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ON LACUNARY WEAK CONVERGENCE OF DOUBLE SEQUENCES DEFINED BY ORLICZ FUNCTIONS: AN ANALYSIS OF TOPOLOGICAL AND ALGEBRAIC STRUCTURES

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Abstract. This paper provides a comprehensive study of lacunary weak convergence for double sequences, defined through Orlicz functions. It delves into the examination of significant topological and algebraic properties, such as solidity, symmetry, and monotonicity, within the framework of these spaces. To enhance the theoretical foundation, the study includes a range of illustrative examples that highlight instances where certain conditions fail. Furthermore, the paper investigates and establishes inclusion relationships between the newly defined spaces and other existing spaces in the literature. The findings significantly contribute to the broader understanding of sequence spaces, particularly focusing on their structural and convergence characteristics. These results not only enhance the mathematical framework but also provide a foundation for future research into the applications and implications of lacunary weak convergence in double sequences.

1. Introduction

The foundational work on lacunary sequences was carried out by Freedman et al. [7]. They studied Cesàro summable sequences and strongly lacunary convergent sequences, considering a general lacunary sequence θ, and established connections between the classes of these two types of sequences. Fridy and Orhan [8] explored the concept of lacunary statistical convergence through the use of lacunary sequences. Their publication had a profound impact on various scientific fields. Çakan and Altay [3] highlighted the multidimensional parallels in the results presented by Fridy and Orhan [8]. The concept of difference sequence spaces was first presented by Kızmaz [10]. He investigated the characteristics of the difference sequence spaces $X(\Delta) = \{\mathfrak{p} = (\mathfrak{p}_u) \in w : (\Delta\mathfrak{p}_u) \in X\}$, $\Delta\mathfrak{p}_u = \mathfrak{p}_{u+1} - \mathfrak{p}_u$ for all $u \in \mathbb{N}$, and specifically examined cases where X includes spaces such as l_∞, c, c_0. Tripathy and Esi [17] introduced lacunary generalized difference paranormed sequence spaces $[N_\theta, M, A, \Delta^m, p]_0$, $[N_\theta, M, A, \Delta^m, p]_0$, $[N_\theta, M, A, \Delta^m, p]_\infty$ based on Orlicz functions, with $A = (a_{nk})$ representing a non-negative matrix. In parallel, Tripathy and Et [18] introduced the concepts of Δ^m-lacunary statistical convergence and Δ^m-lacunary strong convergence, providing a comprehensive analysis of their foundational properties. Building on this, Tripathy and Mahanta [20] extended the theory by defining the generalized lacunary difference sequence spaces $[N_\theta, M, \Delta^m]_0$, $[N_\theta, M, \Delta^m]_0$, $[N_\theta, M, \Delta^m]_\infty$ using m^{th}-difference operators. Their study focused on key properties such as completeness, solidness, and symmetry, and they explored important inclusion relationships between these spaces and other sequence spaces like Cesro summable and strongly Cesro summable sequences. In parallel, Tamuli and Tripathy [15, 16] developed the concept of generalized difference lacunary weak convergence and its connection to lacunary weak convergence in Orlicz function spaces. Their work paved the way for a multidimensional approach to refining lacunary sequences, as seen in [13], where the authors utilized this concept to present the dual inclusion theorem, further enriching the field's theoretical framework. Savaş and Patterson [14] obtained some new double sequence

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spaces using the Orlicz function. Yaying and Hazarika [21] put forward the notion of lacunary arithmetic statistical convergence, thereby introducing a novel perspective to the study of sequence convergence by incorporating lacunary structures into these convergence concepts. The convergence of complex uncertain sequences within a given uncertainty space has gained significant attention in recent years. In these spaces, new sequence spaces have been defined, and their convergence properties have been thoroughly examined. Some of the pioneering works in this area can be found in references [4, 5, 6, 12].

Banach [1] introduced the notion of weak convergence in normed linear spaces and explored the concept from this perspective. While this approach is intriguing, it has certain limitations. Many results derived from these ideas are generally applicable only to separable spaces. In recent years, Tripathy and Mahanta [19] have examined vector-valued sequence spaces, along with many other related topics.

This paper explores the topological and algebraic properties of lacunary weak convergence spaces for double sequences, offering a thorough examination of key concepts such as solidity, symmetry, and monotonicity. By analyzing these properties, the study highlights how they shape the structure and convergence behavior of sequences within these spaces. Additionally, the paper establishes inclusion relationships between the newly defined spaces and others found in the literature, further clarifying their topological and algebraic interconnections. These findings not only deepen the understanding of sequence spaces but also shed light on their structural characteristics, justifying the title's focus on the analysis of both topological and algebraic structures.

