

BOAS-TYPE THEOREMS FOR THE LINEAR CANONICAL STURM-LIOUVILLE TRANSFORM

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ABSTRACT. We introduce generalized Lipschitz classes $\text{Lip}^M(\eta)$ and $\text{lip}^M(\eta)$ of functions associated with the canonical Sturm-Liouville operator

$$L^M := \frac{d^2}{dx^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x \frac{A'(x)}{A(x)} + i\frac{a}{b} \right),$$

where A is a nonnegative function satisfying certain conditions; and we prove two versions of Boas-type theorems for the canonical Sturm-Liouville transform \mathcal{F}^M . An application to the canonical Sturm-Liouville multipliers is given. Boas-type results for the canonical Fourier-Bessel transform and the canonical Fourier-Jacobi transform are special cases of this work.

1. INTRODUCTION

In connection with Titchmarsh's results ([28], Theorems 84 and 85), Boas found necessary and sufficient conditions on the Fourier coefficients of a function to belong to a generalized Lipschitz class, which is one of the classical topics in harmonic analysis and approximation theory. In 1967, Boas proved the first characterisation of this type, see [3]. In [19], Móricz has studied the continuity and smoothness properties of a function f with absolutely convergent Fourier series. Next, in [20], the author extended these results as in the following. If $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, and for some $0 < \eta \leq k$, $k \in \mathbb{N}$, we have

$$\int_{|\lambda| < s} |\lambda|^k |\widehat{f}(\lambda)| d\lambda = O(s^{k-\eta}) \quad \text{for all } s > 0,$$

then $\widehat{f} \in L^1(\mathbb{R})$ and f satisfies the smooth Lipschitz condition of order k . Here \widehat{f} stands for the Fourier transform of f .

There are many analogues of Boas-type theorems are also proved by Loualid et al. [15] for the Fourier-Bessel transform, by Volosivets [30] for the generalized Fourier-Bessel transform, by Rakhimi et al. [23] for the Fourier-Laguerre transform, and by Mannai et al. [17] for the (K, n) -Fourier transform.

Here, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix in $SL(2, \mathbb{R})$ such that $b > 0$. We define the canonical Sturm-Liouville operator L^M on \mathbb{R}_+^* by

$$L^M := \frac{d^2}{dx^2} + \left(\frac{A'(x)}{A(x)} - 2i\frac{a}{b}x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x \frac{A'(x)}{A(x)} + i\frac{a}{b} \right),$$

where A is a nonnegative function satisfying certain conditions.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator L^M is reduced to the Sturm-Liouville operator L :

$$L := \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.$$

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The classical Sturm-Liouville operator L plays an important role in analysis [31, 2]. In particular, the two references [4, 29] investigate standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with the operator L .

Using the Sturm-Liouville harmonic analysis [4, 29], for all $\lambda \in \mathbb{C}$, the system

$$\begin{cases} L^M u = -(\frac{\lambda^2}{b^2} + \rho^2)u \\ u(0) = e^{\frac{id}{2b}\lambda^2}, \quad u'(0) = 0 \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} \varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+,$$

where $\varphi_\lambda(x)$ is the Sturm-Liouville kernel [2].

In this paper, we introduce the canonical Sturm-Liouville transform \mathcal{F}^M :

$$\mathcal{F}^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^M(x) f(x) A(x) dx, \quad \lambda \in \mathbb{R}_+.$$

The canonical Sturm-Liouville transform \mathcal{F}^M can be regarded as a generalization of the Sturm-Liouville transform \mathcal{F} (see [6, 7, 16, 25, 26, 27]):

$$\mathcal{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) A(x) dx, \quad \lambda \in \mathbb{R}_+.$$

The main objective of this work is to prove two versions of Boas-type theorems for the canonical Sturm-Liouville transform \mathcal{F}^M . These versions are taken for $f \in L^1(\mathbb{R}_+, A(x)dx) \cap C(\mathbb{R}_+)$ and for $\mathcal{F}^M(f)$ meets certain conditions of integrability. These results are proven by means of the canonical Sturm-Liouville translation defined later in the Section 3. We conclude this work by providing an application for Theorem 4.3, which involves the study of a family of canonical Sturm-Liouville multipliers. Boas-type results for the canonical Fourier-Bessel transform and the canonical Fourier-Jacobi transform are special cases of our Theorem 4.3.

Note that if $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$, the operator L^M is reduced to the canonical Bessel operator L_α^M :

$$L_\alpha^M := \frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b} \right).$$

In this case $\varphi_\lambda^M(x) = \varphi_{\lambda,\alpha}^M(x) = e^{\frac{i}{2}(\frac{d}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha(\frac{\lambda x}{b})$, where j_α is the spherical Bessel function. The canonical transform \mathcal{F}^M is the canonical Fourier-Bessel transform \mathcal{F}_α^M :

$$\mathcal{F}_\alpha^M(f)(\lambda) := \int_0^\infty \varphi_{\lambda,\alpha}^M(x) f(x) x^{2\alpha+1} dx, \quad \lambda \in \mathbb{R}_+.$$

Recently, the canonical Fourier-Bessel transform \mathcal{F}_α^M is the goal of many applications in the harmonic analysis (see [9, 24, 13, 14, 18]).

