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ON THE REPRESENTATION OF CHANGEABLE SETS IN THE FORM OF AN AUTOMULTIIMAGE

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This article is dedicated to the memory of academician Yu.M. Berezansky

Abstract. From an intuitive point of view, the multi-image construction procedure resembles the procedure of constructing the evolution scenario of some system in all possible reference frames, if we know its scenario of evolution in the given (fixed) reference frame. In the present paper it is investigated the problem of representation of a changeable set in the form of automultiimage that is in the form of the multi-image of some its reference frame. In particular we prove the necessary and sufficient condition for evolutionarily visible changeable set to be representable as automultiimage. Also using the last result we give the example of evolutionarily visible changeable set, which can not be represented as automultiimage.

1. Introduction

The topic of this article is closely related to the theory of changeable sets. From an intuitive point of view, changeable sets are sets of objects which, unlike elements of ordinary (static) sets, may be in the process of continuous transformations, and which may change properties depending on the point of view on them (that is depending on the reference frame). The main motivation for introduction of the changeable set notion was the famous sixth Hilbert problem, that is the problem of mathematically strict formulation for the fundamentals of theoretical physics. The last problem was posed in 1900 (see [21]), but it remains very relevant today, [35, 11, 6, 1, 31, 41, 7, 10, 40]. The problem of constructing the mathematical theory of changeable sets (that is the "sets" possessing the properties listed above) in different forms was emerged in many papers (for example see [29, 30, 3, 4, 28][1]). On the mathematically strict level the theory of changeable sets was developed in the papers [12, 13, 14] etc. The most complete and systematic presentation of this theory can be found in the preprint [18] as well as in the dissertation [20] (for readers who know Ukrainian language).

The concept of the multi-image (in a mathematically precise form) first appeared in [17]. From an intuitive point of view, the multi-image construction procedure resembles the procedure of constructing the evolution scenario of some system in all possible reference frames, if we know its scenario of evolution in the given (fixed) reference frame. In the present paper it is investigated the problem of representation of a changeable set in the form of self-multiimage that is in the form of the multi-image of some its reference frame. In particular we prove the necessary and sufficient condition for evolutionarily visible

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[1]In the papers [3, 4, 28] it was used the term "variable sets" instead of "changeable sets". We do not use the term "variable sets", because there is not unambiguous interpretation of the last term in the scientific literature. For example in the programming and data sciencies this term means "the group of related variables" (see [27, 24], see also [2, 23]).

changeable set to be representable as self-multiimage. Also using the last result we give the example of changeable sets, which can not be represented as self-multiimage.

2. Main Notions and Some Facts from the Theory of Changeable Sets

For the convenience of readers, in this section we recall some important concepts and facts of the theory of changeable sets necessary for further presentation.

2.1. **Definition and main properties of changeable sets.** Definition of changeable set will be made in two steps. In the first step we formulate the definition of base changeable set.

Let $\mathbb{T} = (\mathbf{T}, \leq)$ be any linearly (totally) ordered set (the sense of [5, p. 12]) and let $\mathcal{X}$ be any nonempty set. For any ordered pair $\omega = (t, x) \in \mathbf{T} \times \mathcal{X}$ we use the notations:

$$\mathsf{bs}\,(\omega) := x, \quad \mathsf{tm}\,(\omega) := t.$$

Definition 2.1 ([15] [2]). The ordered triple of kind $\mathcal{B} = (\mathbf{B}, \mathbb{T}, \Leftarrow)$, where $\mathbf{B} \subseteq \mathbf{T} \times \mathcal{X}$, is named by **base changeable set** if and only if the following conditions are satisfied:

(1) $\mathbf{B} \neq \emptyset$ and $\Leftarrow$ is reflexive binary relation on $\mathbf{B}$ (that is $\forall \omega \in \mathbf{B}\ \omega \Leftarrow \omega$);
(2) for arbitrary $\omega, \omega_2 \in \mathbf{B}$ the conditions $\omega_2 \Leftarrow \omega_1$ and $\omega_1 \neq \omega_2$ cause the inequality $\mathsf{tm}\,(\omega_1) < \mathsf{tm}\,(\omega_2)$, where $<$ is the strict order relation, generated by the non-strict order $\leq$ of linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$. [3]

Remark 2.2. For an arbitrary base changeable set $\mathcal{B} = (\mathbf{B}, \mathbb{T}, \Leftarrow) = (\mathbf{B}, (\mathbf{T}, \leqslant), \Leftarrow)$ (where $\mathbf{B} \subseteq \mathbf{T} \times \mathcal{X}$) we use the following *notations and terminology*:

$$\mathbb{B}\mathfrak{s}(\mathcal{B}) := \mathbf{B};\ \underset{\mathcal{B}}{\leftarrow} := \Leftarrow;\ \mathbb{T}\mathbf{m}(\mathcal{B}) := \mathbb{T};\ \mathbf{Tm}(\mathcal{B}) := \mathbf{T};\ \leq_{\mathcal{B}} := \leqslant$$

$$\mathfrak{B}\mathfrak{s}(\mathcal{B}) := \{x \in \mathcal{X} \mid \exists\, \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\ (\mathsf{bs}\,(\omega) = x)\} = \{\mathsf{bs}\,(\omega)\ |\, \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\}\,. \tag{2.1}$$

- For $t, \tau \in \mathbf{Tm}(\mathcal{B})$ we write $t <_{\mathcal{B}} \tau$ if and only if $t \leq_{\mathcal{B}} \tau$ and $t \neq \tau$.
- The set $\mathfrak{B}\mathfrak{s}(\mathcal{B})$ is named by the basic set or the set of all elementary states of $\mathcal{B}$.
- The set $\mathbb{B}\mathfrak{s}(\mathcal{B})$ is named by the set of all elementary-time states of $\mathcal{B}$.
- The set $\mathbf{Tm}(\mathcal{B})$ is named by the set of time points of $\mathcal{B}$.
- The relation $\underset{\mathcal{B}}{\leftarrow}$ is named by the base of elementary processes of $\mathcal{B}$ [4].

Note that from the equality (2.1) and introduced notations for any base changeable set $\mathcal{B}$ we deduce:

$$\mathbb{T}\mathbf{m}(\mathcal{B}) = (\mathbf{Tm}(\mathcal{B}), \leq_{\mathcal{B}}) \tag{2.2}$$

$$\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B}) \tag{2.3}$$

Remark 2.3. In the cases, when the base changeable set $\mathcal{B}$ is known in advance we use the notations $\leftarrow, \leq, <$ instead of the notations $\underset{\mathcal{B}}{\leftarrow}, \leq_{\mathcal{B}}, <_{\mathcal{B}}$.

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[2] In some papers it can be found definition of base changeable set, that uses the notion of primitive changeable set, which is different from Definition 2.1 (see for example [13, 18]). As it was proven in [14, 18], the both definitions are equivalent.

[3] Recall [5] that the (non strict) linear order relation $\leq$ generates the strict order relation $<$ on $\mathbf{T}$ by the following rule:

$$t_1 < t_2 \text{ holds if and only if } t_1 \leq t_2 \text{ and } t_1 \neq t_2\ (\forall t_1, t_2 \in \mathbf{T}).$$

[4] In some papers we use the notation $\overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$ instead of $\underset{\mathcal{B}}{\leftarrow}$ to distinguish the base of elementary processes from directing relation of changes, which sometimes is denoted by $\overset{\mathfrak{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}}$. (For elementary states $x_1, x_2 \in \mathfrak{B}\mathfrak{s}(\mathcal{B})$ we use the notation $x_2 \overset{\mathfrak{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}} x_1$, or more briefly $x_2 \underset{\mathcal{B}}{\leftarrow} x_1$, if an only if there exist $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ such that $\mathsf{bs}\,(\omega_1) = x_1$, $\mathsf{bs}\,(\omega_2) = x_2$ and $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$.)

For the elements $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ the record $\omega_2 \leftarrow \omega_1$ should be interpreted as "the elementary-time state ω_2 is the result of transformations (or the transformation prolongation) of the elementary-time state ω_1".

Definition 2.4. We say that elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ are ***united by fate*** in the base changeable set $\mathcal{B}$ iff at least one of the correlations $\omega_2 \leftarrow \omega_1$ or $\omega_1 \leftarrow \omega_2$ is valid. In the case, when elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ are united by fate in $\mathcal{B}$ we use the notation

$$\omega_1 \underset{\mathcal{B}}{\leftrightarrow} \omega_2.$$

Remark 2.5. Directly from Definition 2.1 and notations, introduced in Remark 2.2 it follows that for any base changeable set $\mathcal{B}$ the following statements are valid:

(1) *If* $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$, $\omega_2 \leftarrow \omega_1$ *and* $\omega_1 \neq \omega_2$ *then* $\mathsf{tm}(\omega_1) < \mathsf{tm}(\omega_2)$.
(2) *If* $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ *and* $\omega_2 \leftarrow \omega_1$ *then* $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$.
(3) *If* $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$, $\omega_2 \underset{\mathcal{B}}{\leftrightarrow} \omega_1$ *and* $\omega_1 \neq \omega_2$ *then* $\mathsf{tm}(\omega_1) \neq \mathsf{tm}(\omega_2)$.

The main method of generation base changeable sets is connected with systems of abstract trajectories.

Definition 2.6. Let M be an arbitrary set and $\mathbb{T} = (\mathbf{T}, \leq)$ be any linearly ordered set.
1. Any mapping $r : \mathfrak{D}(r) \to M$, where $\mathfrak{D}(r) \subseteq \mathbf{T}$, will be referred to as an **abstract trajectory** from $\mathbb{T}$ to M (here $\mathfrak{D}(r)$ is the domain of the abstract trajectory r).
2. Any set $\mathcal{R}$, which consists of abstract trajectories from $\mathbb{T}$ to M will be named by **system of abstract trajectories** from $\mathbb{T}$ to M.

Theorem 2.7 ([13], see also [18])**.** *Let $\mathcal{R}$ be a system of abstract trajectories from $\mathbb{T} = (\mathbf{T}, \leq)$ to M. Then there exists a unique base changeable set $\mathcal{B} = \mathcal{A}t(\mathbb{T}, \mathcal{R})$, such, that:*
1) $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T}$;
2) $\mathbb{B}\mathfrak{s}(\mathcal{B}) = \bigcup_{r \in \mathcal{R}} r$;
3) For arbitrary $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B})$ *the condition* $\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1$ *is satisfied if and only if* $\mathsf{tm}(\omega_1) \leq \mathsf{tm}(\omega_2)$ *and there exists an abstract trajectory* $r \in \mathcal{R}$ *such, that* $\omega_1, \omega_2 \in r$

Remark 2.8. Note, that in Theorem 2.7 any trajectory $r \in \mathcal{R}$ can be interpreted as the set: $r = \{(t, r(t)) \mid t \in \mathfrak{D}(r)\}$. So, in accordance with item 2) of this theorem, for the base changeable set $\mathcal{B} = \mathcal{A}t(\mathbb{T}, \mathcal{R})$ we have, $\mathfrak{B}\mathfrak{s}(\mathcal{B}) = \bigcup_{r \in \mathcal{R}} \mathfrak{R}(r) \subseteq M$, where $\mathfrak{R}(r)$ is the range of trajectory $r \in \mathcal{R}$.

Conversely, it can be proven, that any base changeable set can be generated by some system of abstract trajectories ([13], see also [18]).

Other further important method of generation new base changeable sets is creation of image of existing base changeable set.

Definition 2.9. An ordered triple $(\mathbb{T}, \mathcal{X}, U)$ is referred to as **evolution projector** for base changeable set $\mathcal{B}$ if and only if:
1. $\mathbb{T} = (\mathbf{T}, \leq)$ is a linearly ordered set;
2. $\mathcal{X}$ is any set;
3. U is a mapping from $\mathbb{B}\mathfrak{s}(\mathcal{B})$ into $\mathbf{T} \times \mathcal{X}$ $(U : \mathbb{B}\mathfrak{s}(\mathcal{B}) \to \mathbf{T} \times \mathcal{X})$.

Theorem 2.10 (theorem on image, published in [17], see also [18])**.** *Let $(\mathbb{T}, \mathcal{X}, U)$ be any evolution projector for a base changeable set $\mathcal{B}$. Then there exists only one base changeable set $\mathcal{B}_1 := U[\mathcal{B}, \mathbb{T}]$, satisfying the following conditions:*

(1) $\mathbb{T}\mathbf{m}(\mathcal{B}_1) = \mathbb{T}$;
(2) $\mathbb{B}\mathfrak{s}(\mathcal{B}_1) = U(\mathbb{B}\mathfrak{s}(\mathcal{B})) = \{U(\omega) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\}$;

(3) *Let* $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{Bs}(\mathcal{B}_1)$ *and* $\mathsf{tm}(\widetilde{\omega}_1) \neq \mathsf{tm}(\widetilde{\omega}_2)$. *Then* $\widetilde{\omega}_1$ *and* $\widetilde{\omega}_2$ *are united by fate in* $\mathcal{B}_1$ *if and only if, there exist united by fate in* $\mathcal{B}$ *elementary-time states* $\omega_1, \omega_2 \in \mathbb{Bs}(\mathcal{B})$ *such, that* $\widetilde{\omega}_1 = U(\omega_1)$, $\widetilde{\omega}_2 = U(\omega_2)$.

$U[\mathcal{B}, \mathbb{T}]$ is named by ***image of base changeable set*** $\mathcal{B}$ relatively the mapping U and time scale $\mathbb{T}$. In the case where $\mathbb{T} = \mathbb{T}\mathbf{m}(\mathcal{B})$ we use the notation $U[\mathcal{B}]$ instead of $U[\mathcal{B}, \mathbb{T}]$:

$$U[\mathcal{B}] = U[\mathcal{B}, \mathbb{T}\mathbf{m}(\mathcal{B})].$$

Denotation 1. *Let* M *be any set. Further we note by* $\mathbb{I}_M$ *the identity mapping on* M $(\mathbb{I}_M(x) = x \quad (\forall x \in M))$.

It is apparently that the triple of kind $\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathfrak{Bs}(\mathcal{B}), \mathbb{I}_{\mathbb{Bs}(\mathcal{B})}\right)$ is an evolution projector for any base changeable set $\mathcal{B}$.

Assertion 2.11 (see [20], see also [18], Remark 1.11.3)**.** *For an arbitrary base changeable set* $\mathcal{B}$ *it holds the equality* $\mathbb{I}_{\mathbb{Bs}(\mathcal{B})}[\mathcal{B}] = \mathcal{B}$.