2. Preliminaries

Lindenstrauss and Tzafriri [11] applied the concept of the Orlicz function to develop a sequence space, which is defined as follows:

$$\ell_{\mathfrak{R}} := \left\{ \mathfrak{p} = (p_u) \in \omega : \sum_{u=1}^{\infty} \mathfrak{L}\left(\frac{|\mathfrak{p}_u|}{\gamma}\right) < \infty, \text{ for some } \gamma > 0 \right\},$$

where ω represents the set of all sequences.

Definition 2.1. [16] An Orlicz function $\mathfrak{L}$ satisfies the Δ_2 condition if $\exists$ a constant $U > 0$, such that for all $f \geq 0$ the inequality $\mathfrak{R}(2f) \leq U\mathcal{L}(f)$ holds.

Definition 2.2. [2] A sequence $(\mathfrak{p}_u) \in \mathbb{X}$ is weakly convergent if there is an element $\mathfrak{p}_0 \in \mathbb{X}$ so that

$$\lim_{u\to\infty} \mathfrak{g}(\mathfrak{p}_u - \mathfrak{p}_0) = 0, \text{ for all } \mathfrak{g} \in \mathbb{X}'$$

where $\mathbb{X}'$ is the algebraic dual of the normed linear space $\mathbb{X}$.

Definition 2.3. [16] A sequence $(\mathfrak{p}_u)$ is considered lacunary weakly convergent to $\mathfrak{p}_0 \in \mathbb{X}$ provided that

$$\lim_{\mathfrak{a}\to\infty} \frac{1}{\mathfrak{h}_{\mathfrak{a}}} \sum_{u\in I_{\mathfrak{a}}} \mathfrak{g}(\mathfrak{p}_u - \mathfrak{p}_0) = 0,$$

for all $g \in \mathbb{X}'$. The set of lacunary weak convergent is indicated by $\mathcal{D}_\theta^\omega$. The subset of those sequences in $\mathcal{D}_\theta^\omega$ for which $\mathfrak{p}_0 = 0$ is denoted by $[\mathcal{D}_\theta^\omega]_0$, and $[\mathcal{D}_\theta^\omega]_\infty$ represents the set of lacunary weakly bounded sequences.

Definition 2.4. [9] A sequence space $\mathbb{M}$ is referred to as solid if for every sequence $(p_u) \in \mathbb{M}$, the sequence $(\alpha_u p_u)$ belongs to $\mathbb{M}$ whenever $|\alpha_u| \leq 1, \forall u \in \mathbb{N}$, where α_u is scalar.

Definition 2.5. [9] Consider $\mathfrak{J} = \{\mathfrak{i}_1 < \mathfrak{i} < \cdots < \mathfrak{i}_u < ...\} \subset \mathbb{N}$. For a sequence $(\mathfrak{p}_u) \in \omega$, the $\mathfrak{J}$-step space associated with the sequence space $\mathbb{M} \subset \omega$ is given by

$$\lambda_{\mathfrak{J}}^{\mathbb{M}} = \{(\mathfrak{p}_{\mathfrak{i}_u}) \in \omega : (\mathfrak{p}_u) \in \mathbb{M}\}.$$

Definition 2.6. [9] The canonical pre-image $(\mathfrak{q}_u)$ corresponding to a sequence $(\mathfrak{p}_u) \in \mathbb{M}$, where $\mathfrak{J}$-step space $\lambda_{\mathfrak{J}}^{\mathbb{M}}$ is taken into account, can be expressed as:

$$\mathfrak{q}_u = \begin{cases} \mathfrak{p}_u, & \text{if } u \in \mathbb{N} \\ 0, & \text{if not.} \end{cases}$$

Definition 2.7. [9] If the sequence space $\mathbb{M}$ is a subset of ω and includes every pre-image of its step spaces, it is termed as monotone.

Definition 2.8. [9] The sequence space $\mathbb{M} \subset \omega$ is termed symmetric if, for any sequence $(\mathfrak{p}_u) \in \mathbb{M}$, the
permuted sequence $\left(\mathfrak{p}_{\pi(i)}\right) \in \mathbb{M}$, where π represents a permutation of $\mathbb{N}$.