This paper is organized as follows. In Section 2, we recall some results about the Sturm-Liouville transform \mathcal{F} and the Sturm-Liouville translation T_y . In Section 3, we introduce the canonical Sturm-Liouville operator L^M , and we investigate the properties of the canonical Sturm-Liouville transform \mathcal{F}^M and the canonical Sturm-Liouville translation T_y^M associated with this operator. In Section 4, we prove two versions of Boas-type results for the canonical Sturm-Liouville transform \mathcal{F}^M , and we give an application to canonical Sturm-Liouville multiplier operators. In Section 5, we discuss the special cases of Boas-type results for the canonical Fourier-Bessel transform and the canonical Fourier-Jacobi transform. In the last section, we summarize the obtained results and we describe the future work.

2. THE STURM-LIOUVILLE OPERATOR

We consider the second-order differential operator L defined on \mathbb{R}_+^* by

$$L := \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on \mathbb{R} such that $B(0) = 1$. Moreover we assume that A satisfies the following conditions:

- i) A is increasing and $\lim_{x \rightarrow \infty} A(x) = \infty$,
- ii) $\frac{A'}{A}$ is decreasing and $\lim_{x \rightarrow \infty} \frac{A'(x)}{A(x)} = 2\rho \geq 0$,
- iii) there exists a constant $\delta > 0$, such that

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x} D(x), \quad \text{if } \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha + 1}{x} + e^{-\delta x} D(x), \quad \text{if } \rho = 0,$$

where D is an infinitely differentiable function on \mathbb{R}_+^* , bounded and with bounded derivatives on all intervals $[x_0, \infty)$, for $x_0 > 0$.

This operator was studied in [4, 29], and the following results have been established:

For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} Lu = -(\lambda^2 + \rho^2)u \\ u(0) = 1, \quad u'(0) = 0 \end{cases}$$

admits a unique solution, denoted by φ_λ , with the following properties:

- for $x \in \mathbb{R}_+$, the function $\lambda \rightarrow \varphi_\lambda(x)$ is analytic on \mathbb{C} ,
- for $\lambda \in \mathbb{C}$, the function $x \rightarrow \varphi_\lambda(x)$ is even and infinitely differentiable on \mathbb{R} .

For nonzero $\lambda \in \mathbb{C}$, the equation

$$Lu = -(\lambda^2 + \rho^2)u,$$

has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}} V(x, \lambda),$$

with

$$\lim_{x \rightarrow \infty} V(x, \lambda) = 1.$$

Consequently there exists a function (spectral function) $\lambda \rightarrow c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \in \mathbb{R}_+,$$

for nonzero $\lambda \in \mathbb{C}$.

Moreover there exist positive constants k_1, k_2, k , such that

$$k_1 |\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2 |\lambda|^{2\alpha+1},$$

for all λ such that $\text{Im}\lambda \leq 0$ and $|\lambda| \geq k$.

Lemma 2.1. (See [26]). *The Sturm-Liouville function $\varphi_\lambda(x)$; $\lambda, x \in \mathbb{R}_+$, possesses the following properties.*

- i) $|\varphi_\lambda(x)| \leq 1$,
- ii) $1 - \varphi_\lambda(x) \leq \frac{1}{2}(\lambda^2 + \rho^2)x^2$.

We denote by

- μ the measure defined on \mathbb{R}_+ by

$$d\mu(x) := A(x)dx,$$

and by $L^p(\mu)$, $1 \leq p \leq \infty$, the space of measurable functions f on \mathbb{R}_+ , such that

$$\|f\|_{L^p(\mu)} := \left[\int_0^\infty |f(x)|^p d\mu(x) \right]^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty(\mu)} := \operatorname{ess\,sup}_{x \in \mathbb{R}_+} |f(x)| < \infty,$$

- ν the measure defined on \mathbb{R}_+ by

$$d\nu(\lambda) := \frac{d\lambda}{2\pi|c(\lambda)|^2},$$

and by $L^p(\nu)$, $1 \leq p \leq \infty$, the space of measurable functions f on \mathbb{R}_+ , such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm-Liouville transform is the Fourier transform associated with the operator L and is defined for $f \in L^1(\mu)$ by

$$\mathcal{F}(f)(\lambda) := \int_0^\infty \varphi_\lambda(x) f(x) d\mu(x), \quad \lambda \in \mathbb{R}_+. \quad (2.1)$$

Theorem 2.2. (See [4, 29, 31, 2]).

