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THE BEREZANSKY METHOD IN THE MOMENTS PROBLEM

MYKOLA DUDKIN

The work is dedicated to the 100th anniversary of the birth of Yu. M. Berezansky 05/08/1925 – 06/07/2019.

Abstract. In the review of the main modern positions of the moment problem, the importance of using the method of Berezansky Yu. M. - the expansion by generalized eigenvectors – is discussed. This approach is currently the only correct one for solving the moment problem in different statements using the operator theory.

1. Introduction

The problem of moments is known from the ancient works of Chebyshev P. L., Markov A. A., and Stilties T. The largest list of works by these authors is collected in the monograph [1].

In the simplest modern formulation the classical moment problem consists in finding the measure $d\rho(x)$ on the real line ($x \in \mathbb{R}$) for a given sequence of real numbers (s_n), $s_n \in \mathbb{R}$, $n \in \mathbb{N}_0 = \{0, 1, ...\}$, such that the representation

$$s_n = \int\limits_{\mathbb{R}} x^n \, d\rho(x) \tag{1.1}$$

hold true.

The solution of the classical problem using the method of function theory is given in the book by Akhiezer N.I. [2], which is called a "diamond" in the article by Simon B. [3].

However, only in the IVth chapter of the book by Akhiezer N.I. we see some observations by methods of operator theory. These observations are often mistakenly used as a solution method, since they have some logical gap.

The solution traditionally starts with positive definiteness, which is always a necessary condition for the existence of a solution (or many solutions). That is, if the representation (1.1) is satisfied, then

$$\sum_{n,m \in \mathbb{N}_0} s_{n+m} f_n \bar{f}_m \geq 0, \tag{1.2}$$

for an arbitrary finite sequence (f_n), $f_n \in \mathbb{C}$, $n \in \mathbb{N}_0$. This fact gives grounds to introduce the scalar product on monomials x^n, $n \in \mathbb{N}_0$ according to the rule

$$(x^n, x^m) = s_{n+m}, \quad n, m \in \mathbb{N}_0, \tag{1.3}$$

which defines a Hilbert space $\mathcal{H}$ (some L_2 on $\mathbb{R}$) with the scalar product $(\cdot, \cdot)$ defined on a total set $\mathfrak{D} := \{x^n\}$ in $\mathcal{H}$. The existence of such a space is guaranteed by the famous

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Koranyi A. theorem (see for example [4]). Then the shift operator J (see (3.12) below) has spectral measure dE_λ, which leads to the representation (1.1)

$$s_n = \int\limits_{\mathbb{R}} x^n \, d(E_\lambda e_0, e_0), \tag{1.4}$$

where e_0 is a cyclic vector. The representation (1.4) is unique in this sequence of arguments if the operator is essentially self-adjoint. This fact is guaranteed, for example, by the Carleman criterion, i.e., if the series $\sum\limits_{n=0}^{\infty} s_{2n}^{-\frac{1}{2n}}$ diverges [5, 6, 7].

Suppose that the operator J is generally bounded. Then, according to the above considerations, a solution exists and is unique. But the problem is that the space $\mathcal{H}$ is not uniquely defined by the set $\mathfrak{D}$ if the scalar product $(\cdot,\cdot)$, and hence the norm, are defined only on $\mathfrak{D}$. Now the operator, even if it is bounded, is different in each space. And these operators must have different spectral functions in general. Explanations, that these operators in different spaces are one operator, that is, have the same spectral function, cannot be simple task. Krein M., Langer G. used not the operator approach but methods of the function theory in similar problems. Examples of two different Hilbert spaces whose norms coincide on a dense (in both spaces) set were first constructed in the works of Koshmanenko V. D., using the study of singularly perturbed operators [8, 9]. The corresponding description is in the paragraph entitled "singularity phenomenon". The prerequisites for the emergence of such examples are in the work [10].

Particularly interesting is the example of Nizhnik L. P., where it is simply shown that spaces also have non-equivalent norms.