Lemma 2.9. [9] *In the setting of sequence space $\mathbb{M}$, if the space exhibits solidity, it automatically suggests monotonicity. Nevertheless, the reverse does not always hold true.*

Let θ_1 and θ_2 be arbitrary lacunary sequences, and define the sets

$$I_{\mathfrak{a}} = \{\mathfrak{m} : \mathfrak{m}_{\mathfrak{a}-1} < \mathfrak{m} \leq \mathfrak{m}_a\}, I_{\mathfrak{b}} = \{\mathfrak{n} : \mathfrak{n}_{\mathfrak{b}-1} < \mathfrak{n} \leq \mathfrak{n}_{\mathfrak{b}}\}.$$

A set $\mathcal{A} \subseteq \mathbb{N}^2$ has a double lacunary density $\delta_2^{\theta}(\mathcal{A})$ (see [3]) if

$$\lim_{\mathfrak{a},b\to\infty} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} |\{u \in I_{\mathfrak{a}}, w \in I_{\mathfrak{b}} : (u,w) \in \mathcal{A}\}| = 0,$$

exists, where $\mathfrak{h}_{\mathfrak{a},\mathfrak{b}} = \mathfrak{h}_{\mathfrak{a}}\overline{\mathfrak{h}}_{\mathfrak{b}}$, and

$$\mathfrak{m}_0 = 0,\ \mathfrak{h}_{\mathfrak{a}} = \mathfrak{m}_{\mathfrak{a}} - \mathfrak{m}_{\mathfrak{b}-1},\ \text{as } \mathfrak{a} \to \infty, \mathfrak{n}_0 = 0,\ \overline{\mathfrak{h}}_{\mathfrak{b}} = \mathfrak{n}_{\mathfrak{b}} - \mathfrak{n}_{\mathfrak{b}-1},\ \text{as } \mathfrak{b} \to \infty.$$

Example 2.10. [13] Let $\theta_1 = (2^{\mathfrak{a}} - 1)$ and $\theta_2 = \left(3^{\mathfrak{b}} - 1\right)$ and $\mathcal{A} = \left\{\left(u^2, w^2\right) : u, w \in \mathbb{N}\right\}$. Then, $\delta_2^{\theta}(\mathcal{A}) = 0$.

Example 2.11. [13] Let $\theta_1 = (2^{\mathfrak{a}} - 1)$ and $\theta_2 = \left(3^{\mathfrak{b}} - 1\right)$ and $\mathcal{A} = \{(u, 3w) : u, w \in \mathbb{N}\}$. Then, $\delta_2^{\theta}(\mathcal{A}) = 0$.

A sequence $\mathfrak{p} = (\mathfrak{p}_{u,w})$ is considered lacunary statistically convergent to a number $\mathfrak{p}_0$ if for every $\gamma > 0$, the set $\{(u,w) : |p_{u,w} - \mathfrak{p}_0| \geq \gamma\}$ has double lacunary density zero. In this case, we express it as $S_2^{\theta} - \lim \mathfrak{p}_{u,w} = \mathfrak{p}_0$.

3. Main Results

Let $\mathfrak{q} = (\mathfrak{q}_{\mathfrak{u},\mathfrak{w}})$ represent a sequence of strictly positive real numbers, and $\mathcal{L}$ be an Orlicz function. In this context, we introduce certain classes of sequences:

$$[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0 := \left\{ \mathfrak{p} = (\mathfrak{p}_{u,w}) \in \omega : \lim_{\mathfrak{a},\mathfrak{b}\to\infty} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}} = 0, \right.$$
$$\left. \text{for some } \gamma > 0 \right\},$$

$$[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1 := \left\{ \mathfrak{p} = (\mathfrak{p}_{u,w}) \in \omega : \lim_{\mathfrak{a},\mathfrak{b}\to\infty} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w}-\mathfrak{p}_0)|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}} = 0, \right.$$
$$\left. \text{for some } \mathfrak{p}_0,\ \text{and } \gamma > 0 \right\},$$

$$[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_\infty := \left\{ \mathfrak{p} = (\mathfrak{p}_{u,w}) \in \omega : \sup_{\mathfrak{a},\mathfrak{b}} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}} < \infty, \right.$$
$$\left. \text{for some } \gamma > 0 \right\}.$$

Here, $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$, $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$, $[\mathcal{D}_\theta^\omega, \mathcal{L}, \mathfrak{q}]_\infty$ are lacunary weak convergence sequence spaces that are, respectively, zero-convergent, convergent, and bounded for double sequences defined by Orlicz functions.

When $\mathfrak{L}(\mathfrak{p}) = \mathfrak{p}$, the spaces $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$, $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$, $[\mathcal{D}_\theta^\omega, \mathcal{L}, \mathfrak{q}]_\infty$ are denoted as $[\mathcal{D}_\theta^\omega, \mathfrak{q}]_0$, $[\mathcal{D}_\theta^\omega, \mathfrak{q}]_1$, $[\mathcal{D}_\theta^\omega, \mathfrak{q}]_\infty$ respectively.
Moreover, if $\mathfrak{q}_{u,w} = 1$, for all $u, w = 1$, these spaces simplify to $[\mathcal{D}_\theta^\omega, \mathfrak{L}]_0$, $[\mathcal{D}_\theta^\omega, \mathfrak{L}]_1$, $[\mathcal{D}_\theta^\omega, \mathfrak{L}]_\infty$, respectively.
Example 3.1, 3.2, and 3.3 serve as illustrations of the spaces $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$, $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$, $[\mathcal{D}_\theta^\omega, \mathcal{L}, \mathfrak{q}]_\infty$.