- Plancherel theorem.* The Sturm-Liouville transform \mathcal{F} extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular, we have

$$\|f\|_{L^2(\mu)} = \|\mathcal{F}(f)\|_{L^2(\nu)}.$$

- Inversion theorem.* Let $f \in L^1(\mu)$, such that $\mathcal{F}(f) \in L^1(\nu)$. Then

$$f(x) = \int_0^\infty \varphi_\lambda(x) \mathcal{F}(f)(\lambda) d\nu(\lambda), \quad \text{a.e. } x \in \mathbb{R}_+.$$

The Sturm-Liouville function φ_λ satisfies the product formula [4, 29]

$$\varphi_\lambda(x) \varphi_\lambda(y) = \int_0^\infty \varphi_\lambda(z) w(x, y, z) d\mu(z) \quad \text{for } x, y \in \mathbb{R}_+; \quad (2.2)$$

where $w(x, y, \cdot)$ is a measurable positive function on \mathbb{R}_+ , with support in $[|x - y|, x + y]$, satisfying

$$\int_0^\infty w(x, y, z) d\mu(z) = 1,$$

$$w(x, y, z) = w(y, x, z) \quad \text{for } z \in \mathbb{R}_+, \quad (2.3)$$

$$w(x, y, z) = w(x, z, y) \quad \text{for } z > 0. \quad (2.4)$$

We now define the generalized translation operator induced by (2.2). For $f \in L^1(\mu)$, the linear operator

$$T_y f(x) := \int_0^\infty f(z) w(x, y, z) d\mu(z), \quad x, y \in \mathbb{R}_+, \quad (2.5)$$

will be called Sturm-Liouville translation.

As a first remark, we note that the relation (2.3) means that

$$T_y f(x) = T_x f(y), \quad x, y \in \mathbb{R}_+.$$

Theorem 2.3. (See [26, 27]).

- For all $y \in \mathbb{R}_+$ and $f \in L^p(\mu)$, $1 \leq p \leq \infty$, we have*

$$\|T_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

ii) For $f \in L^1(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathcal{F}(T_y f)(\lambda) = \varphi_\lambda(y) \mathcal{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

3. THE CANONICAL STURM-LIOUVILLE OPERATOR

Throughout this paper, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix in $SL(2, \mathbb{R})$ such that $b > 0$. We define the canonical Sturm-Liouville operator L^M on \mathbb{R}_+^* by

$$L^M := \frac{d^2}{dx^2} + \left(\frac{A'(x)}{A(x)} - 2i \frac{a}{b} x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2} x^2 + i \frac{a}{b} x \frac{A'(x)}{A(x)} + i \frac{a}{b} \right),$$

where A is the nonnegative function given above in Section 2.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator L^M is reduced to the Sturm-Liouville operator L :

$$L := \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx}.$$

For all $\lambda \in \mathbb{C}$, the equation

$$\begin{cases} L^M u = -\left(\frac{\lambda^2}{b^2} + \rho^2\right)u \\ u(0) = e^{\frac{id}{2b}\lambda^2}, \quad u'(0) = 0 \end{cases}$$

admits a unique solution, denoted by φ_λ^M and given by

$$\varphi_\lambda^M(x) = e^{\frac{i}{2}\left(\frac{\lambda}{b}\lambda^2 + \frac{a}{b}x^2\right)} \varphi_{\frac{\lambda}{b}}(x), \quad x \in \mathbb{R}_+.$$

For $f \in L^1(\mu)$, we define the canonical Sturm-Liouville transform $\mathcal{F}^M(f)$ by

$$\mathcal{F}^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^M(x) f(x) d\mu(x), \quad \lambda \in \mathbb{R}_+.$$

This transform can be written as

$$\mathcal{F}^M(f)(\lambda) = e^{\frac{id}{2b}\lambda^2} \mathcal{F}\left(e^{\frac{ia}{2b}x^2} f\right)\left(\frac{\lambda}{b}\right), \quad f \in L^1(\mu), \quad (3.6)$$

where \mathcal{F} is the Sturm-Liouville transform given by (2.1).

We denote by ν_b , $b > 0$ the measure defined on \mathbb{R}_+ by

$$d\nu_b(\lambda) := \frac{d\lambda}{2\pi b |c(\frac{\lambda}{b})|^2},$$

and by $L^p(\nu_b)$, $1 \leq p \leq \infty$, the space of measurable functions f on \mathbb{R}_+ , such that $\|f\|_{L^p(\nu_b)} < \infty$.

Theorem 3.1.

i) Let $f \in L^1(\mu)$, such that $\mathcal{F}^M(f) \in L^1(\nu_b)$. Then

$$f(x) = \int_0^\infty \varphi_\lambda^N(x) \mathcal{F}^M(f)(\lambda) d\nu_b(\lambda), \quad a.e. \ x \in \mathbb{R}_+,$$

where N is the matrix given by $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$.

ii) For $f \in L^2(\mu)$ we have

$$\|\mathcal{F}^M(f)\|_{L^2(\nu_b)} = \|f\|_{L^2(\mu)}.$$

Proof.

i) follows from Theorem 2.2 ii) and relation (3.6).

ii) follows from Theorem 2.2 i) and relation (3.6). □

For $f \in L^1(\mu)$, we define the canonical Sturm-Liouville translation operators by

$$T_y^M f(x) := e^{\frac{ia}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{-\frac{ia}{2b}z^2} w(x, y, z) d\mu(z), \quad x, y \in \mathbb{R}_+.$$

It is easy to prove the following results.