Definition 2.12. Let $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ be any indexed family of base changeable sets (where $\mathcal{A} \neq \emptyset$ is some set of indexes). The system of mappings $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ of kind $\mathfrak{U}_{\beta\alpha} : 2^{\mathbb{Bs}(\mathcal{B}_\alpha)} \longrightarrow 2^{\mathbb{Bs}(\mathcal{B}_\beta)}$ $(\alpha, \beta \in \mathcal{A})$ is referred to as **unification of perception** on $\overleftarrow{\mathcal{B}}$ if and only if the following conditions are satisfied:

(1) $\mathfrak{U}_{\alpha\alpha} A = A$ for any $\alpha \in \mathcal{A}$ and $A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$.
(Here and further we denote by $\mathfrak{U}_{\beta\alpha} A$ *the action of the mapping* $\mathfrak{U}_{\beta\alpha}$ *to the set* $A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$*, that is* $\mathfrak{U}_{\beta\alpha} A := \mathfrak{U}_{\beta\alpha}(A)$.*)*
(2) Any mapping $\mathfrak{U}_{\beta\alpha}$ is a monotonous mapping of sets, IE for any $\alpha, \beta \in \mathcal{A}$ and $A, B \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$ the condition $A \subseteq B$ assures $\mathfrak{U}_{\beta\alpha} A \subseteq \mathfrak{U}_{\beta\alpha} B$.
(3) For any $\alpha, \beta, \gamma \in \mathcal{A}$ and $A \subseteq \mathbb{Bs}(\mathcal{B}_\alpha)$ the following inclusion holds:

$$\mathfrak{U}_{\gamma\beta} \mathfrak{U}_{\beta\alpha} A \subseteq \mathfrak{U}_{\gamma\alpha} A. \tag{2.4}$$

In this case the mappings $\mathfrak{U}_{\beta\alpha}$ $(\alpha, \beta \in \mathcal{A})$ we name by **unification mappings**, and the triple of kind $\mathcal{Z} = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$ we name by **changeable set**.

Remark 2.13 (on notations)**.** Let $\mathcal{Z} = \left(\mathcal{A}, \overleftarrow{\mathcal{B}}, \overleftarrow{\mathfrak{U}}\right)$ be a changeable set, where $\overleftarrow{\mathcal{B}} = (\mathcal{B}_\alpha \mid \alpha \in \mathcal{A})$ is an indexed family of base changeable sets and $\overleftarrow{\mathfrak{U}} = (\mathfrak{U}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{A})$ is an unification of perception on $\overleftarrow{\mathcal{B}}$. Further we use the following terms and notations:

1) The set $\mathcal{A}$ is named by the ***index set*** of the changeable set $\mathcal{Z}$, and it is denoted by $\mathcal{I}nd(\mathcal{Z})$.

2) For any index $\alpha \in \mathcal{I}nd(\mathcal{Z})$ the pair

$$\mathbf{lk}_\alpha(\mathcal{Z}) = (\alpha, \mathcal{B}_\alpha)$$

is referred to as ***reference frame*** of the changeable set $\mathcal{Z}$.

3) The set of all reference frames of $\mathcal{Z}$ we denote by $\mathcal{L}k(\mathcal{Z})$:

$$\mathcal{L}k(\mathcal{Z}) := \{(\alpha, \mathcal{B}_\alpha) \mid \alpha \in \mathcal{I}nd(\mathcal{Z})\} = \{\mathbf{lk}_\alpha(\mathcal{Z}) \mid \alpha \in \mathcal{I}nd(\mathcal{Z})\}. \tag{2.5}$$

Typically, reference frames we denote by small Gothic letters ($\mathfrak{l}, \mathfrak{m}, \mathfrak{k}, \mathfrak{p}$ and so on).

4) For $\mathfrak{l} = (\alpha, \mathcal{B}_\alpha) \in \mathcal{L}k(\mathcal{Z})$ we introduce the following denotations:

$$\mathtt{ind}(\mathfrak{l}) := \alpha; \quad \mathfrak{l}^\wedge := \mathcal{B}_\alpha.$$

Thus:

$$\mathfrak{l} = (\mathtt{ind}(\mathfrak{l}), \mathfrak{l}^\wedge) \qquad (\forall \mathfrak{l} \in \mathcal{L}k(\mathcal{Z}));\tag{2.6}$$

$$\mathcal{I}nd(\mathcal{Z}) = \{\mathtt{ind}(\mathfrak{l}) \mid \mathfrak{l} \in \mathcal{L}k(\mathcal{Z})\} \tag{2.7}$$

and for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ the object $\mathfrak{l}^{\hat{}}$ is a base changeable set. Further, when it does not cause confusion, for any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ the symbol "$\hat{}$" will be omitted in the denotations $\mathfrak{Bs}(\mathfrak{l}^{\hat{}})$, $\mathbb{Bs}(\mathfrak{l}^{\hat{}})$, $\mathbf{Tm}(\mathfrak{l}^{\hat{}})$, $\mathbb{Tm}(\mathfrak{l}^{\hat{}})$, $\underset{\mathfrak{l}^{\hat{}}}{\leftarrow}$, $\leq_{\mathfrak{l}^{\hat{}}}$, $<_{\mathfrak{l}^{\hat{}}}$ and the denotations $\mathfrak{Bs}(\mathfrak{l})$, $\mathbb{Bs}(\mathfrak{l})$, $\mathbf{Tm}(\mathfrak{l})$, $\mathbb{Tm}(\mathfrak{l})$, $\underset{\mathfrak{l}}{\leftarrow}$, $\leq_{\mathfrak{l}}$, $<_{\mathfrak{l}}$ will be used.

5) For any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ the mapping $\mathfrak{U}_{\mathtt{ind}(\mathfrak{m}),\mathtt{ind}(\mathfrak{l})}$ is denoted by $\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle$. Hence:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle = \mathfrak{U}_{\mathtt{ind}(\mathfrak{m}),\mathtt{ind}(\mathfrak{l})}.$$

In the case, when the changeable $\mathcal{Z}$ set is known in advance, the symbol $\mathcal{Z}$ in the above notation will be omitted, and the denotation "$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle$" will be used instead.

6) In the case, when it does not cause confusion, we use the denotations $\leftarrow$, $\leq$, $<$ instead of the denotations $\underset{\mathfrak{l}}{\leftarrow}$, $\leq_{\mathfrak{l}}$, $<_{\mathfrak{l}}$.

7) For any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ we reserve the terminology, introduced in Remark 2.2 (where the symbol $\mathcal{B}$ should be replaced by the symbol "$\mathfrak{l}$" and the phrase "base changeable set" should be replaced by the phrase "reference frame").

Directly from the system of notations, introduced in Remark 2.13, we deliver the following statement.

Assertion 2.14 ([17], see also [18], Assertion 1.10.1)**.** *Let, $\mathcal{Z}_1$, $\mathcal{Z}_2$ be arbitrary changeable sets such that:*

(1) $\mathcal{L}k(\mathcal{Z}_1) = \mathcal{L}k(\mathcal{Z}_2)$.
(2) *For arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z}_1) = \mathcal{L}k(\mathcal{Z}_2)$ it is true the equality:* $\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z}_1 \rangle = \langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z}_2 \rangle$.

Then, $\mathcal{Z}_1 = \mathcal{Z}_2$.

Definition 2.15. Let $\mathcal{Z}$ be a changeable set and $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ be any reference frame of $\mathcal{Z}$. We say that elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathfrak{l})$ are ***united by fate*** in $\mathfrak{l}$ iff ω_1 and ω_2 are united by fate in the base changeable set $\mathfrak{l}^{\hat{}}$, that is if and only if $\omega_2 \underset{\mathfrak{l}}{\leftarrow} \omega_1$ or $\omega_1 \underset{\mathfrak{l}}{\leftarrow} \omega_2$. We use the notation

$$\omega_1 \underset{\mathfrak{l}}{\leftrightarrow} \omega_2$$

to indicate the fact that elementary-time states $\omega_1, \omega_2 \in \mathbb{Bs}(\mathfrak{l})$ are united by fate in $\mathfrak{l}$.

Definition 2.16. We say, that a changeable set $\mathcal{Z}$ is **precisely visible** if and only if for any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ and for any element $\omega \in \mathbb{Bs}(\mathfrak{l})$ there exist a unique element $\omega' \in \mathbb{Bs}(\mathfrak{m})$ such, that $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \{\omega\} = \{\omega'\}$. [5]

Let $\mathcal{Z}$ be any precisely visible changeable set and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ be any reference frames of $\mathcal{Z}$. For any $\omega \in \mathbb{Bs}(\mathfrak{l})$ we denote by $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle \omega$ (or by $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \omega$) the unique (in accordance with Definition 2.16) element $\omega' \in \mathbb{Bs}(\mathfrak{m})$ such, that $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \{\omega\} = \{\omega'\}$. Hence, we have $\forall \omega \in \mathbb{Bs}(\mathfrak{l})\ \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \{\omega\} = \{\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \omega\}$. The mapping $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle : \mathbb{Bs}(\mathfrak{l}) \to \mathbb{Bs}(\mathfrak{m})$ we name as the ***precise unification mapping*** of $\mathcal{Z}$.

Assertion 2.17 ([14], see also [18])**.** *Let $\mathcal{Z}$ be any precisely visible changeable set, and $\mathfrak{l}, \mathfrak{m}, \mathfrak{p} \in \mathcal{L}k(\mathcal{Z})$ be arbitrary reference frames of $\mathcal{Z}$. Then:*

(1) $\forall\, \omega \in \mathbb{Bs}(\mathfrak{l}) \quad \langle !\, \mathfrak{l} \leftarrow \mathfrak{l} \rangle \omega = \omega$;
(2) $\forall\, A \subseteq \mathbb{Bs}(\mathfrak{l}) \quad \langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle A = \{\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \omega \mid \omega \in A\}$;
(3) $\forall\, \omega \in \mathbb{Bs}(\mathfrak{l}) \quad \langle !\, \mathfrak{p} \leftarrow \mathfrak{m} \rangle \langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \omega = \langle !\, \mathfrak{p} \leftarrow \mathfrak{l} \rangle \omega$.

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[5] In some papers (see, for example, [18, Definition I.12.3]) it had been given another, different, definition of precisely visible changeable set notion. Using [18, Corollary I.12.5 and Assertion I.12.11] it can be proved, that Definition 2.16 is equivalent to the definition, given in [18].

Let $\mathcal{Z}$ be any precisely visible changeable set and $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ be any reference frames of $\mathcal{Z}$. Using Assertion 2.17, for any elementary-time states $\omega_0 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$, $\omega_1 \in \mathbb{B}\mathfrak{s}(\mathfrak{m})$ we obtain:

$$\langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle \langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \omega_0 = \langle !\, \mathfrak{l} \leftarrow \mathfrak{l} \rangle \omega_0 = \omega_0$$
$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle \langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle \omega_1 = \langle !\, \mathfrak{m} \leftarrow \mathfrak{m} \rangle \omega_1 = \omega_1.$$

Thus we obtain the following corollary:

Corollary 2.18. *For any reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z})$ of any precisely visible changeable set $\mathcal{Z}$ the precise unification mapping $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle$ is a bijection from $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ onto $\mathbb{B}\mathfrak{s}(\mathfrak{m})$. Moreover the mapping $\langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle$ is inverse to $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle$, that is*

$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle^{[-1]} = \langle !\, \mathfrak{l} \leftarrow \mathfrak{m} \rangle \; .$$

2.2. Theorem on Multi-image for Changeable Sets. Evolutionarily Visible Changeable Sets.

Definition 2.19.

(1) An evolution projector $(\mathbb{T}, \mathcal{X}, U)$ (where $\mathbb{T} = (\mathbf{T}, \leq)$) for a base changeable set $\mathcal{B}$ is named as **injective** if and only if the mapping U is injection from $\mathbb{B}\mathfrak{s}(\mathcal{B})$ to $\mathbf{T} \times \mathcal{X}$ (that is bijection from $\mathbb{B}\mathfrak{s}(\mathcal{B})$ onto the set $\mathfrak{R}(U) \subseteq \mathbf{T} \times \mathcal{X}$) [6].

(2) Any indexed family $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ (where $\mathcal{A} \neq \emptyset$) of injective evolution projectors for the base changeable set $\mathcal{B}$ we name by **evolution multi-projector** for $\mathcal{B}$.

Theorem 2.20 (published in [17], see also [18], Theorem 1.11.2)**.** *Let, $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for a base changeable set $\mathcal{B}$. Then only one changeable set $\mathcal{Z}$ exists, satisfying the following conditions:*

(1) *$\mathcal{L}k(\mathcal{Z}) = \{(\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \mid \alpha \in \mathcal{A}\}$.*

(2) *For any reference frames $\mathfrak{l} = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha]) \in \mathcal{L}k(\mathcal{Z})$, $\mathfrak{m} = (\beta, U_\beta[\mathcal{B}, \mathbb{T}_\beta]) \in \mathcal{L}k(\mathcal{Z})$ $(\alpha, \beta \in \mathcal{A})$ and an arbitrary set $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha(\mathbb{B}\mathfrak{s}(\mathcal{B}))$ the following equality holds:*

$$\langle \mathfrak{m} \leftarrow \mathfrak{l}, \mathcal{Z} \rangle A = U_\beta\left(U_\alpha^{[-1]}(A)\right) = \left\{U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \mid \omega \in A\right\},$$

*where $U_\alpha^{[-1]}$ is the mapping, **inverse** to U_α.*

Definition 2.21. Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for a base changeable set $\mathcal{B}$. Changeable set $\mathcal{Z}$, satisfying conditions 1,2 of Theorem 2.20 is referred to as **evolution multi-image** of base changeable set $\mathcal{B}$ relatively the evolution multi-projector $\mathfrak{P}$. This evolution multi-image will be denoted by $\mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}]$:

$$\mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}] := \mathcal{Z}.$$

The following corollary follows directly from Theorem 2.20 and formula (2.7):

Corollary 2.22. *Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for base changeable set $\mathcal{B}$ and $\mathcal{Z} = \mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}]$ then:*

(1) *$\mathcal{I}nd(\mathcal{Z}) = \mathcal{A}$.*

(2) *$\mathbf{lk}_\alpha(\mathcal{Z}) = (\alpha, U_\alpha[\mathcal{B}, \mathbb{T}_\alpha])$ (for any $\alpha \in \mathcal{I}nd(\mathcal{Z})$).*

Assertion 2.23 (published in [17], see also [18], Corollary 1.12.7)**.** *Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for a base changeable set $\mathcal{B}$. Then*

[6] Here $\mathfrak{R}(U)$ means the ***range*** of (arbitrary) mapping U.

the changeable set $\mathcal{Z} = \mathcal{Z}\mathsf{im}\,[\mathfrak{P},\mathcal{B}]$ is precisely visible. Moreover, for arbitrary reference frames $\mathfrak{l} = (\alpha, U_\alpha\,[\mathcal{B},\mathbb{T}_\alpha]) \in \mathcal{L}k\,(\mathcal{Z})$, $\mathfrak{m} = (\beta, U_\beta\,[\mathcal{B},\mathbb{T}_\beta]) \in \mathcal{L}k\,(\mathcal{Z})$ ($\alpha,\beta \in \mathcal{A}$) the following equality is performed:

$$\langle !\,\mathfrak{m} \leftarrow \mathfrak{l},\ \mathcal{Z}\rangle\,\omega = U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \quad (\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha\,(\mathbb{B}\mathfrak{s}(\mathcal{B}))).$$

Assertion 2.23 and Corollary 2.22 immediatelly lead to the following corollary.

