let $\mathcal{V}^{''} = \{V \in \mathcal{V} : V \cap O \neq \emptyset\}$. Then $\mathcal{V}^{''}$ is a countable semi-cover of $X$. For each $V \in \mathcal{V}^{''}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq U_V$. Then $\{U_V : V \in \mathcal{V}^{''}\}$ is a countable subcover of $\mathcal{U}$, then $X \subseteq \bigcup U_V$ which shows that $X$ is semi Lindelöf. □

**Theorem 3.8.** *If a space $X$ is semi neighborhood semi star-Menger meta semi-Lindelöf space then $X$ is semi-Lindelöf.*

*Proof.* Let $X$ be a $sNsSM$ meta semi-Lindelöf space and Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of $X$. Let $\mathcal{V}_n$ be a be a point-countable semi-open refinement of $\mathcal{U}_n$. Since $X$ is semi-neighborhood semi star-Menger, one can choose a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of $X$ such that for every semi-open $(O_n \supset F_n)$ we have $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{V}_n) = X$. Denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ that intersect $F_n \subset O_n$. As $\mathcal{V}_n$ is point countable and $F_n$ is finite then $\mathcal{W}_n$ is countable . So, the collection $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of $X$. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \in U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Then $X$ is a semi-Lindelöf space $(sL)$. □

In a similar way one can prove the following two theorems.

**Theorem 3.9.** *If a space $X$ is semi neighborhood semi star-Rothberger meta semi-Lindelöf space then $X$ is semi-Lindelöf.*

**Theorem 3.10.** *If a space $X$ is semi neighborhood semi star-Hurewicz meta semi-Lindelöf space then $X$ is semi-Lindelöf.*

**Definition 3.11.** Let $T$ be a subspace of space $X$. Then

(1) $T$ is relatively $sNsSL$ in $X$ if for every semi-open covers $\mathcal{U}$ of $X$, there exists a countable $A$ of $X$, such that for each semi-open $O \supset A$, $\bigcup\{St(O,\mathcal{U})\} \supset T$.
(2) $T$ is relatively closed $sNsSL$ spaces if it is closed and relatively $sNsSL$ in $X$.

**Proposition 3.12.** *If $X = \bigcup\{T_k : k \in \mathbb{N}\}$, and each $T_k$ is relatively $sNsSL$ in $X$, then $X$ is $sNsSL$.*

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