Theorem 3.2. *The operators T_y^M , $y \in \mathbb{R}_+$, satisfy:*

- i) $T_y^M f(x) = T_x^M f(y)$, $x, y \in \mathbb{R}_+$.
- ii) $T_y^M f(x) = e^{\frac{ia}{2b}(x^2+y^2)} T_y \left(f(z) e^{-\frac{ia}{2b}z^2} \right) (x)$,
where T_y is the Sturm-Liouville translation given by (2.5).
- iii) $T_y^M \varphi_\lambda^M(x) = e^{-\frac{id}{2b}\lambda^2} \varphi_\lambda^M(x) \varphi_\lambda^M(y)$.

Theorem 3.3.

- i) For all $y \in \mathbb{R}_+$ and $f \in L^p(\mu)$, $1 \leq p \leq \infty$, we have

$$\|T_y^M f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

- ii) For $f \in L^1(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathcal{F}^M(T_y^N f)(\lambda) = e^{\frac{id}{2b}\lambda^2} \varphi_\lambda^N(y) \mathcal{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+,$$

$$\text{where } N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}.$$

Proof.

- i) follows from Theorem 2.3 i) and Theorem 3.2 ii).
- ii) Let $f \in L^1(\mu)$. Then

$$\begin{aligned} \mathcal{F}^M(T_y^N f)(\lambda) &= \int_0^\infty T_y^N f(x) \varphi_\lambda^M(x) d\mu(x) \\ &= \int_0^\infty \left[e^{-\frac{ia}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{\frac{ia}{2b}z^2} w(x, y, z) d\mu(z) \right] \varphi_\lambda^M(x) d\mu(x). \end{aligned}$$

By using Fubini's theorem, (2.3) and (2.4) we obtain

$$\mathcal{F}^M(T_y^N f)(\lambda) = e^{-\frac{ia}{2b}y^2} \int_0^\infty f(z) e^{\frac{ia}{2b}z^2} \left[\int_0^\infty \varphi_\lambda^M(x) e^{-\frac{ia}{2b}x^2} w(z, y, x) d\mu(x) \right] d\mu(z).$$

And by Theorem 3.2 iii) we deduce that

$$\mathcal{F}^M(T_y^N f)(\lambda) = e^{\frac{id}{2b}\lambda^2} \varphi_\lambda^N(y) \mathcal{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

The theorem is proved. □

Let $N = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix}$. The finite difference Δ_y^N is defined as follows

$$\Delta_y^N f(x) := T_y^N f(x) - e^{-\frac{ia}{2b}y^2} f(x), \quad y \in \mathbb{R}_+. \quad (3.7)$$

Theorem 3.4.

- i) For $f \in L^1(\mu)$, we have

$$\mathcal{F}^M(\Delta_y^N f)(\lambda) = e^{-\frac{ia}{2b}y^2} (\varphi_{\frac{\lambda}{b}}(y) - 1) \mathcal{F}^M(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

- ii) For $f \in L^1(\mu)$, such that $\mathcal{F}^M(f) \in L^1(\nu_b)$, we have

$$\Delta_y^N f(x) = \int_0^\infty e^{-\frac{ia}{2b}y^2} (\varphi_{\frac{\lambda}{b}}(y) - 1) \mathcal{F}^M(f)(\lambda) \varphi_\lambda^N(x) d\nu_b(\lambda), \quad \text{a.e. } x \in \mathbb{R}_+.$$

Proof.

- i) follows from Theorem 3.3 ii) and (3.7).
 ii) follows from i) and Theorem 3.1 i).

□

4. INVESTIGATION ON BOAS-TYPE THEOREMS

In this section we prove two versions of Boas-type theorems for the canonical Sturm-Liouville transform \mathcal{F}^M , and we give an application to canonical Sturm-Liouville multiplier operators.

Let g be a non-negative, measurable function defined on \mathbb{R}_+ .

Lemma 4.1. *If $0 < \eta \leq 2$, $\left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) \in L^1(\nu_b) \cap Loc(\mathbb{R}_+)$ and*

$$\int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) d\nu_b(\lambda) = O(s^{2-\eta}) \quad \text{for all } s > 0, \quad (4.8)$$

then $g\chi_{[s,\infty)} \in L^1(\nu_b)$ and

$$\int_s^\infty g(\lambda) d\nu_b(\lambda) = O(s^{-\eta}) \quad \text{for all } s > 0. \quad (4.9)$$

Proof. By (4.8), there exists a constant $C > 0$ such that for all $i \in \mathbb{Z}$, we have

$$\int_{2^i}^{2^{i+1}} \left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) d\nu_b(\lambda) \leq \int_0^{2^{i+1}} \left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) d\nu_b(\lambda) \leq C2^{(2-\eta)(i+1)}.$$

It is clear that

$$\frac{2^{2i}}{b^2} \int_{2^i}^{2^{i+1}} g(\lambda) d\nu_b(\lambda) \leq \int_0^{2^{i+1}} \left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) d\nu_b(\lambda) \leq C2^{(2-\eta)(i+1)},$$

whence it follows that

$$\int_{2^i}^{2^{i+1}} g(\lambda) d\nu_b(\lambda) \leq Cb^22^{2-\eta}2^{-i\eta}. \quad (4.10)$$