The only correct operator approach known today is the method of Berezansky Yu. M., which uses the generalized eigenvector expansions, described in [11, 12, 13]. Some modern mathematicians use different approaches to solving the moment problems. The latest monographs [14, 15] are devoted to the moment problems in different settings and related issues.

Further program of this article: preliminary information on the theory of generalized eigenvectors expansions for solving the classical moment problem; the actual solution of the classical moment problem; examples of non-uniqueness of the construction of a Hilbert space by a given dense set.

2. Preliminaries

Let A be a self-adjoint operator defined on $\mathfrak{D}(A)$ in a separable Hilbert space $\mathcal{H}$. Consider the (rigging) equipment of the space $\mathcal{H}$:

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D}, \tag{2.5}$$

where $\mathcal{H}_+$ is the Hilbert space embedded in $\mathcal{H}$ topologically and quasi-nuclearly (topologically means densely and continuously, quasi-nuclearly means, that the embedding operator is of the Hilbert-Schmidt type); $\mathcal{H}_-$ is the space dual of $\mathcal{H}_+$ with respect to the space $\mathcal{H}$; $\mathcal{D}$ is a linear topological space, topologically embedded into $\mathcal{H}_+$.

A self-adjoint defined in $\mathcal{H}$ operator A is called standardly connected with the rigging (2.5), if $\mathcal{D} \subset \mathfrak{D}(A)$ and the restriction $A \upharpoonright \mathcal{D}$ acts continuously from $\mathcal{D}$ into $\mathcal{H}_+$.

We also recall that the vector $\Omega \in \mathcal{D}$ is called strong cyclic for the operator A if for all $n \in \mathbb{N}$ it holds $\Omega \in \mathfrak{D}(A^n) \subset \mathcal{D}$ and the set of all vectors $A^n\Omega$, for $n = \mathbb{N}_0$ is total in the space $\mathcal{H}_+$ (and hence, also in $\mathcal{H}$).

Assuming that a strong cyclic vector exists, we formulate some abbreviated version of the projective spectral theorem (see [12], Chapter 3, Theorem 2.7, or [11] Chapter 5, or [4], Chapter 15 and Chapter 1, Section 1.11).

Theorem 2.1. *For a self-adjoint operator A with a strong cyclic vector in a separable Hilbert space $\mathcal{H}$, there exists a Borel measure $d\rho(\lambda)$ on the real axis, such that for ρ-almost all $\lambda \in \mathbb{R}$ there exists a generalized eigenvector $\xi_\lambda \in \mathcal{H}_-$, i.e. for every $f \in \mathcal{D}$*

$$(\xi_\lambda, Af)_{\mathcal{H}} = \lambda(\xi_\lambda, f)_{\mathcal{H}}, \quad \xi_\lambda \neq 0. \tag{2.6}$$

The corresponding Fourier transform F acts according to the rule:

$$\mathcal{H} \supset \mathcal{H}_+ \ni f \mapsto (Ff)(\lambda) = \hat{f}(\lambda) = (f, \xi_\lambda)_{\mathcal{H}} \in L_2(\mathbb{R}, d\rho(\lambda)), \tag{2.7}$$

is an isometric operator with unit norm (after its closure) acting from the space $\mathcal{H}$ into $L^2(\mathbb{R}, d\rho(\lambda))$. The image of the operator A under the transformation F is the operator of multiplication by the independent variable λ in $L^2(\mathbb{R}, d\rho(\lambda))$, namely $(FAf)(\lambda) = \lambda\hat{f}(\lambda)$.

Remark that the generalized scalar product in (2.7): $(f, \xi)_{\mathcal{H}} \in L^2(\mathbb{R}, d\rho(\lambda))$ for $f \in \mathcal{H}$, we understand as the limit of fundamental with respect to $L^2(\mathbb{R}, d\rho(\lambda))$ sequences $(f_n, \xi)_{\mathcal{H}}$, $\mathcal{H}_+ \ni f_n \to f$ in $\mathcal{H}$.