Corollary 2.24. *Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for base changeable set $\mathcal{B}$ and $\mathcal{Z} = \mathcal{Z}\mathsf{im}\,[\mathfrak{P},\mathcal{B}]$ then for any indexes $\alpha,\beta \in \mathcal{A}$ the following equality holds:*

$$\langle !\,\mathbf{lk}_\beta\,(\mathcal{Z}) \leftarrow \mathbf{lk}_\alpha\,(\mathcal{Z})\,,\ \mathcal{Z}\rangle\,\omega = U_\beta\left(U_\alpha^{[-1]}(\omega)\right) \quad (\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) = U_\alpha\,(\mathbb{B}\mathfrak{s}(\mathcal{B}))).$$

Definition 2.25. **1**. Let $\mathcal{Z}$ be a precisely visible changeable set and $\mathfrak{l},\mathfrak{m} \in \mathcal{L}k\,(\mathcal{Z})$ be any reference frames of $\mathcal{Z}$. We say that the reference frame $\mathfrak{l}$ is **evolutionarily visible** from the reference frame $\mathfrak{m}$ iff for arbitrary $\omega_1,\omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ such, that $\omega_1 \underset{\mathfrak{l}}{\leftrightarrow} \omega_2$ and $\mathsf{tm}\,(\langle !\,\mathfrak{m} \leftarrow \mathfrak{l}\rangle\,\omega_1) \neq \mathsf{tm}\,(\langle !\,\mathfrak{m} \leftarrow \mathfrak{l}\rangle\,\omega_2)$ we have $\langle !\,\mathfrak{m} \leftarrow \mathfrak{l}\rangle\,\omega_1 \underset{\mathfrak{m}}{\leftrightarrow} \langle !\,\mathfrak{m} \leftarrow \mathfrak{l}\rangle\,\omega_2$.

2. Changeable set $\mathcal{Z}$ is named as **evolutionarily visible** iff $\mathcal{Z}$ is precisely visible and any reference frame $\mathfrak{l} \in \mathcal{L}k\,(\mathcal{Z})$ is evolutionarily visible from an arbitrary reference frame $\mathfrak{m} \in \mathcal{L}k\,(\mathcal{Z})$.

From the physical point of view the evolutionary visibility concept can be interpreted as invariance of evolutionary (transformation) processes in different reference frames.

The next statement shows that any multi-image is an evolutionarily visible changeable set.

Theorem 2.26 (published in [16], see also [18], Assertion 3.27.8)**.** *Let $\mathfrak{P} = ((\mathbb{T}_\alpha, \mathcal{X}_\alpha, U_\alpha) \mid \alpha \in \mathcal{A})$ be an evolution multi-projector for a base changeable set $\mathcal{B}$. Then the changeable set $\mathcal{Z}\mathsf{im}\,[\mathfrak{P},\mathcal{B}]$ is evolutionarily visible.*

3. On Mathematical Problems Connected with Evolutionary Visibility. Statement of the main problem

Theorem 2.26 generates the following "inverse" problem.

Problem 3.1. Can every evolutionarily visible changeable set be represented as a multi-image? That is, does there exist such base changeable set $\mathcal{B}$ and evolution multi-projector $\mathfrak{P}$ for $\mathcal{B}$ that

$$\mathcal{Z} = \mathcal{Z}\mathsf{im}\,[\mathfrak{P},\mathcal{B}]$$

for every evolutionarily visible changeable set $\mathcal{Z}$?

The first idea, which comes to mind, when we are trying to solve Problem 3.1, is to try to represent an arbitrary evolutionarily visible changeable set $\mathcal{Z}$ as a "self-multiimage", that is in the form of a multi-image of some its reference frame by means of the precise unification mappings. Below we will formulate more precisely, what we mean, that is, we will give the mathematically strict definition of the self-multiimage notion.

Let $\mathcal{Z}$ be a precisely visible changeable set and $\mathfrak{l}_0 \in \mathcal{L}k\,(\mathcal{Z})$ be any (fixed) reference frame of $\mathcal{Z}$. Then, according to Remark 2.13 (item **4)**), $(\mathfrak{l}_0)^\wedge$ is a base changeable set. Moreover, by Corollary 2.18, the precise unification mapping $\langle !\,\mathfrak{l} \leftarrow \mathfrak{l}_0\rangle : \mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0) \to \mathbb{B}\mathfrak{s}(\mathfrak{l})$ is a bijection between $\mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0) = \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge)$ and $\mathbb{B}\mathfrak{s}(\mathfrak{l})$ for any reference frame $\mathfrak{l} \in \mathcal{L}k\,(\mathcal{Z})$. Thus, for any (fixed) reference frame $\mathfrak{l}_0 \in \mathcal{L}k\,(\mathcal{Z})$ the following statements are valid:

- *The ordered triple $(\mathbb{T}\mathbf{m}\,(\mathfrak{l})\,, \mathfrak{B}\mathfrak{s}\,(\mathfrak{l})\,, \langle !\,\mathfrak{l} \leftarrow \mathfrak{l}_0\rangle\,)$ is an injective evolution projector for the the base changeable set $(\mathfrak{l}_0)^\wedge$.*

- *The indexed family:*

$$\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle} = \left(\left(\mathbb{T}\mathbf{m}\left(\mathbf{lk}_\alpha\left(\mathcal{Z}\right)\right), \mathfrak{B}\mathfrak{s}\left(\mathbf{lk}_\alpha\left(\mathcal{Z}\right)\right), \langle !\, \mathbf{lk}_\alpha\left(\mathcal{Z}\right) \leftarrow \mathfrak{l}_0 \rangle\right) \mid \alpha \in \mathcal{I}nd\left(\mathcal{Z}\right)\right) \tag{3.8}$$

is an evolution multi-projector for the base changeable set $(\mathfrak{l}_0)^\wedge$.

Definition 3.2.

- We say that a precisely visible changeable set $\mathcal{Z}$ is a **multi-image of reference frame** $\mathfrak{l}_0 \in \mathcal{L}k\left(\mathcal{Z}\right)$, iff the following equality holds:
$$\mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}, (\mathfrak{l}_0)^\wedge\right] = \mathcal{Z}.$$
- We say that the changeable set $\mathcal{Z}$ is a **self-multiimage**, iff there exists a reference frame $\mathfrak{l}_0 \in \mathcal{L}k\left(\mathcal{Z}\right)$, such that $\mathcal{Z}$ is a multi-image of $\mathfrak{l}_0$ (IE $\mathcal{Z} = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}, (\mathfrak{l}_0)^\wedge\right]$).

Thus, we can mathematically strictly formulate the next problem, generanted by Problem 3.1.

Problem 3.3. Can every evolutionarily visible changeable set $\mathcal{Z}$ be represented as a self-multiimage, that is can $\mathcal{Z}$ be represented in a form $\mathcal{Z} = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}, (\mathfrak{l}_0)^\wedge\right]$ for a some reference frame $\mathfrak{l}_0 \in \mathcal{L}k\left(\mathcal{Z}\right)$?

The main aim of the present paper is to give the negative general solution of Problem 3.3. In Section 5 we give the example of a changeable set that cannot be represented as a self-multiimage. For this aim we prove the theorem which gives neccessary and sufficient condition for changeable set to be represented as a self-multiimage. But, first of all, in the next section we will show that for some changeable sets Problem 3.3 may have the positive solution.

4. Some examples of changeable sets, which can be represented as a Self-multiimage

The following theorem describes the quite wide diapason of cases, where changeable set can be represented as a self-multiimage.

Theorem 4.1. ***Let*** *$\mathcal{Z}$ be any changeable set of the kind*

$$\mathcal{Z} = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}, \mathcal{B}\right],$$

where $\mathfrak{P} = \left(\left(\mathbb{T}_\alpha, \mathcal{X}_\alpha, \mathcal{U}_\alpha\right) \mid \alpha \in \mathcal{A}\right)$ is an evolution multi-projector for a base changeable set $\mathcal{B}$, satisfying the following additional condition:

$$\left(\mathbb{T}\mathbf{m}(\mathcal{B}), \mathfrak{B}\mathfrak{s}(\mathcal{B}), \mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}\right) = \left(\mathbb{T}_{\alpha_0}, \mathcal{X}_{\alpha_0}, \mathcal{U}_{\alpha_0}\right) \quad \text{for some } \alpha_0 \in \mathcal{A}. \tag{4.9}$$

Then *$\mathcal{Z}$ is a self-multiimage.*

Remark 4.2. In other words, Condition (4.9) means that the identity mapping is one of the "components" of the multi-projector $\mathfrak{P}$.

Proof of Theorem 4.1. Let $\mathcal{Z}$ be the changeable set, satisfying conditions of the theorem. Without loss of generality we can consider that

$$\mathcal{X}_\alpha = \mathsf{bs}\left(\mathfrak{R}\left(\mathcal{U}_\alpha\right)\right) = \left\{\mathsf{bs}\left(\mathcal{U}_\alpha\left(\omega\right)\right) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\right\} \qquad \left(\forall \alpha \in \mathcal{A}\right). \tag{4.10}$$

Indeed, if condition (4.10) is not performed we may consider the evolution multi-projector of kind:

$$\mathfrak{P}^\sim = \left(\left(\mathbb{T}_\alpha, \mathcal{X}^\sim_\alpha, \mathcal{U}_\alpha\right) \mid \alpha \in \mathcal{A}\right), \quad \text{where}$$
$$\mathcal{X}^\sim_\alpha = \mathsf{bs}\left(\mathfrak{R}\left(\mathcal{U}_\alpha\right)\right) = \left\{\mathsf{bs}\left(\mathcal{U}_\alpha\left(\omega\right)\right) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\right\} \quad \left(\alpha \in \mathcal{A}\right).$$

Condition (4.10), for the multi-projector $\mathfrak{P}^\sim$, surely is satisfied. Moreover, applying Assertion 2.14, it is not hard to verify that $\mathcal{Z}\mathsf{im}\left[\mathfrak{P}^\sim, \mathcal{B}\right] = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}, \mathcal{B}\right] = \mathcal{Z}$.

Thus, further we assume that condition (4.10) is performed.

Consider the reference frame:

$$\mathfrak{l}_0 := \mathbf{lk}_{\alpha_0}(\mathcal{Z}) \in \mathcal{L}k(\mathcal{Z}). \tag{4.11}$$

According to Assertion 2.11, we get, $\mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}[\mathcal{B}, \mathbb{T}\mathbf{m}(\mathcal{B})] = \mathcal{B}$. So, by Corollary 2.22 and condition 4.9, we have:

$$\mathfrak{l}_0 = (\alpha_0, \mathcal{U}_{\alpha_0}[\mathcal{B}, \mathbb{T}_{\alpha_0}]) = \left(\alpha_0, \mathbb{I}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}[\mathcal{B}, \mathbb{T}\mathbf{m}(\mathcal{B})]\right) = (\alpha_0, \mathcal{B}).$$

Thence, by the notation system, accepted in Remark 2.13, we have:

$$\mathfrak{l}_0\hat{\ } = \mathcal{B}. \tag{4.12}$$

Moreover, using Corollary 2.22, notation (4.11), Corollary 2.24, condition (4.9), formula (2.1) and convention (4.10), we deliver:

$$\begin{aligned}
\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle} &= \left(\left(\mathbb{T}\mathbf{m}\left(\mathbf{lk}_\alpha(\mathcal{Z})\right), \mathfrak{B}\mathfrak{s}\left(\mathbf{lk}_\alpha(\mathcal{Z})\right), \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0\rangle\right) \mid \alpha \in \mathcal{I}nd(\mathcal{Z})\right) = \\
&= \left(\left(\mathbb{T}\mathbf{m}\left(\mathbf{lk}_\alpha(\mathcal{Z})\right), \mathfrak{B}\mathfrak{s}\left(\mathcal{U}_\alpha[\mathcal{B}, \mathbb{T}_\alpha]\right), \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathbf{lk}_{\alpha_0}(\mathcal{Z})\rangle\right) \mid \alpha \in \mathcal{A}\right) = \\
&= \left(\left(\mathbb{T}\mathbf{m}\left(\mathcal{U}_\alpha[\mathcal{B}, \mathbb{T}_\alpha]\right), \mathfrak{B}\mathfrak{s}\left(\mathcal{U}_\alpha[\mathcal{B}, \mathbb{T}_\alpha]\right), \mathcal{U}_\alpha\left(\mathcal{U}_{\alpha_0}^{[-1]}\right)\right) \mid \alpha \in \mathcal{A}\right) = \\
&= \left(\left(\mathbb{T}_\alpha, \mathfrak{B}\mathfrak{s}\left(\mathcal{U}_\alpha[\mathcal{B}, \mathbb{T}_\alpha]\right), \mathcal{U}_\alpha\left(\mathbb{I}^{[-1]}_{\mathbb{B}\mathfrak{s}(\mathcal{B})}\right)\right) \mid \alpha \in \mathcal{A}\right) = \\
&= \left(\left(\mathbb{T}_\alpha, \left\{\mathsf{bs}\left(\mathcal{U}_\alpha(\omega)\right) \mid \omega \in \mathbb{B}\mathfrak{s}(\mathcal{B})\right\}, \mathcal{U}_\alpha\right) \mid \alpha \in \mathcal{A}\right) = \left((\mathbb{T}_\alpha, \mathcal{X}_\alpha, \mathcal{U}_\alpha) \mid \alpha \in \mathcal{A}\right) = \mathfrak{P}.
\end{aligned} \tag{4.13}$$

Using equalities (4.12) and (4.13), we obtain:

$$\mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}, \mathfrak{l}_0\hat{\ }\right] = \mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}] = \mathcal{Z}. \tag{4.14}$$

Thus, according to Definition 3.2 and equality (4.14), the changeable set $\mathcal{Z} = \mathcal{Z}\mathsf{im}[\mathfrak{P}, \mathcal{B}]$ is a self-multiimage. □

Example 4.3. Let $\mathcal{B}$ be a base changeable set such, that

$$\mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R}^3, \quad \mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{R}_{ord} = (\mathbb{R}, \leq), \tag{4.15}$$

where $\leq$ is the standard linear order relation on the real numbers. Such base changeable set $\mathcal{B}$ must exist, because, for example, we may take $\mathcal{B} := \mathcal{A}t(\mathbb{R}_{ord}, \mathcal{R})$, where $\mathcal{R}$ is a system of abstract trajectories from $\mathbb{R}_{ord}$ to the subset $M \subseteq \mathbb{R}^3$. Let us consider Poincare group $P(1,3,c)$, defined on the 4-dimensional space-time $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$, that is the group of affine transformations[7] of the space $\mathbb{R}^4$, which are satisfying the following conditions:

(1) Any transformation $\boldsymbol{P} \in P(1,3,c)$ leaves unchanged values of the Lorentz-Minkowski pseudo-distance on $\mathbb{R}^4$:

$$\mathsf{M}_c(\boldsymbol{P}\mathrm{w}_1 - \boldsymbol{P}\mathrm{w}_2) = \mathsf{M}_c(\mathrm{w}_1 - \mathrm{w}_2), \quad (\forall\, \mathrm{w}_1, \mathrm{w}_2 \in \mathbb{R}^4), \quad \text{where;}$$

$$\mathsf{M}_c(\mathrm{w}) = \sum_{j=1}^{3} w_i^2 - c^2 w_0^2 \quad \text{and}$$

$$\mathrm{w} - \widetilde{\mathrm{w}} = (w_0 - \widetilde{w}_0,\ w_1 - \widetilde{w}_1,\ w_2 - \widetilde{w}_2,\ w_3 - \widetilde{w}_3)$$

$$\left(\mathrm{w} = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4,\ \widetilde{\mathrm{w}} = (\widetilde{w}_0, \widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3) \in \mathbb{R}^4\right).$$

Here the number c means any fixed positive real constant, which has the physical content of the speed of light in vacuum.