Example 3.1. Consider $\theta_1 = (2^a - 1), \theta_2 = \left(2^b - 1\right)$ be any lacunary sequences and $\mathfrak{L}(\mathfrak{p}) = \mathfrak{p}^2$, and $\mathfrak{q}_{u,w} = 1, \forall u, w \in \mathbb{N}$ and $\mathfrak{g}(\mathfrak{p}) = \mathfrak{p}$. Consider the sequence $\mathfrak{p} = (\mathfrak{p}_{u,w})$ where $\mathfrak{p}_{u,w} = \frac{1}{uw}$. Then, $\mathfrak{p}_{u,w} \in [\mathcal{D}_\theta^\omega, \mathcal{L}, \mathfrak{q}]_0$.

Example 3.2. Consider $\theta_1 = (2^a - 1), \theta_2 = \left(2^b - 1\right)$ be any lacunary sequences and $\mathfrak{L}(\mathfrak{p}) = \mathfrak{p}^2$, and $\mathfrak{q}_{u,w} = 1, \forall u, w \in \mathbb{N}$ and $\mathfrak{g}(\mathfrak{p}) = \mathfrak{p}$. Consider the sequence $\mathfrak{p} = (\mathfrak{p}_{u,w})$ where $\mathfrak{p}_{u,w} = \sin(uw)$. Then, $\mathfrak{p}_{u,w} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$.

Example 3.3. Consider $\theta_1 = (2^a - 1), \theta_2 = \left(2^b - 1\right)$ be any lacunary sequences and $\mathcal{L}(\mathfrak{p}) = e^{\mathfrak{p}} - 1$, and $\mathfrak{q}_{u,w} = 1, \forall u, w \in \mathbb{N}$ and $\mathfrak{g}(\mathfrak{p}) = \mathfrak{p}$. Consider the sequence $\mathfrak{p} = (\mathfrak{p}_{u,w})$ where $\mathfrak{p}_{u,w} = (-1)^{u+w}$. Then, $\mathfrak{p}_{u,w} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_\infty$.

Theorem 3.4. *The classes of sequences* $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$, $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$ *and* $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_\infty$ *form linear spaces.*

Proof. The proof is provided only for the class $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$; the other cases can be established using a similar approach. Let $\mathfrak{p}, \mathfrak{r} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$ and $\mathfrak{y}, \mathfrak{z} \in \mathbb{C}$. To prove the result, it is necessary to find some $\gamma_3 > 0$ such that

$$\lim_{\mathfrak{a},\mathfrak{b}\to\infty} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}\left(\mathfrak{y}\mathfrak{p}_{u,w} + \mathfrak{z}\mathfrak{r}_{u,w}\right)|}{\gamma_3}\right)\right]^{\mathfrak{q}_{u,w}} = 0.$$

Since, $\mathfrak{p}, \mathfrak{r} \in [\mathcal{D}_\theta^\omega, \mathfrak{R}, \mathfrak{q}]_0$, then $\exists \gamma_1, \gamma_2 > 0$, such that

$$\sup_{\mathfrak{a},\mathfrak{b}} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}\left(\mathfrak{p}_{u,w}\right)|}{\gamma_1}\right)\right]^{\mathfrak{q}_{u,w}} > 0$$

and

$$\sup_{\mathfrak{a},\mathfrak{b}} \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}\left(\mathfrak{r}_{u,w}\right)|}{\gamma_2}\right)\right]^{\mathfrak{q}_{u,w}} > 0.$$

We set $\gamma_3 = \max\left(2|\mathfrak{y}|\gamma_1, 2|\mathfrak{z}|\gamma_2\right)$. Given that $\mathcal{L}$ is both convex and non-decreasing, it follows that

$$\begin{aligned}
&\tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{y}\mathfrak{p}_{u,w}+\mathfrak{z}\mathfrak{r}_{u,w})|}{\gamma_3}\right)\right]^{\mathfrak{q}_{u,w}}\\
&\le \tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{y}\mathfrak{p}_{u,w})|}{\gamma_3} + \tfrac{|\mathfrak{g}(\mathfrak{z}\mathfrak{r}_{u,w})|}{\gamma_3}\right)\right]^{\mathfrak{q}_{u,w}}\\
&\le \tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \tfrac{1}{2^{\mathfrak{q}_{u,w}}} \left[\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{y}\mathfrak{p}_{u,w})|}{\gamma_1} + \tfrac{|\mathfrak{g}(\mathfrak{z}\mathfrak{r}_{u,w})|}{\gamma_2}\right)\right]^{\mathfrak{q}_{u,w}}\\
&\le \mathcal{T}\tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma_1}\right)\right]^{\mathfrak{q}_{u,w}} + \tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{r}_{u,w})|}{\gamma_2}\right)\right]^{\mathfrak{q}_{u,w}}\\
&\to 0, \text{ as } \mathfrak{a}, \mathfrak{b} \to \infty,
\end{aligned}$$