Recall also that for a self-adjoint operator A defined on $\mathfrak{D}(A)$ in the space $\mathcal{H}$, the vector $f \in \bigcap_{n=0}^{\infty} \mathfrak{D}(A^n)$ is called quasi-analytic if the class $\mathrm{C}\{m_n\}$ is quasi-analytic, where in this case $m_n = \|A^n f\|_{\mathcal{H}}$, The class of functions on $[a, b] \subset \mathbb{R}$ is defined by the expression

$$\mathrm{C}(\{m_n\}) = \{g \in C^\infty([a, b]) \ \exists K > 0, |g^{(n)}(t)| \leq K^n m_n, \ t \in [a, b], \ n \in \mathbb{N}_0\},$$

hence it is quasi-analytic if

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\|A^n f\|_{\mathcal{H}}}} = \infty.$$

The quasi-analyticity of a vector is used in the self-adjointness criterion.

3. The classical moment problem by the Berezansky method

So once again, the classical moment problem is understood, as mentioned above, as the problem of finding conditions for the sequence (s_n), $n \in \mathbb{N}_0$ ($s_n \in \mathbb{R}$), under which there exists a Borel measure $d\rho(\lambda)$ on the real axis $\mathbb{R}$, such that

$$s_n = \int_{\mathbb{R}} \lambda^n \, d\rho(\lambda), \ n \in \mathbb{N}_0. \tag{3.8}$$

The solution of the classical moment problem is contained in the following theorem.

Theorem 3.1. *A sequence of real numbers (s_n), $n \in \mathbb{N}_0$ has a representation (3.8) if and only if it is positive definite, i.e.*

$$\sum_{m,n=0}^{\infty} s_{n+m} f_m \bar{f}_n \geq 0 \tag{3.9}$$

on arbitrary finite sequences $(f_n)_{n=0}^{\infty}$ of numbers $f_n \in \mathbb{C}$.

The representation (3.8) exists and the measure $d\rho(\lambda)$ is unique if the sequence of numbers (s_n), $n \in \mathbb{N}_0$ is positive definite and additionally

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[2p]{s_{2p}}} = \infty. \tag{3.10}$$

Note that the condition (3.9) is necessary and sufficient for the existence of the representation (3.8), but there can be many representations (measures). The condition (3.9) together with (3.10) guarantee the uniqueness (measure) of the representation (3.8).

Proof. Let us prove the necessity of the condition (3.9). If the sequence $(s_n)_{n=0}^{\infty}$ has a representation (3.8), then for an arbitrary finite sequence $f=(f_n)_{n=0}^{\infty}$, $f_n \in \mathbb{C}$, we have

$$\sum_{m,n=0}^{\infty} s_{n+m} f_m \bar{f}_n = \int_{\mathbb{R}} \left| \sum_{n=0}^{\infty} \lambda^n f_n \right|^2 d\rho(\lambda) \geq 0,$$

which proves the necessity.

Let us proceed to the proof of sufficiency. The linear space $\mathbb{C}^{\infty}$ of all sequences $f=(f_n)_{n=0}^{\infty}$, $f_n \in \mathbb{C}$, will be denoted by l. Let $\delta_n = (0,...,0,1,0,...)$, $n \in \mathbb{N}_0$ be a δ-sequence (1 is in the n-th place), then for each vector $f \in l_{\text{fin}}$, we have $f=\sum_{n=0}^{\infty} f_n\delta_n$. So

$$\forall f \in l_{\text{fin}}, \quad f=(f_n)_{n=0}^{\infty} = \sum_{n=0}^{\infty} f_n \delta_n. \tag{3.11}$$

Consider a linear expression in the space l:

$$\begin{aligned} &\forall f \in l, \quad Jf = J(f_0, f_1, ...) = (0, f_0, f_1, ...); \\ &J : (Jf)_n = f_{n-1}, \quad n \in \mathbb{N}, \quad (Jf)_0 = 0, \quad f \in l. \end{aligned} \tag{3.12}$$

This expression is a "birth-type operator": for a δ-sequence we have the relation

$$J\delta_n = \delta_{n+1}, \ n \in \mathbb{N}_0 \tag{3.13}$$

holds true.