[7] Recall that the affine transformation over the linear space L is the mapping $\mathbf{U} : L \to L$, acting by the formula $\mathbf{U}x = \mathbf{A}x + \mathbf{a}$ $(x \in L)$, where $\mathbf{A}$ is some linear operator over L and $\mathbf{a}$ is some (fixed) element of L.

(2) Any transformation $\boldsymbol{P} \in P(1,3,c)$ has positive direction of time, that is $\boldsymbol{P}\mathrm{w}_2 - \boldsymbol{P}\mathrm{w}_1 \in \mathcal{M}_{c,+}\left(\mathbb{R}^3\right)$ for any $\mathrm{w}_1, \mathrm{w}_2 \in \mathbb{R}^4$ such, that $\mathrm{w}_2 - \mathrm{w}_1 \in \mathcal{M}_{c,+}\left(\mathbb{R}^3\right)$, where

$$\mathcal{M}_{c,+}\left(\mathbb{R}^3\right) = \left\{\mathrm{w} = (w_0, w_1, w_2, w_3) \in \mathbb{R}^4 \mid w_0 > 0,\ \mathsf{M}_c(\mathrm{w}) < 0\right\}$$

(Cf. [42, 34, 25, 8, 33]).

From formulas (2.3), (2.2) and (4.15) it follows that

$$\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbf{Tm}(\mathcal{B}) \times \mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4.$$

Therefore, for every transformation $\boldsymbol{P} \in P(1,3,c)$ the triple $\left(\mathbb{R}_{ord}, \mathbb{R}^3, \boldsymbol{P}_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}\right)$ is an evolution projector for $\mathcal{B}$ (where $\boldsymbol{P}_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}$ is the restriction of the mapping $\boldsymbol{P}$ to the set $\mathbb{B}\mathfrak{s}(\mathcal{B})$). So, the family:

$$\mathfrak{P}_{[P(1,3,c)]} = \left(\left(\mathbb{R}_{ord}, \mathbb{R}^3, \boldsymbol{P}_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}\right) \mid \boldsymbol{P} \in P(1,3,c)\right)$$

is an evolution multi-projector for $\mathcal{B}$.

Since the identity mapping $\mathbb{I}_{\mathbb{R}^4}$ is the element of the Poincare group $P(1,3,c)$, we see that the changeable set

$$\mathcal{Z}P(1,3,\mathcal{B},c) = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}_{[P(1,3,c)]}, \mathcal{B}\right]$$

is a self-multiimage (by Theorem 4.1). Note that the changeable set $\mathcal{Z}P(1,3,\mathcal{B},c)$ represents a mathematically strict model of the kinematics of special relativity theory in the inertial frames of reference.

Example 4.4. Let $(\mathfrak{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a Hilbert space over Real field such, that $\dim(\mathfrak{H}) \geq 1$, where $\dim(\mathfrak{H})$ is dimension of the space $\mathfrak{H}$. Emphasize, that the condition $\dim(\mathfrak{H}) \geq 1$ should be interpreted in a way that the space $\mathfrak{H}$ may be infinite-dimensional. Let $\mathcal{L}(\mathfrak{H})$ be the space of (homogeneous) linear continuous operators over the space $\mathfrak{H}$. Denote by $\mathcal{L}^{\times}(\mathfrak{H})$ the space of all operators of affine transformations over the space $\mathfrak{H}$, that is $\mathcal{L}^{\times}(\mathfrak{H}) = \left\{\mathbf{A}_{[\mathbf{a}]} \mid \mathbf{A} \in \mathcal{L}(\mathfrak{H}), \mathbf{a} \in \mathfrak{H}\right\}$, where $\mathbf{A}_{[\mathbf{a}]}x = \mathbf{A}x + \mathbf{a}$, $x \in \mathfrak{H}$. The ***Minkowski space*** over the Hilbert space $\mathfrak{H}$ is defined as the Hilbert space $\mathcal{M}(\mathfrak{H}) = \mathbb{R} \times \mathfrak{H} = \{(t,x) \mid t \in \mathbb{R},\ x \in \mathfrak{H}\}$, equipped by the inner product and norm: $\langle \mathrm{w}_1, \mathrm{w}_2\rangle = \langle \mathrm{w}_1, \mathrm{w}_2\rangle_{\mathcal{M}(\mathfrak{H})} = t_1 t_2 + \langle x_1, x_2\rangle$, $\|\mathrm{w}_1\| = \|\mathrm{w}_1\|_{\mathcal{M}(\mathfrak{H})} = \left(t_1^2 + \|x_1\|^2\right)^{1/2}$ (where $\mathrm{w}_i = (t_i, x_i) \in \mathcal{M}(\mathfrak{H})$, $i \in \{1,2\}$).

Denote via $\mathbf{Pk}(\mathfrak{H})$ the set of all operators $\mathbf{S} \in \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$, which has the continuous inverse operator $\mathbf{S}^{-1} \in \mathcal{L}^{\times}(\mathcal{M}(\mathfrak{H}))$. Operators $\mathbf{S} \in \mathbf{Pk}(\mathfrak{H})$ we name as ***(affine) coordinate transform operators***.

Let, $\mathcal{B}$ be any base changeable set such, that $\mathfrak{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathfrak{H}$ and $\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{R}_{ord} = (\mathbb{R}, \leq)$. From formulas (2.3) and (2.2) it follows that

$$\mathbb{B}\mathfrak{s}(\mathcal{B}) \subseteq \mathbb{R} \times \mathfrak{H} = \mathcal{M}(\mathfrak{H}).$$

So any operator $\mathbf{S} \in \mathbf{Pk}(\mathfrak{H})$ generates the evolution projector $\left(\mathbb{R}_{ord}, \mathfrak{H}, \mathbf{S}_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}\right)$. Therefore any set of operators $\mathbb{S} \subseteq \mathbf{Pk}(\mathfrak{H})$ generates the evolution multi-projector

$$\mathfrak{P}_{[\mathbb{S},\mathfrak{H}]} = \left(\left(\mathbb{R}_{ord}, \mathfrak{H}, \mathbf{S}_{\restriction \mathbb{B}\mathfrak{s}(\mathcal{B})}\right) \mid \mathbf{S} \in \mathbb{S}\right)$$

for $\mathcal{B}$. Denote:

$$\mathcal{Z}\mathsf{im}(\mathbb{S}, \mathcal{B}, \mathfrak{H}) := \mathcal{Z}\mathsf{im}\left[\mathfrak{P}_{[\mathbb{S},\mathfrak{H}]}, \mathcal{B}\right].$$

Let c be any fixed positive real constant, which has the physical content of the speed of light in vacuum. In the papers [14, 18, 19] the following classes of operators were introduced:

$$\mathfrak{PT}(\mathfrak{H},c) \subseteq \mathbf{Pk}(\mathfrak{H});$$
$$\mathfrak{PT}_+(\mathfrak{H},c) \subseteq \mathbf{Pk}(\mathfrak{H});$$
$$\mathfrak{P}(\mathfrak{H},c) \subseteq \mathbf{Pk}(\mathfrak{H});$$
$$\mathfrak{P}_+(\mathfrak{H},c) \subseteq \mathbf{Pk}(\mathfrak{H});$$
$$\mathfrak{PT}^{\mp}_{\mathsf{fin}}(\mathfrak{H},c) \subseteq \mathbf{Pk}(\mathfrak{H}).$$

Remark 4.5. Using results of the papers [14, 18, 19] it can be proven that all above classes of operators are subclasses of the class $\mathfrak{PT}(\mathfrak{H},c)$, that is $\mathfrak{PT}_+(\mathfrak{H},c), \mathfrak{P}(\mathfrak{H},c), \mathfrak{P}_+(\mathfrak{H},c), \mathfrak{PT}^{\mp}_{\mathsf{fin}}(\mathfrak{H},c) \subseteq \mathfrak{PT}(\mathfrak{H},c)$. In the case $\mathfrak{H} = \mathbb{R}^3$ operators from the class $\mathfrak{PT}(\mathfrak{H},c)$ represent generalized Lorentz-Poincare transformations introduced in the papers of E. Recami, V. Olkhovsky and R. Goldoni (see [37, 36, 38, 39, 9], see also [43]). These transformations were later rediscovered in [22, 32](see also [26]). In the papers [14, 18, 19] generalized Lorentz-Poincare transformations in the sense of E. Recami, V. Olkhovsky and R. Goldoni were extend to the case of general Hilbert Space.

The introduced above families of operators generate the following changeable sets:

$$\mathcal{Z}\mathfrak{P}\mathcal{T}_0(\mathfrak{H},\mathcal{B},c) := \mathcal{Z}\mathsf{im}\left(\mathfrak{PT}(\mathfrak{H},c),\mathcal{B},\mathfrak{H}\right);$$
$$\mathcal{Z}\mathfrak{P}\mathcal{T}(\mathfrak{H},\mathcal{B},c) := \mathcal{Z}\mathsf{im}\left(\mathfrak{PT}_+(\mathfrak{H},c),\mathcal{B},\mathfrak{H}\right);$$
$$\mathcal{Z}\mathfrak{P}_0(\mathfrak{H},\mathcal{B},c) := \mathcal{Z}\mathsf{im}\left(\mathfrak{P}(\mathfrak{H},c),\mathcal{B},\mathfrak{H}\right);$$
$$\mathcal{Z}\mathfrak{P}(\mathfrak{H},\mathcal{B},c) := \mathcal{Z}\mathsf{im}\left(\mathfrak{P}_+(\mathfrak{H},c),\mathcal{B},\mathfrak{H}\right);$$
$$\mathcal{Z}\mathfrak{P}\mathcal{T}^{\mp}(\mathfrak{H},\mathcal{B},c) := \mathcal{Z}\mathsf{im}\left(\mathfrak{PT}^{\mp}_{\mathsf{fin}}(\mathfrak{H},c),\mathcal{B},\mathfrak{H}\right).$$

The introduced above changeable sets are connected with kinematics of special relativity theory as well as its tachyon extensions in inertial reference frames [14]. Since the identity operator is an element of each from the sets of operators $\mathfrak{PT}(\mathfrak{H},c), \mathfrak{PT}_+(\mathfrak{H},c), \mathfrak{P}(\mathfrak{H},c), \mathfrak{P}_+(\mathfrak{H},c), \mathfrak{PT}^{\mp}_{\mathsf{fin}}(\mathfrak{H},c)$, we conclude that all changeable sets $\mathcal{Z}\mathfrak{P}\mathcal{T}_0(\mathfrak{H},\mathcal{B},c), \mathcal{Z}\mathfrak{P}\mathcal{T}(\mathfrak{H},\mathcal{B},c), \mathcal{Z}\mathfrak{P}_0(\mathfrak{H},\mathcal{B},c), \mathcal{Z}\mathfrak{P}(\mathfrak{H},\mathcal{B},c), \mathcal{Z}\mathfrak{P}\mathcal{T}^{\mp}(\mathfrak{H},\mathcal{B},c)$ are self-multiimages (similarly to example 4.3).

From examples 4.3 and 4.4 we may conclude that the most significant changeable sets, connected with kinematics of special relativity theory as well as its tachyon extensions in inertial reference frames are self-multiimages.

5. Criterion for Representation of a Changeable Set as a Self-multiimage. Example of a Changeable Set that cannot be Represented as a Self-multiimage

The main goal of this section is to give an example of an evolutionarily visible changeable set that is not a self-multiimage. For this aim we will prove the simple for verification necessary and sufficient condition for evolutionarily visible changeable set to be representable as self-multiimage.