where $\mathcal{T} = \max\left(1, 2^{Q-1}\right)$, and $Q = \sup \mathfrak{q}_{u,w} < \infty$. So, $\mathfrak{y}\mathfrak{p} + \mathfrak{z}\mathfrak{r} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$. As a result, the class $[\mathcal{D}_\theta^\omega, \mathfrak{L}, q]_0$ constitutes a linear space. ☐

Theorem 3.5. *Let $\mathcal{L}$ be any Orlicz function, and $\mathfrak{q} = (\mathfrak{q}_{u,w})$ represent a bounded sequence of strictly positive real numbers. The space $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0$ is a topological linear space, totally paranormed by*

$$\varpi(\mathfrak{p}) = \inf\left\{\gamma^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}/Q} : \left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} < 1,\ \mathfrak{a},\mathfrak{b} = 1,2,...\right\},$$

where $Q = \max\left(1, \sup_{u,w}\mathfrak{q}_{u,w}\right)$.

Proof. It is evident that, $\varpi(\mathfrak{p}) = \varpi(-\mathfrak{p})$. Applying Theorem 3.4 with $\mathfrak{y} = \mathfrak{z} = 1$, we obtain $\varpi(\mathfrak{p}+\mathfrak{r}) \le \varpi(\mathfrak{p}) + \varpi(\mathfrak{r})$. Additionally, since $\mathcal{L}(0) = 0$, it follows that $\inf\left\{\gamma^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}/Q}\right\} = 0$ when $\mathfrak{p} = 0$.
Conversely, assume $\varpi(\mathfrak{p}) = 0$. This yields that

$$\inf\left\{\gamma^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}/Q} : \left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} \le 1,\ \mathfrak{a},\mathfrak{b} = 1,2,...\right\}.$$

For any $\sigma > 0$, there exists a value γ_σ satisfying $0 < \gamma_\sigma < \sigma$, such that

$$\left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma_\sigma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} \le 1.$$

So, we have

$$\begin{aligned}&\left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\sigma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q}\\ &\le \left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma_\sigma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} \le 1.\end{aligned}$$

Assume there exists an $(u,w) \in I_{\mathfrak{a},\mathfrak{b}}$ such that $\mathfrak{p}_{u,w} \ne 0$. As $\sigma \to 0$, we have $\left(\frac{|\mathfrak{p}_{u,w}|}{\sigma}\right) \to \infty$. Thus

$$\left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma_\sigma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} \to \infty.$$

This leads to a contradiction. Therefore, $\mathfrak{p}_{u,w} = 0$ for all u, w.
In summary, we prove that scalar multiplication is continuous. Let $\lambda \in \mathbb{C}$. According to the definition,

$$\varpi(\lambda\mathfrak{p}) = \inf\left\{\gamma^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}/Q} : \left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} \le 1,\ \mathfrak{a},\mathfrak{b} = 1,2,...\right\}.$$

Then,

$$\varpi(\lambda\mathfrak{p}) = \inf\left\{(|\lambda|\zeta)^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}/Q} : \left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\zeta}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q} \le 1,\ \mathfrak{a},\mathfrak{b} = 1,2,...\right\}.$$

Let $\zeta=\gamma/|\lambda|$, and as $|\lambda|^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}}\leq\max\left(1,|\lambda|^{\sup\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}}\right)$, we have

$$\varpi(\lambda\mathfrak{p})\leq\left(\max\left(1,|\lambda|^{\sup\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}}\right)\right)^{1/Q}\inf\left\{\zeta^{\mathfrak{q}_{\mathfrak{a},\mathfrak{b}}/Q}\left(\tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\zeta}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q}\leq 1,\quad \mathfrak{a},\mathfrak{b}=1,2,\dots\right\},$$

which tends to zero as $\mathfrak{p}$ approaches zero in $[\mathcal{D}_\theta^\omega,\mathfrak{R},\mathfrak{q}]_0$.
Now, suppose $\lambda_{\mathfrak{a},\mathfrak{b}}\to 0$ and $\mathfrak{p}$ is fixed in $[\mathcal{D}_\theta^\omega,\mathfrak{L},\mathfrak{q}]_0$. For any $\sigma>0$, take $\mathcal{P}>0$ such that

$$\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}\left(\mathfrak{p}_{u,w}\right)|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}<\left(\frac{\sigma}{2}\right)^Q,\ \text{for some }\gamma>0\text{ and all }\mathfrak{a},\mathfrak{b}>\mathcal{P}.$$