Then by (4.10) we get

$$\begin{aligned} \int_{2^i}^\infty g(\lambda) d\nu_b(\lambda) &= \sum_{j=i}^\infty \int_{2^j}^{2^{j+1}} g(\lambda) d\nu_b(\lambda) \\ &\leq Cb^22^{2-\eta} \sum_{j=i}^\infty 2^{-j\eta} = O(2^{-i\eta}). \end{aligned}$$

When $0 < s < \infty$, let $i \in \mathbb{Z}$ such that $2^i \leq s < 2^{i+1}$. It follows that

$$\begin{aligned} \int_s^\infty g(\lambda) d\nu_b(\lambda) &\leq \int_{2^i}^\infty g(\lambda) d\nu_b(\lambda) \\ &\leq C2^{-i\eta} = C2^\eta 2^{-(i+1)\eta} \\ &\leq C2^\eta s^{-\eta}. \end{aligned}$$

This proves (4.9) in the general case. □

Lemma 4.2. *If $0 < \eta \leq 2$, $\left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) \in L^1(\nu_b) \cap Loc(\mathbb{R}_+)$ and*

$$\int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2\right)g(\lambda) d\nu_b(\lambda) = o(s^{2-\eta}) \quad \text{as } s \rightarrow \infty, \quad (4.11)$$

then $g\chi_{[s,\infty)} \in L^1(\nu_b)$ for large s , and

$$\int_s^\infty g(\lambda) d\nu_b(\lambda) = o(s^{-\eta}) \quad \text{as } s \rightarrow \infty. \quad (4.12)$$

Proof. By (4.11), for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{Z}$ such that for all $i \geq i_0$, we have

$$\int_{2^i}^{2^{i+1}} \left(\frac{\lambda^2}{b^2} + \rho^2 \right) g(\lambda) d\nu_b(\lambda) \leq \int_0^{2^{i+1}} \left(\frac{\lambda^2}{b^2} + \rho^2 \right) g(\lambda) d\nu_b(\lambda) \leq \varepsilon 2^{(2-\eta)(i+1)}.$$

Let $i \geq i_0$. It is clear that

$$\frac{2^{2i}}{b^2} \int_{2^i}^{2^{i+1}} g(\lambda) d\nu_b(\lambda) \leq \int_0^{2^{i+1}} \left(\frac{\lambda^2}{b^2} + \rho^2 \right) g(\lambda) d\nu_b(\lambda) \leq \varepsilon 2^{(2-\eta)(i+1)},$$

whence it follows that

$$\int_{2^i}^{2^{i+1}} g(\lambda) d\nu_b(\lambda) \leq \varepsilon b^2 2^{2-\eta} 2^{-i\eta}. \quad (4.13)$$

Then by (4.13) we get

$$\begin{aligned} \int_{2^i}^{\infty} g(\lambda) d\nu_b(\lambda) &= \sum_{j=i}^{\infty} \int_{2^j}^{2^{j+1}} g(\lambda) d\nu_b(\lambda) \\ &\leq \varepsilon b^2 2^{2-\eta} \sum_{j=i}^{\infty} 2^{-j\eta} \\ &\leq C\varepsilon 2^{-i\eta}. \end{aligned}$$

When $0 < s < \infty$, let $i \in \mathbb{Z}$ such that $2^i \leq s < 2^{i+1}$. It follows that

$$\begin{aligned} \int_s^{\infty} g(\lambda) d\nu_b(\lambda) &\leq \int_{2^i}^{\infty} g(\lambda) d\nu_b(\lambda) \\ &\leq C\varepsilon 2^{-i\eta} = C\varepsilon 2^\eta 2^{-(i+1)\eta} \\ &\leq C\varepsilon 2^\eta s^{-\eta}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then

$$\int_s^{\infty} g(\lambda) d\nu_b(\lambda) = o(s^{-\eta}).$$

This proves (4.12) in the general case. \square

At present we define, the Lipschitz class $\text{Lip}^M(\eta)$ and the little Lipschitz class $\text{lip}^M(\eta)$. A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said belong to $\text{Lip}^M(\eta)$ for $\eta > 0$ if

$$|\Delta_y^N f(x)| = O(y^\eta) \quad \text{as } y \rightarrow 0,$$

and is said belong to $\text{lip}^M(\eta)$ for $\eta > 0$ if

$$|\Delta_y^N f(x)| = o(y^\eta) \quad \text{as } y \rightarrow 0.$$

Theorem 4.3. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $0 < \eta \leq 2$. Suppose that $f \in L^1(\mu) \cap C(\mathbb{R}_+)$.*

i) *If*

$$\int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) = O(s^{2-\eta}) \quad \text{for all } s > 0, \quad (4.14)$$

then $\mathcal{F}^M(f) \in L^1(\nu_b)$ and $f \in \text{Lip}^M(\eta)$.

ii) *If*

$$\int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) = o(s^{2-\eta}) \quad \text{as } s \rightarrow \infty, \quad (4.15)$$

then $\mathcal{F}^M(f) \in L^1(\nu_b)$ and $f \in \text{lip}^M(\eta)$.

Proof.

i) From Lemma 4.1, statement (4.14) implies that

$$\int_s^\infty |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) < \infty.$$

Since $\mathcal{F}^M(f) \in C(\mathbb{R}_+)$, then $\mathcal{F}^M(f) \in L^1(\nu_b)$.