Denote:

$$\mathbf{T} := \mathcal{X} := \mathbb{R}; \qquad \mathbb{T} := (\mathbf{T}, \leq) = \mathbb{R}_{ord} = (\mathbb{R}, \leq)$$

with the standard linear order relation $\leq$ on Real field $\mathbb{R}$. Let us put:

$$\mathbf{B} := \mathbf{T} \times \mathcal{X} = \mathbb{R}^2.$$

Consider the binary relation $\Leftarrow \subseteq \mathbf{B}^2$ on the set $\mathbf{B}$ such that for arbitrary $\omega_1 = (t_1, x_1) \in \mathbf{B} = \mathbb{R}^2$, $\omega_2 = (t_2, x_2) \in \mathbf{B}$ it holds the following statement:

$$\omega_2 \Leftarrow \omega_1 \quad \text{if and only if} \quad \omega_1 = \omega_2 \quad \text{or} \quad t_1 < t_2, \tag{5.16}$$

where $<$ is the standard strong linear order on the Real field $\mathbb{R}$. From (5.16) it follows that the binary relation $\Leftarrow$ is reflexive on $\mathbf{B}$ and for any $\omega_1, \omega_2 \in \mathbf{B}$ the conditions $\omega_2 \Leftarrow \omega_1$ and $\omega_1 \neq \omega_2$ stipulate the inequality $\mathsf{tm}\,(\omega_1) < \mathsf{tm}\,(\omega_2)$. Hence, the relation $\Leftarrow$ satisfies conditions of Definition 2.1. Therefore the triple $\mathcal{B} = (\mathbf{B}, \mathbb{T}, \Leftarrow)$ is the base changeable set such that:

$$\mathbb{T}\mathbf{m}(\mathcal{B}) = \mathbb{T} = \mathbb{R}_{ord} = (\mathbb{R}, \leq)\,, \qquad \mathbb{B}\mathfrak{s}(\mathcal{B}) = \mathbf{B} = \mathbb{R}^2, \qquad \overset{\mathbb{B}\mathfrak{s}}{\underset{\mathcal{B}}{\leftarrow}} = \Leftarrow\,. \tag{5.17}$$

From (5.16) and (5.17) we conclude that:

$$\forall\, \omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathcal{B}) \ \big(\omega_2 \underset{\mathcal{B}}{\leftarrow} \omega_1 \quad \text{if and only if} \quad \omega_1 = \omega_2 \quad \text{or} \quad \mathsf{tm}\,(\omega_1) < \mathsf{tm}\,(\omega_2)\big), \tag{5.18}$$

Further, we consider the following set of indexes:

$$\mathcal{A}_{p\tau} := (\mathbb{R} \setminus \{-1, 1\}) \times \mathbb{R}^2 \tag{5.19}$$

and the following set of mappings:

$$\mathscr{P}\tau := \{\mathbf{P}_\alpha \mid \alpha \in \mathcal{A}_{p\tau}\}\,,$$

where for each $\alpha = (\lambda, \mathbf{t}_0, \mathbf{x}_0) \in \mathcal{A}_{p\tau}$ the mapping $\mathbf{P}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the formula:

$$\mathbf{P}_\alpha(t, x) = \left(\frac{t - \lambda x}{\sqrt{|1 - \lambda^2|}} + \mathbf{t}_0,\ \frac{x - \lambda t}{\sqrt{|1 - \lambda^2|}} + \mathbf{x}_0 \right) \quad \big((t, x) \in \mathbb{R}^2\big). \tag{5.20}$$

By direct verification, it can be established that every mapping $\mathbf{P}_\alpha$ ($\alpha = (\lambda, \mathbf{t}_0, \mathbf{x}_0) \in \mathcal{A}_{p\tau}$) is a bijection in $\mathbb{R}^2$, moreover, the inverse mapping $\mathbf{P}_\alpha^{[-1]}$ is determined by the formula:

$$\begin{aligned}
\mathbf{P}_\alpha^{[-1]}(\tau, \xi) &= \frac{\operatorname{sign}\,(1 - |\lambda|)}{\sqrt{|1 - \lambda^2|}} \left(\tau + \lambda\xi - (\mathbf{t}_0 + \lambda \mathbf{x}_0)\,,\ \xi + \lambda\tau - (\mathbf{x}_0 + \lambda \mathbf{t}_0)\right) = \\
&= \mathbf{P}_{\alpha_*}\left(\mathcal{I}_\alpha(\tau, \xi)\right), \qquad \text{where} \\
&\qquad \alpha_* = \left(-\lambda, \operatorname{sign}\,(|\lambda| - 1)\, \frac{\mathbf{t}_0 + \lambda \mathbf{x}_0}{\sqrt{|1 - \lambda^2|}}, \operatorname{sign}\,(|\lambda| - 1)\, \frac{\mathbf{x}_0 + \lambda \mathbf{t}_0}{\sqrt{|1 - \lambda^2|}} \right); \\
&\quad \mathcal{I}_\alpha(\tau, \xi) = \left(\operatorname{sign}\,(1 - |\lambda|)\,\tau,\ \operatorname{sign}\,(1 - |\lambda|)\,\xi\right) = \operatorname{sign}\,(1 - |\lambda|)\,(\tau, \xi)\,. \\
&\qquad\qquad \big((\tau, \xi) \in \mathbb{R}^2,\ \alpha = (\lambda, \mathbf{t}_0, \mathbf{x}_0) \in \mathcal{A}_{p\tau}\big)
\end{aligned} \tag{5.21}$$

Remark 5.1. In fact transformations from the class $\mathscr{P}\tau$ are simplified form of generalized Lorentz-Poincare transforms from the class $\mathfrak{P}\mathfrak{T}_+(\mathfrak{H}, c)$ (cf. [14, 18, 19]) in the case $\mathfrak{H} = \mathbb{R}$ with the speed of light parameter c being equal to 1 ($c = 1$).

Next we present the formula for the composition of two mappings from $\mathscr{P}\tau$. For this aim for each index $\alpha = (\lambda, \mathbf{t}_0, \mathbf{x}_0) \in \mathcal{A}_{p\tau}$ we introduce the following notations:

$$\begin{aligned}
\overline{\alpha}^0 &:= (\lambda, 0, 0) \in \mathcal{A}_{p\tau}; \\
{}^0\overline{\alpha} &:= (0, \mathbf{t}_0, \mathbf{x}_0) \in \mathcal{A}_{p\tau} \\
\alpha^{(0)} &:= \lambda \in \mathbb{R}; \\
{}^{(0)}\alpha &:= (\mathbf{t}_0, \mathbf{x}_0) \in \mathbb{R}^2.
\end{aligned}$$

Assertion 5.2. *For arbitrary $\alpha, \beta \in \mathcal{A}_{p\tau}$ the following formula is valid:*

$$\mathbf{P}_\alpha\left(\mathbf{P}_\beta(t,x)\right) = \begin{cases} \operatorname{sign}\left(1+\alpha^{(0)}\beta^{(0)}\right)\mathbf{P}_{\Lambda_{\alpha,\beta}}(t,x) + \mathbf{P}_\alpha\left({}^{(0)}\beta\right), & 1+\alpha^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha^{(0)}+\beta^{(0)}\right)\mathbf{P}_{(\infty,0,0)}(t,x) + \mathbf{P}_\alpha\left({}^{(0)}\beta\right), & 1+\alpha^{(0)}\beta^{(0)} = 0, \end{cases} \tag{5.22}$$

$$\text{where}\quad \mathbf{P}_{(\infty,0,0)}(t,x) = \lim_{\nu\to+\infty}\mathbf{P}_{(\nu,0,0)}(t,x) = -(x,t) \quad \left((t,x)\in\mathbb{R}^2\right); \tag{5.23}$$

$$\Lambda_{\alpha,\beta} = \left(\frac{\alpha^{(0)}+\beta^{(0)}}{1+\alpha^{(0)}\beta^{(0)}}, 0, 0\right) \in \mathcal{A}_{p\tau}. \tag{5.24}$$

Remark 5.3. Note that we have $\Lambda_{\alpha,\beta} \in \mathcal{A}_{p\tau}$ (for arbitrary $\alpha,\beta \in \mathcal{A}_{p\tau}$ such that $1+\alpha^{(0)}\beta^{(0)} \neq 0$). Indeed, if $\alpha,\beta\in\mathcal{A}_{p\tau}$ then $\alpha^{(0)},\beta^{(0)} \notin \{-1,1\}$. Hence, $\left(\alpha^{(0)}\right)^2 - 1 \neq 0$, $\left(\beta^{(0)}\right)^2 - 1 \neq 0$. Thence for $1+\alpha^{(0)}\beta^{(0)} \neq 0$ we obtain:

$$\left(\frac{\alpha^{(0)}+\beta^{(0)}}{1+\alpha^{(0)}\beta^{(0)}}\right)^2 - 1 = \frac{\left(\alpha^{(0)}\right)^2 + \left(\beta^{(0)}\right)^2 - 1 - \left(\alpha^{(0)}\beta^{(0)}\right)^2}{\left(1+\alpha^{(0)}\beta^{(0)}\right)^2} = \\ = -\frac{\left(\left(\alpha^{(0)}\right)^2-1\right)\left(\left(\beta^{(0)}\right)^2-1\right)}{\left(1+\alpha^{(0)}\beta^{(0)}\right)^2} \neq 0.$$

Remark 5.4. Note that for $\alpha,\beta\in\mathcal{A}_{p\tau}$ and $1+\alpha^{(0)}\beta^{(0)} = 0$ it holds the inequality $\alpha^{(0)}+\beta^{(0)} \neq 0$. Indeed, the system of equations $\begin{cases} 1+\lambda\mu = 0 \\ \lambda+\mu = 0 \end{cases}$ has only the following solutions:

$$(\lambda,\mu) = (-1,1) \text{ and } (\lambda,\mu) = (1,-1).$$

So if we assume $\alpha^{(0)}+\beta^{(0)} = 0$ and $1+\alpha^{(0)}\beta^{(0)} = 0$, we obtain $\begin{cases} \alpha^{(0)} &= 1 \\ \beta^{(0)} &= -1 \end{cases}$ or $\begin{cases} \alpha^{(0)} &= -1 \\ \beta^{(0)} &= 1 \end{cases}$. But in the both cases we have $\alpha,\beta\notin\mathcal{A}_{p\tau}$.

Proof of Assertion 5.2. It is easy to see that for each $\alpha\in\mathcal{A}_{p\tau}$ it is valid the equality:

$$\mathbf{P}_\alpha(t,x) = \mathbf{P}_{\overline{\alpha}^0}(t,x) + {}^{(0)}\alpha \quad \left((t,x)\in\mathbb{R}^2\right),$$

where $\mathbf{P}_{\overline{\alpha}^0}$ is a **linear** operator over the space $\mathbb{R}^2$. Therefore for arbitrary indexes $\alpha = (\lambda, \mathbf{t}_0, \mathbf{x}_0) \in \mathcal{A}_{p\tau}$, $\beta = (\mu, \tau_0, \mathbf{y}_0) \in \mathcal{A}_{p\tau}$ and arbitrary vector $(t,x)\in\mathbb{R}^2$ we obtain:

$$\mathbf{P}_\alpha\left(\mathbf{P}_\beta(t,x)\right) = \mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_\beta(t,x)\right) + {}^{(0)}\alpha = \mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x) + {}^{(0)}\beta\right) + {}^{(0)}\alpha = \\ = \mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x)\right) + \mathbf{P}_{\overline{\alpha}^0}\left({}^{(0)}\beta\right) + {}^{(0)}\alpha = \mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x)\right) + \mathbf{P}_\alpha\left({}^{(0)}\beta\right). \tag{5.25}$$

Note that:

$$\mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x)\right) = \mathbf{P}_{\overline{\alpha}^0}\left(\widetilde{t},\widetilde{x}\right), \tag{5.26}$$

where $\left(\widetilde{t},\widetilde{x}\right) = \mathbf{P}_{\overline{\beta}^0}(t,x)$, and taking into account that $\beta = (\mu,\tau_0,\mathbf{y}_0)$, according to formula (5.20), we get:

$$\widetilde{t} = \frac{t-\mu x}{\sqrt{|1-\mu^2|}}; \qquad \widetilde{x} = \frac{x-\mu t}{\sqrt{|1-\mu^2|}}. \tag{5.27}$$

In accordance with formulas (5.26) and (5.20) we deliver:

$$\mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x)\right) = \left(\frac{\widetilde{t}-\lambda\widetilde{x}}{\sqrt{|1-\lambda^2|}}, \frac{\widetilde{x}-\lambda\widetilde{t}}{\sqrt{|1-\lambda^2|}}\right) \tag{5.28}$$

Further, using (5.27), we derive:

$$\left.\begin{aligned}
\frac{\widetilde{t}-\lambda\widetilde{x}}{\sqrt{|1-\lambda^2|}} &= \frac{\frac{t-\mu x}{\sqrt{|1-\mu^2|}}-\lambda\frac{x-\mu t}{\sqrt{|1-\mu^2|}}}{\sqrt{|1-\lambda^2|}} = \frac{t-\mu x-\lambda x+\lambda\mu t}{\sqrt{|1-\lambda^2|}\sqrt{|1-\mu^2|}} = \\
&= \frac{t\,(1+\lambda\mu)-x\,(\lambda+\mu)}{\sqrt{|1-\lambda^2-\mu^2+\lambda^2\mu^2|}} = \frac{t\,(1+\lambda\mu)-x\,(\lambda+\mu)}{\sqrt{\left|(1+\lambda\mu)^2-(\lambda+\mu)^2\right|}}; \\
\frac{\widetilde{x}-\lambda\widetilde{t}}{\sqrt{|1-\lambda^2|}} &= \frac{\frac{x-\mu t}{\sqrt{|1-\mu^2|}}-\lambda\frac{t-\mu x}{\sqrt{|1-\mu^2|}}}{\sqrt{|1-\lambda^2|}} = \frac{x-\mu t-\lambda t+\lambda\mu x}{\sqrt{|1-\lambda^2|}\sqrt{|1-\mu^2|}} = \\
&= \frac{x\,(1+\lambda\mu)-t\,(\lambda+\mu)}{\sqrt{\left|(1+\lambda\mu)^2-(\lambda+\mu)^2\right|}}.
\end{aligned}\right\} \tag{5.29}$$

Thence for the case $1+\lambda\mu\neq 0$ we deduce:

$$\frac{\widetilde{t}-\lambda\widetilde{x}}{\sqrt{|1-\lambda^2|}} = \frac{t-x\frac{\lambda+\mu}{1+\lambda\mu}}{\frac{1}{1+\lambda\mu}\sqrt{\left|(1+\lambda\mu)^2-(\lambda+\mu)^2\right|}} = \operatorname{sign}\,(1+\lambda\mu)\,\frac{t-x\frac{\lambda+\mu}{1+\lambda\mu}}{\sqrt{\left|1-\left(\frac{\lambda+\mu}{1+\lambda\mu}\right)^2\right|}};$$

$$\frac{\widetilde{x}-\lambda\widetilde{t}}{\sqrt{|1-\lambda^2|}} = \frac{x-t\frac{\lambda+\mu}{1+\lambda\mu}}{\frac{1}{1+\lambda\mu}\sqrt{\left|(1+\lambda\mu)^2-(\lambda+\mu)^2\right|}} = \operatorname{sign}\,(1+\lambda\mu)\,\frac{x-t\frac{\lambda+\mu}{1+\lambda\mu}}{\sqrt{\left|1-\left(\frac{\lambda+\mu}{1+\lambda\mu}\right)^2\right|}}.$$

Thus, in this case, taking into account (5.28), we obtain:

$$\begin{aligned}
\mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x)\right) &= \left(\frac{\widetilde{t}-\lambda\widetilde{x}}{\sqrt{|1-\lambda^2|}},\frac{\widetilde{x}-\lambda\widetilde{t}}{\sqrt{|1-\lambda^2|}}\right) = \\
&= \operatorname{sign}\,(1+\lambda\mu)\left(\frac{t-x\frac{\lambda+\mu}{1+\lambda\mu}}{\sqrt{\left|1-\left(\frac{\lambda+\mu}{1+\lambda\mu}\right)^2\right|}},\frac{x-t\frac{\lambda+\mu}{1+\lambda\mu}}{\sqrt{\left|1-\left(\frac{\lambda+\mu}{1+\lambda\mu}\right)^2\right|}}\right) = \\
&= \operatorname{sign}\left(1+\alpha^{(0)}\beta^{(0)}\right)\mathbf{P}_{\Lambda_{\alpha,\beta}}(t,x).
\end{aligned} \tag{5.30}$$