This means that for some $\gamma>0$ and each $\mathfrak{a},\mathfrak{b}>\mathcal{P}$

$$\left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}\left(\mathfrak{p}_{u,w}\right)|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q}<\frac{\sigma}{2}.$$

Now, suppose $0<|\lambda|<1$. By applying the convexity of $\mathfrak{L}$, for $\mathfrak{a},\mathfrak{b}>\mathcal{P}$, we obtain

$$\begin{aligned}&\tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\tfrac{|\lambda\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\\&<\tfrac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[|\lambda|\mathfrak{L}\left(\tfrac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}<\left(\tfrac{\sigma}{2}\right)^Q.\end{aligned}$$

Since $\mathfrak{L}$ is continuous on $[0,\infty)$, for $\mathfrak{a},\mathfrak{b}\leq\mathcal{P}$, the function

$$\mathfrak{g}(\mathfrak{f})=\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\ \mathfrak{f}\mathfrak{g}\left(\mathfrak{p}_{u,w}\right)\ |}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}$$

is continuous at 0. So, there is a $\beta\in(0,1)$ such that $|g(\mathfrak{f})|<\left(\frac{\sigma}{2}\right)^Q$ for $0<\mathfrak{f}<\beta$. Let $\mathcal{C}$ be such that $|\lambda_{\mathfrak{h},\mathfrak{j}}|<\sigma$ for $\mathfrak{h},\mathfrak{j}>\mathcal{C}$, then for $\mathfrak{h},\mathfrak{j}>\mathcal{C}$ and $\mathfrak{a},\mathfrak{b}\leq\mathcal{P}$,

$$\left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\ \lambda_{\mathfrak{h},\mathfrak{j}}\left(\mathfrak{p}_{u,w}\right)\ |}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q}<\frac{\sigma}{2}.$$

Thus,

$$\left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\ \lambda_{\mathfrak{h},\mathfrak{j}}\left(\mathfrak{p}_{u,w}\right)\ |}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}\right)^{1/Q}<\sigma,$$

for $u,w>\mathcal{P}$ and all $\mathfrak{a},\mathfrak{b}$, such that $\varpi(\lambda\mathfrak{p})\to 0,(\lambda\to 0)$. □

Theorem 3.6. *The spaces* $[\mathcal{D}_\theta^\omega,\mathfrak{L},\mathfrak{q}]_0,[\mathcal{D}_\theta^\omega,\mathfrak{R},\mathfrak{q}]_1$ *and* $[\mathcal{D}_\theta^\omega,\mathcal{L},\mathfrak{q}]_\infty$ *are solid.*

Proof. Let $(\mathfrak{p}_{u,w})\in[\mathcal{D}_\theta^\omega,\mathfrak{L},\mathfrak{q}]_0$. Then, $\exists\gamma>0$ such that

$$\lim_{\mathfrak{a},\mathfrak{b}\to\infty}\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}}\left[\mathfrak{L}\left(\frac{|\mathfrak{g}\left(\mathfrak{p}_{u,w}\right)|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}=0.$$

Let $(\alpha_{u,w})$ be a scalar sequence such that $|\alpha_{u,w}| \leq 1$. For each $\mathfrak{a}$, $\mathfrak{b}$, we have

$$\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\alpha_{u,w}\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}} \leq \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}}$$
$$\Rightarrow \lim_{\mathfrak{a},b\to\infty} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\alpha_{u,w}\mathfrak{p}_{u,w})|}{\gamma}\right)\right]^{\mathfrak{q}_{u,w}} = 0.$$
$$\Rightarrow (\alpha_{u,w}\mathfrak{p}_{u,w}) \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0 .$$

□

In light of Lemma 2.9, we state the following theorem without proof:

Theorem 3.7. *The spaces* $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0, [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$ *and* $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_\infty$ *are monotone.*

Theorem 3.8. *In general, the spaces* $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_0, [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$ *and* $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_\infty$ *are not symmetric.*

The subsequent example demonstrates this.