Now, let us prove that $f \in \text{Lip}^M(\eta)$. Let $x \in \mathbb{R}_+$ and $y > 0$. By Theorem 3.4 ii) we have

$$|\Delta_y^N f(x)| = \left| \int_0^\infty e^{-\frac{iq}{2b}y^2} (1 - \varphi_{\frac{\lambda}{b}}(y)) \mathcal{F}^M(f)(\lambda) \varphi_\lambda^N(x) d\nu_b(\lambda) \right| \leq I_1 + I_2, \quad (4.16)$$

where

$$I_1 = \int_0^{\frac{1}{y}} (1 - \varphi_{\frac{\lambda}{b}}(y)) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda)$$

and

$$I_2 = \int_{\frac{1}{y}}^\infty (1 - \varphi_{\frac{\lambda}{b}}(y)) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda).$$

By Lemma 2.1 ii) we get

$$I_1 \leq y^2 \int_0^{\frac{1}{y}} \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda), \quad (4.17)$$

and by (4.14) we deduce that

$$I_1 = y^2 O(y^{\eta-2}) = O(y^\eta). \quad (4.18)$$

On the other hand, by Lemma 2.1 i), we have

$$I_2 \leq 2 \int_{\frac{1}{y}}^\infty |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda). \quad (4.19)$$

By Lemma 4.1 and (4.14), we get

$$I_2 = O(y^\eta). \quad (4.20)$$

Combining (4.18) and (4.20) yields $f \in \text{Lip}^M(\eta)$.

ii) From Lemma 4.2, statement (4.15) implies that

$$\int_s^\infty |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) < \infty \quad \text{as } s \rightarrow \infty.$$

Then $\mathcal{F}^M(f) \in L^1(\nu_b)$.

Now, let us prove that $f \in \text{lip}^M(\eta)$. Again, we start with the estimate (4.16). By (4.17) and (4.15) we conclude that

$$I_1 = y^2 o(y^{\eta-2}) = o(y^\eta) \quad \text{as } y \rightarrow 0. \quad (4.21)$$

On the other hand, by Lemma 4.2, (4.15) and (4.19), we obtain

$$I_2 = o(y^\eta) \quad \text{as } y \rightarrow 0 \quad (4.22)$$

Combining (4.21) and (4.22) yields $f \in \text{lip}^M(\eta)$. □

In the next part of this section we give application to canonical Sturm-Liouville multiplier operators. Let X_m be a multiplier operator defined by the relation

$$\mathcal{F}^M(X_m f)(\lambda) = m(\lambda) \mathcal{F}^M(f)(\lambda),$$

where m be a measurable function on \mathbb{R}_+ satisfying, there exist $\varepsilon, c > 0$ such that, for all $\lambda > 0$,

$$|m(\lambda)| \leq c\lambda^\varepsilon. \quad (4.23)$$

Theorem 4.4. *Let m be a multiplier satisfies (4.23), then*

- i) X_m is bounded from $Lip^M(\eta)$ into $Lip^M(\eta - \varepsilon)$, for $\varepsilon < \eta \leq 2$.
- ii) X_m is bounded from $lip^M(\eta)$ into $lip^M(\eta - \varepsilon)$, for $\varepsilon < \eta \leq 2$.

Proof.

i) Let $f \in Lip^M(\eta)$. Then by Theorem 4.3 i) we have

$$\begin{aligned} \int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}^M(X_m f)(\lambda)| d\nu_b(\lambda) &= \int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |m(\lambda)| |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) \\ &\leq c \int_0^s \lambda^\varepsilon \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) \\ &\leq cs^\varepsilon \int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}^M(f)(\lambda)| d\nu_b(\lambda) \\ &= O(s^{2-(\eta-\varepsilon)}) \quad \text{for all } s > 0. \end{aligned}$$

Again using Theorem 4.3 i), we get

$$X_m f \in Lip^M(\eta - \varepsilon).$$

The i) is proved.

- ii) As in the same of i) we prove ii).

□

Example 4.5. Let $\sigma > 0$, we but

$$m_\sigma(\lambda) := \left(\frac{\lambda}{b} \right)^\sigma, \quad \lambda \in \mathbb{R}_+.$$

The operator X_{m_σ} is studied by Bloom and Xu [2] in the Sturm-Liouville case. Then by application of Theorem 4.4, we deduce that

- i) $X_{m_\sigma} : Lip^M(\eta) \rightarrow Lip^M(\eta - \sigma)$ is bounded, for $\sigma < \eta \leq 2$.
- ii) $X_{m_\sigma} : lip^M(\eta) \rightarrow lip^M(\eta - \sigma)$ is bounded, for $\sigma < \eta \leq 2$.

5. SPECIAL CASES

Boas-type results for the canonical Fourier-Bessel transform and the canonical Fourier-Jacobi transform are special cases of Theorem 4.3.