Similarly in the case $1+\lambda\mu=0$, based on (5.29), we deduce:

$$\frac{\widetilde{t}-\lambda\widetilde{x}}{\sqrt{|1-\lambda^2|}} = -\frac{x\,(\lambda+\mu)}{\sqrt{\left|(\lambda+\mu)^2\right|}} = -\frac{x\,(\lambda+\mu)}{|(\lambda+\mu)|} = -x\operatorname{sign}\,(\lambda+\mu)\,;$$

$$\frac{\widetilde{x}-\lambda\widetilde{t}}{\sqrt{|1-\lambda^2|}} = -\frac{t\,(\lambda+\mu)}{\sqrt{\left|(\lambda+\mu)^2\right|}} = -\frac{t\,(\lambda+\mu)}{|(\lambda+\mu)|} = -t\operatorname{sign}\,(\lambda+\mu)\,.$$

According to Remark 5.4, we have $\lambda+\mu\neq 0$ for $1+\lambda\mu=0$ and $\alpha,\beta\in\mathcal{A}_{p\tau}$. Thus, taking into account (5.28) and (5.23), in this case we obtain:

$$\mathbf{P}_{\overline{\alpha}^0}\left(\mathbf{P}_{\overline{\beta}^0}(t,x)\right)=\left(\frac{\widetilde{t}-\lambda\widetilde{x}}{\sqrt{|1-\lambda^2|}},\frac{\widetilde{x}-\lambda\widetilde{t}}{\sqrt{|1-\lambda^2|}}\right)=$$
$$=\operatorname{sign}\,(\lambda+\mu)\left(-\left(x,t\right)\right)=\operatorname{sign}\,\left(\alpha^{(0)}+\beta^{(0)}\right)\mathbf{P}_{(\infty,0,0)}(t,x). \qquad (5.31)$$

Now the formula (5.22) follows from (5.25), (5.30) and (5.31). □

Denote:

$$\mathcal{B}_\alpha:=\mathcal{B}\quad(\alpha\in\mathcal{A}_{p\tau}) \qquad (5.32)$$

(Recall that $\mathcal{B}$ is the base changeable set, satisfying (5.17)). Using (5.17) for any index $\alpha\in\mathcal{A}_{p\tau}$ we get:

$$\mathbb{Bs}\,(\mathcal{B}_\alpha)=\mathbb{Bs}(\mathcal{B})=\mathbb{R}^2.$$

Now we introduce the next notation:

$$\mathbf{U}_{\beta\alpha}(\omega):=\mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right)\qquad\left(\omega=(t,x)\in\mathbb{Bs}\,(\mathcal{B}_\alpha)=\mathbb{R}^2;\ \alpha,\beta\in\mathcal{A}_{p\tau}\right). \qquad (5.33)$$

Since the mappings $\mathbf{P}_\beta$ and $\mathbf{P}_\alpha$ are bijections in $\mathbb{R}^2$ (for any fixed indexes $\alpha,\beta\in\mathcal{A}_{p\tau}$), we see that the mapping $\mathbf{U}_{\beta\alpha}$ also is a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$. Next we introduce the following notations:

$$\mathcal{U}_{\beta\alpha}(A):=\{\mathbf{U}_{\beta\alpha}(\omega)\mid\omega\in A\}=\left\{\mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right)\mid\omega\in A\right\}$$
$$\left(A\subseteq\mathbb{Bs}\,(\mathcal{B}_\alpha)=\mathbb{R}^2;\ \alpha,\beta\in\mathcal{A}_{p\tau}\right)$$
$$\overleftarrow{\mathcal{B}}:=(\mathcal{B}_\alpha\mid\alpha\in\mathcal{A}_{p\tau})\,;$$
$$\overleftarrow{\mathcal{U}}:=(\mathcal{U}_{\beta\alpha}\mid\alpha,\beta\in\mathcal{A}_{p\tau})\,.$$

Note that the mapping $\mathcal{U}_{\beta\alpha}$ acts from $2^{\mathbb{Bs}(\mathcal{B}_\alpha)}=2^{\mathbb{R}^2}$ to $2^{\mathbb{Bs}(\mathcal{B}_\beta)}=2^{\mathbb{R}^2}$ unlike the mapping $\mathbf{U}_{\beta\alpha}$, which acts from $\mathbb{Bs}\,(\mathcal{B}_\alpha)$ to $\mathbb{Bs}\,(\mathcal{B}_\beta)$.

It is not hard to verify that the indexed set of mappings $\overleftarrow{\mathcal{U}}=(\mathcal{U}_{\beta\alpha}\mid\alpha,\beta\in\mathcal{A}_{p\tau})$ satisfies all conditions of Definition 2.12. Therefore, the ordered triple $\left(\mathcal{A}_{p\tau},\overleftarrow{\mathcal{B}},\overleftarrow{\mathfrak{U}}\right)$ is a changeable set. Futher we will denote this changeable set by $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$:

$$\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}):=\left(\mathcal{A}_{p\tau},\overleftarrow{\mathcal{B}},\overleftarrow{\mathfrak{U}}\right).$$

According to system of notations, introduced in Remark 2.13, and formulas (5.17), (5.18) for this changeable set we have the following properties.

Properties 5.5.

(1) *$\mathcal{I}nd\,(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}))=\mathcal{A}_{p\tau}$.*
(2) *$\mathbf{lk}_\alpha\,(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}))=(\alpha,\mathcal{B}_\alpha)=(\alpha,\mathcal{B})$ (for any index $\alpha\in\mathcal{A}_{p\tau}=\mathcal{I}nd\,(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}))$).*
(3) *$\mathcal{L}k\,(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}))=\{\mathbf{lk}_\alpha\,(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}))\mid\alpha\in\mathcal{A}_{p\tau}\}=\{(\alpha,\mathcal{B})\mid\alpha\in\mathcal{A}_{p\tau}\}$.*
(4) *For arbitrary reference frame $\mathfrak{l}=(\alpha,\mathcal{B})\in\mathcal{L}k\,(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}))$ $(\alpha\in\mathcal{A}_{p\tau})$ we have:*

$$\mathbb{Bs}\,(\mathfrak{l})=\mathbb{Bs}(\mathcal{B})=\mathbb{R}^2,\quad\mathbb{Tm}(\mathfrak{l})=\mathbb{Tm}(\mathcal{B})=\mathbb{R}_{ord}=(\mathbb{R},\leq)\,.$$

Moreover for any $\omega_1,\omega_2\in\mathbb{Bs}(\mathfrak{l})=\mathbb{R}^2$ we have that $\omega_2\underset{\mathfrak{l}}{\leftarrow}\omega_1$ if and only if $\omega_2\underset{\mathcal{B}}{\leftarrow}\omega_1$ that is if and only if:

$$\omega_1=\omega_2\quad or\quad\mathsf{tm}\,(\omega_1)<\mathsf{tm}\,(\omega_2)\,.$$

(5) *For arbitrary reference frames* $\mathfrak{l} = \mathbf{lk}_\alpha\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\alpha, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$, $\mathfrak{m} = \mathbf{lk}_\beta\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\beta, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ *(*$\alpha, \beta \in \mathcal{A}_{p\tau}$*) we have:*

$$\langle \mathfrak{m} \leftarrow \mathfrak{l},\, \mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R}) \rangle\, A = \mathcal{U}_{\beta\alpha}(A) = \left\{ \mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right) \mid \omega \in A \right\} \qquad \left(A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{R}^2\right).$$

Using Property 5.5(5) [8], for arbitrary reference frames $\mathfrak{l} = \mathbf{lk}_\alpha\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\alpha, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$, $\mathfrak{m} = \mathbf{lk}_\beta\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\beta, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ $(\alpha, \beta \in \mathcal{A}_{p\tau})$ and arbitrary $\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{R}^2$, we obtain:

$$\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \{\omega\} = \left\{ \mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right) \right\}. \tag{5.34}$$

From the other hand if $\langle \mathfrak{m} \leftarrow \mathfrak{l} \rangle \{\omega\} = \{\omega'\}$ for some $\omega' \in \mathbb{B}\mathfrak{s}(\mathfrak{m})$ then, according to the equality (5.34), we deliver $\omega' = \mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right)$. Therefore, taking to account Definition 2.16 and notations, introduced after this definition, we deduce the following statement:

Assertion 5.6. *Changeable set* $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$ *is precisely visible. Moreover for arbitrary reference frames* $\mathfrak{l} = \mathbf{lk}_\alpha\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\alpha, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$, $\mathfrak{m} = \mathbf{lk}_\beta\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\beta, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ *(*$\alpha, \beta \in \mathcal{A}_{p\tau}$*) it holds the equality:*

$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega = \mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right) \qquad \left(\omega \in \mathbb{B}\mathfrak{s}(\mathfrak{l}) = \mathbb{R}^2\right).$$

In fact there is valid the statement, more powerful then Assertion 5.6.

Theorem 5.7. *Changeable set* $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$ *is evolutionarily visible.*

Proof. Let $\mathfrak{m} \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ be any reference frame of $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$. Then according to Property 5.5(4) for arbitrary $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ such that $\mathsf{tm}(\omega_1) \neq \mathsf{tm}(\omega_2)$ we have $\omega_2 \underset{\mathfrak{l}}{\leftarrow} \omega_1$ or $\omega_1 \underset{\mathfrak{l}}{\leftarrow} \omega_2$, and, by Definition 2.15, for these ω_1, ω_2 we obtain $\omega_1 \underset{\mathfrak{l}}{\leftrightarrow} \omega_2$. Hence:

$$\forall \mathfrak{m} \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) \forall \omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{m}) \quad \Big((\mathsf{tm}(\omega_1) \neq \mathsf{tm}(\omega_2)) \Rightarrow \Big(\omega_1 \underset{\mathfrak{m}}{\leftrightarrow} \omega_2 \Big) \Big). \tag{5.35}$$

Therefore, if we take any frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ and any elements $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$ such, that $\omega_1 \underset{\mathfrak{l}}{\leftrightarrow} \omega_2$ and $\mathsf{tm}(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega_1) \neq \mathsf{tm}(\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega_2)$, then, according to (5.35), we obtain $\langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega_1 \underset{\mathfrak{m}}{\leftrightarrow} \langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega_2$. So, by Definition 2.25, changeable set $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$ is evolutionarily visible. □

Using Assertion 5.6 as well as formulas (5.21) and (5.22) for arbitrary reference frames $\mathfrak{l} = \mathbf{lk}_\alpha\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\alpha, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$, $\mathfrak{m} = \mathbf{lk}_\beta\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right) = (\beta, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ $(\alpha, \beta \in \mathcal{A}_{p\tau})$ we obtain the following equality:

$$\begin{aligned} \langle !\, \mathfrak{m} \leftarrow \mathfrak{l} \rangle\, \omega &= \mathbf{P}_\beta\left(\mathbf{P}_\alpha^{[-1]}(\omega)\right) = \mathbf{P}_\beta\left(\mathbf{P}_{\alpha_*}\left(\mathcal{I}_\alpha(\omega)\right)\right) = \\ &= \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right) \mathbf{P}_{\Lambda_{\alpha_*,\beta}}\left(\mathcal{I}_\alpha(\omega)\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right) \mathbf{P}_{(\infty,0,0)}\left(\mathcal{I}_\alpha(\omega)\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0, \end{cases} \end{aligned} \tag{5.36}$$

where the index $\Lambda_{\alpha_*,\beta} \in \mathcal{A}_{p\tau}$ and the mapping $\mathbf{P}_{(\infty,0,0)}$ are determined by the formulas (5.24) and (5.23) correspondingly.

The main result of this article is the following theorem:

Theorem 5.8. *Changeable set* $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$ *is not a self-multiimage.*

[8] Reference to Property 5.5(5) means reference to item 5 from the group of properties "Properties 5.5".

The proof of Theorem 5.8 will be given in the final part of this section. To prove Theorem 5.8, we first will prove the theorem which gives the criterion (necessary and sufficient condition) for changeable set to be represented as a self-multiimage. To deduce the last criterion below we define some auxiliary notions and prove some auxiliary technical results.

Definition 5.9. Let $\mathcal{Z}$ be any precisely visible changeable set.

- We say that the reference frame $\mathfrak{l}_1 \in \mathcal{L}k(\mathcal{Z})$ is **time-separated** relatively the frame $\mathfrak{l}_0 \in \mathcal{L}k(\mathcal{Z})$ iff for arbitrary $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}_1)$ such that $\omega_2 \underset{\mathfrak{l}_1}{\leftarrow} \omega_1$ and $\omega_1 \neq \omega_2$ it is valid the correlation, $\mathsf{tm}(\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1) \neq \mathsf{tm}(\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2)$.
- We say that the changeable set $\mathcal{Z}$ is **partially time-separated** iff there exists the reference frame $\mathfrak{l}_0 \in \mathcal{L}k(\mathcal{Z})$ such that each reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is time-separated relatively $\mathfrak{l}_0$.

Definition 5.10. Let $\mathcal{Z}$ be any precisely visible changeable set. We say that the reference frame $\mathfrak{l}_1 \in \mathcal{L}k(\mathcal{Z})$ is **strictly evolutionarily visible** from the reference frame $\mathfrak{l}_0 \in \mathcal{L}k(\mathcal{Z})$ iff for arbitrary $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}_1)$ such, that $\omega_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \omega_2$ we have $\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1 \underset{\mathfrak{l}_0}{\leftrightarrow} \langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2$.

Assertion 5.11. *For arbitrary reference frames $\mathfrak{l}_0, \mathfrak{l}_1 \in \mathcal{L}k(\mathcal{Z})$ of evolutionarily visible changeable set $\mathcal{Z}$ the following statements are equivalent:*

(i): $\mathfrak{l}_1$ *is time-separated relatively* $\mathfrak{l}_0$*;*
(ii): $\mathfrak{l}_1$ *is strictly evolutionarily visible from* $\mathfrak{l}_0$*;*
(iii): *it is valid the equality:* $(\mathfrak{l}_1)^{\wedge} = \langle ! \, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\wedge}, \mathbb{T}\mathbf{m}(\mathfrak{l}_1) \right]$.

Proof. Let $\mathcal{Z}$ be the evolutionarily visible changeable set and $\mathfrak{l}_0, \mathfrak{l}_1 \in \mathcal{L}k(\mathcal{Z})$ be any reference frames of $\mathcal{Z}$.