Example 3.9. We will demonstrate this for $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$. Consider $\mathbb{X} = \mathbb{C}$ and the function $\mathfrak{g}(\mathfrak{p}) = \mathfrak{p}$, and $\mathfrak{q}_{u,w} = 1, \forall u, w \in \mathbb{N}$. Let $\theta_1 = (2^{\mathfrak{a}} - 1)$, $\theta_2 = \left(2^{\mathfrak{b}} - 1\right)$ be any lacunary sequences and $\mathfrak{L}(\mathfrak{p}) = \mathfrak{p}$. Let $(\mathfrak{p}_{u,w}) = uw$, for all $u, w \in \mathbb{N}$. Thus, the sequence $(\mathfrak{p}_{u,w})$ for all $u, w \in \mathbb{N}$ is in $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$. Next, we establish the sequence $(\mathfrak{r}_{u,w})$, which is rearrangement of the sequence $(\mathfrak{p}_{u,w})$ defined by

$$(\mathfrak{r}_{u,w}) = (\mathfrak{p}_{1,1,}, \mathfrak{p}_{2,2}, \mathfrak{p}_{4,4}, \mathfrak{p}_{3,3}, \mathfrak{p}_{9,9}, \ldots) .$$

This rearranged sequence $(\mathfrak{r}_{u,w})$ does not belong to $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$. Therefore, $[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$ is not symmetric in general.

Lemma 3.10. *Let* $\mathcal{L}$ *be an Orlicz function that satisfies the* Δ_2*-condition, and take* $0 < \sigma < 1$. *For all* $\mathfrak{p} \geq \sigma$, *there exists a constant* $U > 0$ *so that* $\mathcal{L}(\mathfrak{p}) < U\mathfrak{p}\sigma^{-1}\mathcal{L}(2)$.

Proof. As $\mathfrak{L}$ is non-decreasing and convex, and $\mathfrak{p} < \sigma^{-1}\mathfrak{p} < 1 + \sigma^{-1}\mathfrak{p}$ for $\mathfrak{p} \geq \sigma$, it implies that

$$\mathfrak{L}(\mathfrak{p}) < \mathfrak{L}\left(1 + \sigma^{-1}\mathfrak{p}\right) = \mathfrak{L}\left(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 2\sigma^{-1}\mathfrak{p}\right) < \frac{1}{2}\mathfrak{L}(2) + \frac{1}{2}\mathfrak{L}\left(2\sigma^{-1}\mathfrak{p}\right).$$

Given that $\mathfrak{L}$ satisfies Δ_2-condition, there is a constant $U > 2$ such that $\mathfrak{L}\left(2\sigma^{-1}\mathfrak{p}\right) \leq \frac{1}{2}U\sigma^{-1}\mathfrak{p}\mathcal{L}(2)$. As a result,

$$\mathfrak{L}(\mathfrak{p}) < \frac{1}{2}U\sigma^{-1}\mathfrak{p}\mathfrak{L}(2) + \frac{1}{2}U\sigma^{-1}\mathfrak{p}\mathfrak{L}(2) = U\sigma^{-1}\mathfrak{p}\mathcal{L}(2).$$

Hence, the lemma is established. □

Theorem 3.11. *For an Orlicz function* $\mathcal{L}$ *that satisfies the* Δ_2*-condition, we have the following inclusions:*

$$\mathcal{D}_\theta^\omega \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}]_1, [\mathcal{D}_\theta^\omega]_0 \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}]_0 \ \text{ and } \ [\mathcal{D}_\theta^\omega]_\infty \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}]_\infty .$$

Proof. Let $\mathfrak{p} \in \mathcal{D}_\theta^\omega$. Then, we have

$$A_{\mathfrak{a},b} = \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} |\mathfrak{g}\left(\mathfrak{p}_{u,w} - \mathfrak{p}_0\right)| \to 0, \text{ as } \mathfrak{a}, \mathfrak{b} \to \infty \text{ for some } \mathfrak{p}_0.$$

Consider $\gamma > 0$, and select σ with $0 < \sigma < 1$ such that $\mathfrak{L}(t) < \gamma$ for $0 \leq t \leq \sigma$. So, we can express the following:

$$\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} |\mathfrak{g}(\mathfrak{p}_{u,w} - \mathfrak{p}_0)| = \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{\substack{(u,w)\in I_{\mathfrak{a},\mathfrak{b}} \\ |\mathfrak{g}(\mathfrak{p}_{u,w}-\mathfrak{p}_0)|\leq\sigma}} |\mathfrak{g}(\mathfrak{p}_{u,w} - \mathfrak{p}_0)|$$
$$+\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{\substack{(u,w)\in I_{\mathfrak{a},\mathfrak{b}} \\ |\mathfrak{g}(\mathfrak{p}_{u,w}-\mathfrak{p}_0)|>\sigma}} |\mathfrak{g}(\mathfrak{p}_{u,w} - \mathfrak{p}_0)| < \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} (\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}\gamma) + \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} U\sigma^{-1}\mathfrak{L}(2)\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}A_{\mathfrak{a},b},$$