5.1. The canonical Fourier-Bessel transform. In this case $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$ and $\rho = 0$. The operator L^M is reduced to the canonical Bessel operator L_α^M :

$$L_\alpha^M := \frac{d^2}{dx^2} + \left(\frac{2\alpha+1}{x} - 2i\frac{a}{b}x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + 2i(\alpha+1)\frac{a}{b} \right).$$

In this case

$$\varphi_\lambda^M(x) = \varphi_\lambda^{\alpha, M}(x) = e^{\frac{i}{2}(\frac{a}{b}\lambda^2 + \frac{a}{b}x^2)} j_\alpha\left(\frac{\lambda x}{b}\right),$$

where j_α is the spherical Bessel function. We denote by μ_α the measure defined by $d\mu_\alpha(x) := x^{2\alpha+1} dx$.

The canonical Fourier-Bessel transform \mathcal{F}_α^M is defined for $f \in L^1(\mu_\alpha)$ by

$$\mathcal{F}_\alpha^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^{\alpha, M}(x) f(x) d\mu_\alpha(x), \quad \lambda \in \mathbb{R}_+.$$

The canonical Fourier-Bessel transform \mathcal{F}_α^M can be regarded as a generalization of the Fourier-Bessel transform [10, 11]. Recently, this transform is the goal of many applications in the harmonic analysis (see [9, 24, 13, 14, 18]).

The canonical Fourier-Bessel translation operators are defined for $f \in L^1(\mu_\alpha)$ by

$$T_y^{\alpha, M} f(x) := e^{\frac{ia}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{-\frac{ia}{2b}z^2} w_\alpha(x, y, z) d\mu_\alpha(z), \quad x, y \in \mathbb{R}_+,$$

being $w_\alpha(x, y, \cdot)$ the kernel [1, 21] given by

$$w_\alpha(x, y, z) = a_\alpha \frac{[(x+y)^2 - z^2]^{\alpha-\frac{1}{2}} [z^2 - (x-y)^2]^{\alpha-\frac{1}{2}}}{2^{2\alpha-1}(xyz)^{2\alpha}} \chi_{(|x-y|, x+y)}(z),$$

where $a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$ and $\chi_{(|x-y|, x+y)}$ is the characteristic function of the interval $(|x-y|, x+y)$.

The finite difference $\Delta_y^{\alpha, N}$ is defined as follows

$$\Delta_y^{\alpha, N} f(x) := T_y^{\alpha, N} f(x) - e^{-\frac{ia}{2b}y^2} f(x), \quad y \in \mathbb{R}_+.$$

A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said belong to $\text{Lip}_\alpha^M(\eta)$ for $\eta > 0$ if

$$|\Delta_y^{\alpha, N} f(x)| = O(y^\eta) \quad \text{as } y \rightarrow 0,$$

and is said belong to $\text{lip}_\alpha^M(\eta)$ for $\eta > 0$ if

$$|\Delta_y^{\alpha, N} f(x)| = o(y^\eta) \quad \text{as } y \rightarrow 0.$$

Theorem 5.1. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $0 < \eta \leq 2$. Suppose that $f \in L^1(\mu_\alpha) \cap C(\mathbb{R}_+)$.*

i) *If*

$$\int_0^s \lambda^2 |\mathcal{F}_\alpha^M(f)(\lambda)| d\mu_\alpha(\lambda) = O(s^{2-\eta}) \quad \text{for all } s > 0,$$

then $\mathcal{F}_\alpha^M(f) \in L^1(\mu_\alpha)$ and $f \in \text{Lip}_\alpha^M(\eta)$.

ii) *If*

$$\int_0^s \lambda^2 |\mathcal{F}_\alpha^M(f)(\lambda)| d\mu_\alpha(\lambda) = o(s^{2-\eta}) \quad \text{as } s \rightarrow \infty,$$

then $\mathcal{F}_\alpha^M(f) \in L^1(\mu_\alpha)$ and $f \in \text{lip}_\alpha^M(\eta)$.

5.2. The canonical Fourier-Jacobi transform. In this case

$$A(x) = \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x),$$

$\alpha > \beta \geq -1/2$ and $\rho = \alpha + \beta + 1$. The operator L^M is reduced to the canonical Jacobi operator $L_{\alpha, \beta}^M$:

$$L_{\alpha, \beta}^M := \frac{d^2}{dx^2} + \left(h_{\alpha, \beta}(x) - 2i\frac{a}{b}x \right) \frac{d}{dx} - \left(\frac{a^2}{b^2}x^2 + i\frac{a}{b}x h_{\alpha, \beta}(x) + i\frac{a}{b} \right),$$

where

$$h_{\alpha, \beta}(x) = (2\alpha + 1) \coth(x) + (2\beta + 1) \tanh(x).$$

In this case

$$\varphi_\lambda^M(x) = \varphi_\lambda^{(\alpha, \beta), M}(x) = e^{\frac{i}{2}(\frac{a}{b}\lambda^2 + \frac{a}{b}x^2)} \varphi_\lambda^{(\alpha, \beta)}(x),$$

where $\varphi_\lambda^{(\alpha, \beta)}(x)$ is the Jacobi function given by

$$\varphi_\lambda^{(\alpha, \beta)}(x) = {}_2F_1\left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha + 1, -\sinh^2(x)\right),$$

being ${}_2F_1(a, b, c, z)$ the hypergeometric function.