1) First we are going to prove the implication <u>**(i)⇒(ii)**</u>. Assume that the reference frame $\mathfrak{l}_1$ is time-separated relatively $\mathfrak{l}_0$. Take arbitrary elementary-time states $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}_1)$ such, that $\omega_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \omega_2$.

In the case $\omega_1 = \omega_2$ we have, $\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1 = \langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2$, and so, $\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1 \underset{\mathfrak{l}_0}{\leftarrow} \langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2$ (because the relation $\underset{\mathfrak{l}_0}{\leftarrow}$ is reflexive according to Definition 2.1). Thence, by Definition 2.15, we get, $\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1 \underset{\mathfrak{l}_0}{\leftrightarrow} \langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2$.

Now we consider the case $\omega_1 \neq \omega_2$. Since $\omega_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \omega_2$, then, by Definition 2.15, we have $\omega_2 \underset{\mathfrak{l}_1}{\leftarrow} \omega_1$ or $\omega_1 \underset{\mathfrak{l}_1}{\leftarrow} \omega_2$. Hence, taking into account, that $\omega_1 \neq \omega_2$, according to Definition 5.9, we deduce, $\mathsf{tm}(\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1) \neq \mathsf{tm}(\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2)$. Thus, we see that $\omega_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \omega_2$ and $\mathsf{tm}(\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1) \neq \mathsf{tm}(\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2)$. Thence (since changeable set $\mathcal{Z}$ is evolutionarily visible), by Definition 2.25, we obtain $\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1 \underset{\mathfrak{l}_0}{\leftrightarrow} \langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2$.

Therefore, we see that in the both cases it holds the following statement:

– If $\omega_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \omega_2$ then $\langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_1 \underset{\mathfrak{l}_0}{\leftrightarrow} \langle ! \, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle \, \omega_2$.

According to Definition 5.10, the last statement means that the reference frame $\mathfrak{l}_1$ is strictly evolutionarily visible from the frame $\mathfrak{l}_0$.

2) Now we will prove the implication <u>**(ii)⇒(iii)**</u>. Suppose that the reference frame $\mathfrak{l}_1$ is strictly evolutionarily visible from the frame $\mathfrak{l}_0$. Let us prove that $(\mathfrak{l}_1)^{\wedge} = \langle ! \, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\wedge}, \mathbb{T}\mathbf{m}(\mathfrak{l}_1) \right]$.

2.1. According to system of notations of the theory of changeable sets (see Remark 2.13) we have the equality:

$$\mathbb{T}\mathbf{m}\left((\mathfrak{l}_1)^{\wedge} \right) = \mathbb{T}\mathbf{m}(\mathfrak{l}_1) \tag{5.37}$$

2.2. In accordance with Corollary 2.18, the precise unification mapping $\langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle$ is a bijection between $\mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0)$ and $\mathbb{B}\mathfrak{s}\,(\mathfrak{l}_1)$. Therefore, taking into account system of notations of the theory of changeable sets (see Remark 2.13), we obtain:

$$\begin{aligned} \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_1)^\wedge) &= \mathbb{B}\mathfrak{s}\,(\mathfrak{l}_1) = \{\langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega \mid \omega \in \mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0)\} = \\ &= \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, (\mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0)) = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, (\mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge)) . \end{aligned} \tag{5.38}$$

2.3. Consider any $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_1)^\wedge)$ such, that

$$\mathsf{tm}\,(\widetilde{\omega}_1) \neq \mathsf{tm}\,(\widetilde{\omega}_2) . \tag{5.39}$$

<ins>2.3.1</ins>. Suppose, that $\widetilde{\omega}_1 \underset{(\mathfrak{l}_1)^\wedge}{\leftrightarrow} \widetilde{\omega}_2$. Then, according to system of notations, introduced in Remark 2.13, we have, $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}\,(\mathfrak{l}_1)$ and $\widetilde{\omega}_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \widetilde{\omega}_2$. Let us put:

$$\omega_1 := \langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_1; \qquad \omega_2 := \langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_2.$$

Then, by Corollary 2.18 and system of notations, introduced in Remark 2.13, we obtain:

$$\omega_1,\, \omega_2 \in \mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0) = \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge) . \tag{5.40}$$

And, since reference frame $\mathfrak{l}_1$ is strictly evolutionarily visible from the frame $\mathfrak{l}_0$, by Definition 5.10, from the correlation $\widetilde{\omega}_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \widetilde{\omega}_2$ we deduce the correlation $\omega_1 \underset{\mathfrak{l}_0}{\leftrightarrow} \omega_2$, that is the correlation:

$$\omega_1 \underset{(\mathfrak{l}_0)^\wedge}{\leftrightarrow} \omega_2. \tag{5.41}$$

Moreover, using Assertion 2.17, we deliver:

$$\left.\begin{aligned} &\langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_1 = \widetilde{\omega}_1; \\ &\langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_2 = \widetilde{\omega}_2 \end{aligned}\right] \tag{5.42}$$

Correlations (5.40), (5.41) and (5.42) ensure the following implication:

(⇒2.3.1) If $\widetilde{\omega}_1 \underset{(\mathfrak{l}_1)^\wedge}{\leftrightarrow} \widetilde{\omega}_2$ then the elementary-time states $\omega_1,\, \omega_2 \in \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge)$ exist such that $\widetilde{\omega}_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_1$, $\widetilde{\omega}_2 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_2$ and $\omega_1 \underset{(\mathfrak{l}_0)^\wedge}{\leftrightarrow} \omega_2$.

<ins>2.3.2</ins>. Now we suppose, that there exist $\omega_1,\, \omega_2 \in \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge)$ such that

$$\widetilde{\omega}_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_1, \quad \widetilde{\omega}_2 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_2 \tag{5.43}$$

and $\omega_1 \underset{(\mathfrak{l}_0)^\wedge}{\leftrightarrow} \omega_2$. Then we have, $\omega_1,\, \omega_2 \in \mathbb{B}\mathfrak{s}\,(\mathfrak{l}_0)$ and $\omega_1 \underset{\mathfrak{l}_0}{\leftrightarrow} \omega_2$. Since the changeable set $\mathcal{Z}$ is evolutionarily visible, taking into account (5.43) and (5.39), by Definition 2.25, we obtain, $\widetilde{\omega}_1 \underset{\mathfrak{l}_1}{\leftrightarrow} \widetilde{\omega}_2$, and therefore $\widetilde{\omega}_1 \underset{(\mathfrak{l}_1)^\wedge}{\leftrightarrow} \widetilde{\omega}_2$. Hence, the following implication is valid:

(⇒2.3.2) If there exist $\omega_1,\, \omega_2 \in \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge)$ such, that $\widetilde{\omega}_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_1$, $\widetilde{\omega}_2 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_2$ and $\omega_1 \underset{(\mathfrak{l}_0)^\wedge}{\leftrightarrow} \omega_2$, then $\widetilde{\omega}_1 \underset{(\mathfrak{l}_1)^\wedge}{\leftrightarrow} \widetilde{\omega}_2$.

Implications **(⇒2.3.1)** and **(⇒2.3.2)** lead to the following conclusion:

If $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_1)^\wedge)$ *and* $\mathsf{tm}\,(\widetilde{\omega}_1) \neq \mathsf{tm}\,(\widetilde{\omega}_2)$*, then* $\widetilde{\omega}_1 \underset{(\mathfrak{l}_1)^\wedge}{\leftrightarrow} \widetilde{\omega}_2$ *if and only if there exist* $\omega_1,\, \omega_2 \in \mathbb{B}\mathfrak{s}\,((\mathfrak{l}_0)^\wedge)$ *such, that* $\widetilde{\omega}_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_1$, $\widetilde{\omega}_2 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_2$ *and* $\omega_1 \underset{(\mathfrak{l}_0)^\wedge}{\leftrightarrow} \omega_2$.

The last conclusion together with the equalities (5.37) and (5.38) by Theorem 2.10 lead to the equality $(\mathfrak{l}_1)^\wedge = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, [(\mathfrak{l}_0)^\wedge, \mathbb{T}\mathbf{m}\,(\mathfrak{l}_1)]$, which had to be proven.

3) Finally we will prove the implication **(iii)⇒(i)**. Assume that the equality $(\mathfrak{l}_1)^{\hat{}} = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathfrak{l}_1)\right]$ holds. Consider arbitrary elementary-time states $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}_1)$ such, that $\widetilde{\omega}_2 \underset{\mathfrak{l}_1}{\leftarrow} \widetilde{\omega}_1$ and $\widetilde{\omega}_1 \neq \widetilde{\omega}_2$, that is such, that

$$\widetilde{\omega}_2 \underset{(\mathfrak{l}_1)^{\hat{}}}{\leftarrow} \widetilde{\omega}_1 \quad \text{and} \quad \widetilde{\omega}_1 \neq \widetilde{\omega}_2. \tag{5.44}$$

According to Remark 2.5 (item 1) from (5.44) it follows the inequality $\mathsf{tm}(\widetilde{\omega}_1) <_{(\mathfrak{l}_1)^{\hat{}}} \mathsf{tm}(\widetilde{\omega}_2)$, that is $\mathsf{tm}(\widetilde{\omega}_1) \neq \mathsf{tm}(\widetilde{\omega}_2)$. Also, by Definition 2.4, from (5.44) we deduce $\widetilde{\omega}_2 \underset{(\mathfrak{l}_1)^{\hat{}}}{\leftrightarrow} \widetilde{\omega}_1$. Therefore, we have $\widetilde{\omega}_2 \underset{(\mathfrak{l}_1)^{\hat{}}}{\leftrightarrow} \widetilde{\omega}_1$ and $\mathsf{tm}(\widetilde{\omega}_1) \neq \mathsf{tm}(\widetilde{\omega}_2)$. Thence, based on the equality $(\mathfrak{l}_1)^{\hat{}} = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathfrak{l}_1)\right]$ and Theorem 2.10, we deduce that there must exist $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}\left((\mathfrak{l}_0)^{\hat{}}\right)$ such, that:

$$\omega_1 \underset{(\mathfrak{l}_0)^{\hat{}}}{\leftrightarrow} \omega_2, \tag{5.45}$$

$$\widetilde{\omega}_1 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_1, \quad \widetilde{\omega}_2 = \langle !\, \mathfrak{l}_1 \leftarrow \mathfrak{l}_0 \rangle\, \omega_2. \tag{5.46}$$

Since $\widetilde{\omega}_1 \neq \widetilde{\omega}_2$ (according to (5.44)) then from the equalities (5.46) we deliver $\omega_1 \neq \omega_2$. Since $\omega_1 \neq \omega_2$ and $\omega_1 \underset{(\mathfrak{l}_0)^{\hat{}}}{\leftrightarrow} \omega_2$ (according to (5.45)), then, in accordance with Remark 2.5 (item 3), we obtain:

$$\mathsf{tm}(\omega_1) \neq \mathsf{tm}(\omega_2). \tag{5.47}$$

Using (5.46) and Corollary 2.18 we obtain:

$$\omega_1 = \langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_1; \quad \omega_2 = \langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_2.$$

Hence, taking into account (5.47), we have, $\mathsf{tm}(\langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_1) \neq \mathsf{tm}(\langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_2)$.

Thus, for arbitrary $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l}_1)$ such that $\widetilde{\omega}_2 \underset{\mathfrak{l}_1}{\leftarrow} \widetilde{\omega}_1$ and $\widetilde{\omega}_1 \neq \widetilde{\omega}_2$ it holds the correlation $\mathsf{tm}(\langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_1) \neq \mathsf{tm}(\langle !\, \mathfrak{l}_0 \leftarrow \mathfrak{l}_1 \rangle\, \widetilde{\omega}_2)$. Therefore, by Definition 5.9, the reference frame $\mathfrak{l}_1$ is time-separated relatively $\mathfrak{l}_0$. The implication has been proven. □

Theorem 5.12 (criterion for representation of a changeable set as a self-multiimage)**.** *Let $\mathcal{Z}$ be an evolutionarily visible changeable set. Then the following statements hold:*

1. *$\mathcal{Z}$ is a multi-image of a reference frame $\mathfrak{l}_0 \in \mathcal{L}k(\mathcal{Z})$ if and only if every reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is time-separated relatively $\mathfrak{l}_0$.*
2. *$\mathcal{Z}$ is a self-multiimage if and only if it is partially time-separated.*

Proof. Note that the item 2 of Theorem 5.12 is the trivial consequence of the item 1. So, it is sufficient to prove the item 1.

1. Let $\mathcal{Z}$ be a multi-image of a reference frame $\mathfrak{l}_0 \in \mathcal{L}k(\mathcal{Z})$, that is, by Definition 3.2, it holds the equality:

$$\mathcal{Z} = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z} \rangle}, (\mathfrak{l}_0)^{\hat{}}\right],$$

where (according to (3.8))

$$\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z} \rangle} = \left(\left(\mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z})), \mathfrak{B}\mathfrak{s}(\mathbf{lk}_\alpha(\mathcal{Z})), \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0 \rangle \right) \mid \alpha \in \mathcal{I}nd(\mathcal{Z}) \right).$$

Thence, in accordance with Theorem 2.20, it follows that any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ can be represented in the form:

$$\mathfrak{l} = \left(\alpha, \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z}))\right]\right),$$

where $\alpha \in \mathcal{I}nd(\mathcal{Z})$. From the other hand, according to notations, introduced in Remark 2.13, we have:

$$\mathfrak{l} = \left(\mathtt{ind}(\mathfrak{l}), \mathfrak{l}^{\hat{}}\right).$$

From the last two equalities we have $\mathtt{ind}(\mathfrak{l}) = \alpha$ $(\mathfrak{l} = \mathbf{lk}_\alpha(\mathcal{Z}))$ and:

$$\mathfrak{l}^{\hat{}} = \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z}))\right] = \langle !\, \mathfrak{l} \leftarrow \mathfrak{l}_0 \rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathfrak{l})\right].$$

From the last equality it follows that any reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is time-separated relatively $\mathfrak{l}_0$ (according to Assertion 5.11).