As $\mathfrak{a}, \mathfrak{b} \to \infty$, by applying Lemma 3.10 , it follows that $\mathfrak{p} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}]_1$. A similar reasoning can be used to show that $[\mathcal{D}_\theta^\omega]_0 \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}]_0$ and $[\mathcal{D}_\theta^\omega]_\infty \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}]_\infty$. □

Theorem 3.12. *Let $0 < \mathfrak{q}_{u,w} \leq \mathfrak{f}_{u,w}$ for all u, w, let $(\mathfrak{f}_{u,w}/\mathfrak{q}_{u,w})$ be bounded. Then,*

$$[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{f}]_1 \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1 .$$

Proof. Let $\mathfrak{p} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{f}]_1$. We write $v_{u,w} = \left[\mathfrak{L}\left(\frac{|\mathfrak{g}(\mathfrak{p}_{u,w}-\mathfrak{p}_0)|}{\gamma}\right)\right]^{\mathfrak{f}_{u,w}}$ and $\frac{\mathfrak{q}_{u,w}}{\mathfrak{f}_{u,w}} = \lambda_{u,w}$, where $0 < \lambda < \lambda_{u,w} \leq 1$ with λ being a constant. Now, we define

$$e_{u,w} = v_{u,w}\,(v_{u,w} \geq 1)\,, e_{u,w} = 0\,(v_{u,w} < 1)\,, c_{u,w} = 0\,(v_{u,w} \geq 1)\,, c_{u,w} = v_{u,w}\,(v_{u,w} < 1)\,,$$

so that

$$v_{u,w} = e_{u,w} + c_{u,w}, v_{u,w}^{\lambda_{u,w}} = e_{u,w}^{\lambda_{u,w}} + c_{u,w}^{\lambda_{u,w}}.$$

It follows that $e_{u,w}^{\lambda_{u,w}} \leq e_{u,w} \leq s_{u,w}$ and $c_{u,w}^{\lambda_{u,w}} \leq c_{u,w}^{\lambda}$. Therefore,

$$\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} c_{u,w}^{\lambda} = \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} (\mathfrak{h}_{\mathfrak{a},\mathfrak{b}} c_{u,w})^{\lambda} \left(\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}}\right)^{1-\lambda}$$
$$\leq \left(\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\left(\frac{1}{h_{\mathfrak{a},\mathfrak{b}}} c_{u,w}\right)^{\lambda}\right]^{1/\lambda}\right)^{\lambda} \left(\sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} \left[\left(\frac{1}{h_{\mathfrak{a},\mathfrak{b}}} c_{u,w}\right)^{1-\lambda}\right]^{1/1-\lambda}\right)^{1-\lambda}$$
$$= \left(\frac{1}{h_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} c_{u,w}\right)^{\lambda},$$

(by Holder's inequality).
So,

$$\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} s_{u,w}^{\lambda_{u,w}} \leq \frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} v_{u,w} + \left[\frac{1}{\mathfrak{h}_{\mathfrak{a},\mathfrak{b}}} \sum_{(u,w)\in I_{\mathfrak{a},\mathfrak{b}}} c_{u,w}\right]^{\lambda},$$

and hence $\mathfrak{p} \in [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1$. □

Example 3.13. Consider an Orlicz function $\mathcal{L}(\mathfrak{p}) = \mathfrak{p}^2$. Let $\mathfrak{f}_{u,w} = 2, \mathfrak{q}_{u,w} = 1$ and $\mathfrak{g}(\mathfrak{p}) = \mathfrak{p}$. We define the sequence $\mathfrak{p}_{u,w} = \frac{1}{uw}, \forall u, w$. For this sequence $(\mathfrak{p}_{u,w})$ we can write

$$[\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{f}]_1 \subset [\mathcal{D}_\theta^\omega, \mathfrak{L}, \mathfrak{q}]_1 .$$

4. Conclusion

In this research, the concept of lacunary weak convergence for double sequences, formulated through Orlicz functions, has been introduced and analyzed. The investigation revealed several significant topological and algebraic properties, such as solidity, symmetry, and monotonicity, which are fundamental to understanding the structure of these spaces. By providing illustrative examples, the study also highlighted scenarios where certain properties fail, offering valuable insights into the limitations and scope of the proposed framework.

Furthermore, the inclusion relationships between the newly defined spaces and other related spaces have been established, expanding the theoretical foundation of sequence spaces. These findings not only deepen our understanding of the behavior of double sequences under lacunary weak convergence but also pave the way for further research into more generalized forms of sequence convergence.

The results presented here have potential applications in functional analysis, particularly in the study of sequence spaces and their role in various mathematical contexts. Future research may focus on exploring alternative definitions of lacunary convergence, as well as extending the framework to other types of functions or metrics. This work provides a solid basis for such endeavors, contributing to the ongoing development of the theory of sequence space.

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