We denote by $\mu_{\alpha, \beta}$, $\nu_{\alpha, \beta}$ and $\nu_{\alpha, \beta, b}$ the measures defined respectively by

$$d\mu_{\alpha, \beta}(x) := \sin^{2\alpha+1}(x) \cosh^{2\beta+1}(x) dx, \quad d\nu_{\alpha, \beta}(\lambda) := \frac{d\lambda}{2\pi |c_{\alpha, \beta}(\lambda)|^2},$$

and

$$d\nu_{\alpha,\beta,b}(\lambda) := \frac{d\lambda}{2\pi b |c_{\alpha,\beta}(\frac{\lambda}{b})|^2},$$

where

$$c_{\alpha,\beta}(\lambda) = \frac{\Gamma(i\lambda)\Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\rho+i\lambda)-\beta)}.$$

The canonical Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}^M$ is defined for $f \in L^1(\mu_{\alpha,\beta})$ by

$$\mathcal{F}_{\alpha,\beta}^M(f)(\lambda) := \int_0^\infty \varphi_\lambda^{(\alpha,\beta),M}(x) f(x) d\mu_{\alpha,\beta}(x), \quad \lambda \in \mathbb{R}_+.$$

The canonical Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}^M$ can be regarded as a generalization of the Fourier-Jacobi transform [5, 8].

The canonical Fourier-Jacobi translation operators are defined for $f \in L^1(\mu_{\alpha,\beta})$ by

$$T_y^{(\alpha,\beta),M} f(x) := e^{\frac{ia}{2b}(x^2+y^2)} \int_0^\infty f(z) e^{-\frac{ia}{2b}z^2} w_{\alpha,\beta}(x,y,z) d\mu_{\alpha,\beta}(z), \quad x,y \in \mathbb{R}_+,$$

being $w_{\alpha,\beta}(x,y,\cdot)$ the kernel [12, 22] given by

$$\begin{aligned} w_{\alpha,\beta}(x,y,z) &= a_\alpha \frac{[\cosh(x) \cosh(y) \cosh(z)]^{-(\alpha+\beta+1)}}{[\sinh(x) \sinh(y) \sinh(z)]^{2\alpha}} (1-B^2)^{\alpha-\frac{1}{2}} \\ &\times {}_2F_1(\alpha+\beta, \alpha-\beta, \alpha+\frac{1}{2}, \frac{1}{2}(1-B)) \chi_{(|x-y|, x+y)}(z), \end{aligned}$$

where

$$B = \frac{\cosh^2(x) + \cosh^2(y) + \cosh^2(z) - 1}{2 \cosh(x) \cosh(y) \cosh(z)}.$$

The finite difference $\Delta_y^{(\alpha,\beta),N}$ is defined as follows

$$\Delta_y^{(\alpha,\beta),N} f(x) := T_y^{(\alpha,\beta),N} f(x) - e^{-\frac{ia}{2b}y^2} f(x), \quad y \in \mathbb{R}_+.$$

A function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said belong to $\text{Lip}_{\alpha,\beta}^M(\eta)$ for $\eta > 0$ if

$$|\Delta_y^{(\alpha,\beta),N} f(x)| = O(y^\eta) \quad \text{as } y \rightarrow 0,$$

and is said belong to $\text{lip}_{\alpha,\beta}^M(\eta)$ for $\eta > 0$ if

$$|\Delta_y^{(\alpha,\beta),N} f(x)| = o(y^\eta) \quad \text{as } y \rightarrow 0.$$

Theorem 5.2. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $0 < \eta \leq 2$. Suppose that $f \in L^1(\mu_{\alpha,\beta}) \cap C(\mathbb{R}_+)$.*

i) *If*

$$\int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}_{\alpha,\beta}^M(f)(\lambda)| d\nu_{\alpha,\beta,b}(\lambda) = O(s^{2-\eta}) \quad \text{for all } s > 0,$$

then $\mathcal{F}_{\alpha,\beta}^M(f) \in L^1(\nu_{\alpha,\beta,b})$ and $f \in \text{Lip}_{\alpha,\beta}^M(\eta)$.

ii) *If*

$$\int_0^s \left(\frac{\lambda^2}{b^2} + \rho^2 \right) |\mathcal{F}_{\alpha,\beta}^M(f)(\lambda)| d\nu_{\alpha,\beta,b}(\lambda) = o(s^{2-\eta}) \quad \text{as } s \rightarrow \infty,$$

then $\mathcal{F}_{\alpha,\beta}^M(f) \in L^1(\nu_{\alpha,\beta,b})$ and $f \in \text{lip}_{\alpha,\beta}^M(\eta)$.

6. CONCLUSION AND PERSPECTIVE

In this work we succeeded in generalizing the results of Loualid et al. in [15] for the canonical Sturm-Liouville transform in the classes $\text{Lip}^M(\eta)$ and $\text{lip}^M(\eta)$. The harmonic analysis associated with the canonical Sturm-Liouville operator L^M plays an important role in establishing the results of this paper. Boas-type results for the canonical Fourier-Bessel transform and the canonical Fourier-Jacobi transform are special cases of this work. However, we left the same problem as an open topic for other linear transformations.

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