2. Conversely, suppose that every reference frame $\mathfrak{l} \in \mathcal{L}k(\mathcal{Z})$ is time-separated relatively $\mathfrak{l}_0$. Denote:

$$\mathcal{Z}_1 := \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}, (\mathfrak{l}_0)^{\hat{}}\right],$$

where the evolution multi-projector $\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}$ is determined by formula (3.8). Then, according to (3.8) and Corollary 2.22, we get:

$$\mathcal{I}nd(\mathcal{Z}_1) = \mathcal{I}nd(\mathcal{Z}). \tag{5.48}$$

Next, using Corollary 2.22, for any $\alpha \in \mathcal{I}nd(\mathcal{Z}_1) = \mathcal{I}nd(\mathcal{Z})$ we have:

$$\mathbf{lk}_\alpha(\mathcal{Z}_1) = \left(\alpha, \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z}))\right]\right). \tag{5.49}$$

From the other hand, since the reference frame $\mathbf{lk}_\alpha(\mathcal{Z})$ is time-separated relatively $\mathfrak{l}_0$ (in the changeable set $\mathcal{Z}$), then, according to Assertion 5.11 we obtain:

$$(\mathbf{lk}_\alpha(\mathcal{Z}))^{\hat{}} = \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z}))\right].$$

Thence, taking into account (2.6), we deduce:

$$\begin{aligned}\mathbf{lk}_\alpha(\mathcal{Z}) &= \left(\mathtt{ind}(\mathbf{lk}_\alpha(\mathcal{Z})), (\mathbf{lk}_\alpha(\mathcal{Z}))^{\hat{}}\right) = \left(\alpha, (\mathbf{lk}_\alpha(\mathcal{Z}))^{\hat{}}\right) = \\ &= \left(\alpha, \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z}))\right]\right).\end{aligned} \tag{5.50}$$

From formulas (5.49) and (5.50) we see that for any index $\alpha \in \mathcal{I}nd(\mathcal{Z}_1) = \mathcal{I}nd(\mathcal{Z})$ it is valid the equality, $\mathbf{lk}_\alpha(\mathcal{Z}_1) = \mathbf{lk}_\alpha(\mathcal{Z})$. Therefore, according to formula (2.5), we deliver:

$$\mathcal{L}k(\mathcal{Z}) = \{\mathbf{lk}_\alpha(\mathcal{Z}) \mid \alpha \in \mathcal{I}nd(\mathcal{Z})\} = \{\mathbf{lk}_\alpha(\mathcal{Z}_1) \mid \alpha \in \mathcal{I}nd(\mathcal{Z})\} = \mathcal{L}k(\mathcal{Z}_1). \tag{5.51}$$

Consider arbitrary reference frames $\mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z}_1) = \mathcal{L}k(\mathcal{Z})$. Taking into account (2.5) and (5.48), we see that there must exist indexes $\alpha, \beta \in \mathcal{I}nd(\mathcal{Z}_1) = \mathcal{I}nd(\mathcal{Z})$ such that

$$\mathfrak{l} = \mathbf{lk}_\alpha(\mathcal{Z}_1), \qquad \mathfrak{m} = \mathbf{lk}_\beta(\mathcal{Z}_1). \tag{5.52}$$

Hence, using (5.49), we have:

$$\begin{aligned}\mathfrak{l} &= \left(\alpha, \langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\alpha(\mathcal{Z}))\right]\right), \\ \mathfrak{m} &= \left(\beta, \langle !\, \mathbf{lk}_\beta(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \left[(\mathfrak{l}_0)^{\hat{}}, \mathbb{T}\mathbf{m}(\mathbf{lk}_\beta(\mathcal{Z}))\right]\right).\end{aligned}$$

Thence, applying Theorem 2.20, Corollary 2.18, Assertion 2.17 and formula (5.52), for any set $A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})$ we obtain:

$$\begin{aligned}\langle \mathfrak{m} \leftarrow \mathfrak{l},\ \mathcal{Z}_1\rangle A &= \left\{\langle !\, \mathbf{lk}_\beta(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \left(\langle !\, \mathbf{lk}_\alpha(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle^{[-1]}(\omega)\right) \mid \omega \in A\right\} = \\ &= \{\langle !\, \mathbf{lk}_\beta(\mathcal{Z}) \leftarrow \mathfrak{l}_0,\ \mathcal{Z}\rangle \langle !\, \mathfrak{l}_0 \leftarrow \mathbf{lk}_\alpha(\mathcal{Z}),\ \mathcal{Z}\rangle \omega \mid \omega \in A\} = \\ &= \{\langle !\, \mathbf{lk}_\beta(\mathcal{Z}) \leftarrow \mathbf{lk}_\alpha(\mathcal{Z}),\ \mathcal{Z}\rangle \omega \mid \omega \in A\} = \\ &= \langle \mathbf{lk}_\beta(\mathcal{Z}) \leftarrow \mathbf{lk}_\alpha(\mathcal{Z}),\ \mathcal{Z}\rangle A = \langle \mathfrak{m} \leftarrow \mathfrak{l},\ \mathcal{Z}\rangle A \\ &\qquad (\forall \mathfrak{l}, \mathfrak{m} \in \mathcal{L}k(\mathcal{Z}_1) = \mathcal{L}k(\mathcal{Z})\ \forall A \subseteq \mathbb{B}\mathfrak{s}(\mathfrak{l})).\end{aligned} \tag{5.53}$$

From (5.51) and (5.53)), by virtue of Assertion 2.14 we deduce the equality:

$$\mathcal{Z} = \mathcal{Z}_1 = \mathcal{Z}\mathsf{im}\left[\mathfrak{P}^{(e)}_{\langle \mathfrak{l}_0, \mathcal{Z}\rangle}, (\mathfrak{l}_0)^{\hat{}}\right].$$

Thus, $\mathcal{Z}$ is the multi-image of the reference frame $\mathfrak{l}_0 \in \mathcal{L}k(\mathcal{Z})$.

The theorem has been fully proven. □

Proof of Theorem 5.8. Let $\mathfrak{m} = (\beta, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$, where $\beta \in \mathcal{A}_{p\tau}$ be any reference frame of the changeable set $\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})$ (in accordance with Property 5.5(3)). Consider an arbitrary reference frame $\mathfrak{l} = (\alpha, \mathcal{B}) \in \mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}\tau}(\mathbb{R})\right)$ ($\alpha \in \mathcal{A}_{p\tau}$) such that $\alpha_*^{(0)} + \beta^{(0)} \neq 0$. Consider the following two points in the Cartesian plane $\mathbb{R}^2$:

$$\omega_1 := (0,0); \quad \omega_2 := \begin{cases} \left(\operatorname{sign}\left(1 - \left|\alpha^{(0)}\right|\right), \frac{\operatorname{sign}\left(1-\left|\alpha^{(0)}\right|\right)}{\Lambda^{(0)}_{\alpha_*,\beta}}\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \left(\operatorname{sign}\left(1 - \left|\alpha^{(0)}\right|\right), 0\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0. \end{cases}$$

Since $\alpha_*^{(0)} + \beta^{(0)} \neq 0$, then formula (5.24) leads to the correlation $\Lambda^{(0)}_{\alpha_*,\beta} \neq 0$ for $1 + \alpha_*^{(0)}\beta^{(0)} \neq 0$. That is why the value ω_2 is correctly defined. According to Property 5.5(4), we have, $\omega_1, \omega_2 \in \mathbb{B}\mathfrak{s}(\mathfrak{l})$. According to (5.19), we deduce, $\mathsf{tm}(\omega_1) = 0 \neq \mathsf{tm}(\omega_2)$. Therefore, by formula (5.35) we see that $\omega_1 \underset{\mathfrak{l}}{\leftrightarrow} \omega_2$. Emphasize that:

$$\mathcal{I}_\alpha(\omega_1) = \operatorname{sign}\left(1 - \left|\alpha^{(0)}\right|\right)\omega_1 = \operatorname{sign}\left(1 - \left|\alpha^{(0)}\right|\right)(0,0) = (0,0);$$

$$\mathcal{I}_\alpha(\omega_2) = \operatorname{sign}\left(1 - \left|\alpha^{(0)}\right|\right)\omega_2 = \begin{cases} \left(1, \frac{1}{\Lambda^{(0)}_{\alpha_*,\beta}}\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ (1,0), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0. \end{cases}$$

That is why, using (5.36), (5.20) and (5.23) we deduce:

$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l}\rangle\, \omega_1 = \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right)\mathbf{P}_{\Lambda_{\alpha_*,\beta}}\left(\mathcal{I}_\alpha(\omega_1)\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right)\mathbf{P}_{(\infty,0,0)}\left(\mathcal{I}_\alpha(\omega_1)\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0, \end{cases} =$$

$$= \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right)\mathbf{P}_{\Lambda_{\alpha_*,\beta}}(0,0) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right)\mathbf{P}_{(\infty,0,0)}(0,0) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0, \end{cases} =$$

$$= \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right);$$

$$\langle !\, \mathfrak{m} \leftarrow \mathfrak{l}\rangle\, \omega_2 = \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right)\mathbf{P}_{\Lambda_{\alpha_*,\beta}}\left(\mathcal{I}_\alpha(\omega_2)\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right)\mathbf{P}_{(\infty,0,0)}\left(\mathcal{I}_\alpha(\omega_2)\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0, \end{cases} =$$

$$= \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right)\mathbf{P}_{\Lambda_{\alpha_*,\beta}}\left(1, \frac{1}{\Lambda^{(0)}_{\alpha_*,\beta}}\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right)\mathbf{P}_{(\infty,0,0)}(1,0) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0, \end{cases} =$$

$$= \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right)\left(0, \frac{\frac{1}{\Lambda^{(0)}_{\alpha_*,\beta}} - \Lambda^{(0)}_{\alpha_*,\beta}}{\sqrt{\left|1 - \left(\Lambda^{(0)}_{\alpha_*,\beta}\right)^2\right|}}\right) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & \\ & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right)(0,-1) + \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0, \end{cases} =$$

$$= \mathbf{P}_\beta\left({}^{(0)}\alpha_*\right) +$$

$$+ \begin{cases} \operatorname{sign}\left(1 + \alpha_*^{(0)}\beta^{(0)}\right)\left(0, \frac{\operatorname{sign}\left(1 - \left|\Lambda^{(0)}_{\alpha_*,\beta}\right|\right)}{\Lambda^{(0)}_{\alpha_*,\beta}}\sqrt{\left|1 - \left(\Lambda^{(0)}_{\alpha_*,\beta}\right)^2\right|}\right), & \\ & 1 + \alpha_*^{(0)}\beta^{(0)} \neq 0 \\ \operatorname{sign}\left(\alpha_*^{(0)} + \beta^{(0)}\right)(0,-1), & 1 + \alpha_*^{(0)}\beta^{(0)} = 0. \end{cases}$$

Thence we deliver:

$$\mathsf{tm}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\right)=\mathsf{tm}\left(\mathbf{P}_\beta\left({}^{(0)}\alpha_*\right)\right);$$
$$\mathsf{tm}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2\right)=0+\mathsf{tm}\left(\mathbf{P}_\beta\left({}^{(0)}\alpha_*\right)\right)=\mathsf{tm}\left(\mathbf{P}_\beta\left({}^{(0)}\alpha_*\right)\right),$$

that is, $\mathsf{tm}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\right)=\mathsf{tm}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2\right)$. Moreover, according to Remark 5.3, $\Lambda_{\alpha_*,\beta}\in\mathcal{A}_{p\tau}$, and hence $\Lambda^{(0)}_{\alpha_*,\beta}\notin\{-1,1\}$ (for $1+\alpha_*^{(0)}\beta^{(0)}\neq 0$). That is why, we obtain:

$$\mathsf{bs}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2\right)=\mathsf{bs}\left(\mathbf{P}_\beta\left({}^{(0)}\alpha_*\right)\right)+$$
$$+\begin{cases}\frac{\operatorname{sign}\left(\left(1+\alpha_*^{(0)}\beta^{(0)}\right)\left(1-\left|\Lambda^{(0)}_{\alpha_*,\beta}\right|\right)\right)}{\Lambda^{(0)}_{\alpha_*,\beta}}\sqrt{\left|1-\left(\Lambda^{(0)}_{\alpha_*,\beta}\right)^2\right|}, & 1+\alpha_*^{(0)}\beta^{(0)}\neq 0\\ -\operatorname{sign}\left(\alpha_*^{(0)}+\beta^{(0)}\right), & 1+\alpha_*^{(0)}\beta^{(0)}=0.\end{cases}\neq$$
$$\neq\mathsf{bs}\left(\mathbf{P}_\beta\left({}^{(0)}\alpha_*\right)\right)=\mathsf{bs}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\right).$$

Therefore, $\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\neq\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2$. Thus, we have, $\mathsf{tm}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\right)=\mathsf{tm}\left(\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2\right)$, but $\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\neq\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2$. Hence, according to Remark 2.5 (item 3), elementary-time states $\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1$ and $\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2$ can not be united by fate in the reference frame $\mathfrak{m}\in\mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}_\tau}(\mathbb{R})\right)$. Thus, we have proven that the elementary-time states $\omega_1,\omega_2\in\mathbb{Bs}(\mathfrak{l})$ exist such that $\omega_1\underset{\mathfrak{l}}{\leftrightarrow}\omega_2$, but $\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_1\underset{\mathfrak{m}}{\not\leftrightarrow}\langle !\,\mathfrak{m}\leftarrow\mathfrak{l}\rangle\,\omega_2$. That is why, by Definition 5.10, the reference frame $\mathfrak{l}\in\mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}_\tau}(\mathbb{R})\right)$ is not strictly evolutionarily visible from the reference frame $\mathfrak{m}$. According to Assertion 5.11, this means, that the reference frame $\mathfrak{l}$ is not time-separated relatively relatively the frame $\mathfrak{m}$. Thus, we have proven that for every reference frame $\mathfrak{m}\in\mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}_\tau}(\mathbb{R})\right)$ there exists the reference frame $\mathfrak{l}\in\mathcal{L}k\left(\mathcal{Z}^*_{\mathscr{P}_\tau}(\mathbb{R})\right)$, which is not time-separated relatively relatively $\mathfrak{m}$. According to Definition 5.9, this means, that the changeable set $\mathcal{Z}^*_{\mathscr{P}_\tau}(\mathbb{R})$ is not partially time-separated. So, in accordance with Theorem 5.12, $\mathcal{Z}^*_{\mathscr{P}_\tau}(\mathbb{R})$ is **not** a self-multiimage, which was necessary to prove. □

6. Conclusions and announcement for the future

The most important (in the author's opinion) results of the article are the following:

1. The theorem, which describes the quite wide diapason of cases, where changeable set can be represented as a self-multiimage is proven. Many examples of changeable sets, which can be represented as a self-multiimage are presented.
2. Simple for verification criterion, for evolutionarily visible changeable set to be representable a self-multiimage, is deduced. Using the obtained criterion it is proven the existence of a changeable set, which cannot be represented as a self-multiimage.

The obtained results give us the negative solution of Problem 3.3, which is generanted by Problem 3.1 (as it was mentioned in section 3). But, in fact, these results do not give any solution to Problem 3.1, and only show, that this problem can not be solved by a simple way. In future papers author is going to publish the theorem, which gives the positive solution of Problem 3.1.

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