Its restriction to l_2 will not be a Hermitian operator in the space l_2 but it will be Hermitian in S, which is given by the (quasi)-scalar product

$$(f,g)_S = \sum_{j,k=0}^{\infty} s_{j+k} f_k \bar{g}_j. \tag{3.14}$$

The inequality (3.9) shows that (3.14) is a (quasi-)scalar product.

Let S be a Hilbert space constructed using (3.14). To construct it, we need to go from l_{fin} to the classes $\dot{f} \in \dot{l}_{fin}$, where $\dot{l}_{fin}$ is the factor space, that is, $l_{fin}/\{h \in l_{fin} \mid (h,h)_S = 0\}$ and complete it. From (3.14) and (3.12), we have: $f, g \in l_{\text{fin}}$,

$$\begin{aligned} (Jf,g)_S &= \sum_{j,k=0}^{\infty} s_{j+k}(Jf)_k \bar{g}_j = \sum_{j,k=0}^{\infty} s_{j+k} f_{k-1} \bar{g}_j \\ &= \sum_{j,k=0}^{\infty} s_{j+k+1} f_k \bar{g}_j = \sum_{j,k=0}^{\infty} s_{j+k} f_k \bar{g}_{j-1} = \sum_{j,k=0}^{\infty} s_{j+k} f_k \overline{(Jg)_j} = (f, Jg)_S, \end{aligned} \tag{3.15}$$

(here it is taken into account that according to (3.12) it is assumed that $f_{-1}=g_{-1}=0$).

The hermitianity implies the possibility of correctly defining the operator $\dot{J}$ in the Hermitian space S

$$\dot{J}\dot{f} = (Jf)^{\cdot}, \quad f \in \mathfrak{D}(\dot{J}) = \dot{l}_{fin}.$$

Due to Theorem 3.11 [15], the operator has equal defect numbers (either $(0,0)$ or $(1,1)$). Next, we consider the case $(0,0)$. A more general case is in [12] Capter 8, §1, Subsection 4, or [11] Chapter 5, §5, Subsection 2.

Let A denote either the closure of $\dot{J}$ in the space S if $\dot{J}$ is essentially self-adjoint or some self-adjoint extension of it in S. And Theorem 2.1 will be applied to the operator A.

For simplicity, we assume that the moment sequence s_n is not degenerate in the sense that if $(f,f)_S = 0$ for some $f \in l_{fin}$, then $f = 0$, now $\dot{f} = f$, $\dot{J} = J$.

A more general case is described in [12] Chapter 8, §1, Subsection 4, or [11] Chapter 5, §5, Subsections 1-3.

Now, as the space (2.5) we take

$$(l_2(p))_{-,S} \supset S \supset l_2(p) \supset l_{fin}, \tag{3.16}$$

where $l_2(p)$ is the space l_2 with the weight $p = (p_n)_{n=0}^{\infty}$ and the norm given by the expression

$$\|f\|^2_{l_2(p)} = \sum_{n=0}^{\infty} |f_n|^2 p_n;$$

and $(l_2(p))_{-,S} = \mathcal{H}_-$ is the negative space with respect to the positive space $l_2(p)$ and S, and we put $\mathcal{D} := l_{fin}$.

Lemma 3.2. *There will always be a sufficiently rapidly increasing sequence* $p = (p_n)_{n=0}^{\infty}$, *such that the embedding* $l_2(p) \hookrightarrow S$ *is quasi-nuclear.*

Proof. Indeed, (3.9) means that the matrix $K = (K_{jk})_{j,k=0}^{\infty}$, where $K_{jk} = s_{j+k}$, is non-negative definite, and moreover

$$|s_{j+k}|^2 = |K_{jk}|^2 \leq K_{jj} K_{kk} = s_{2j} s_{2k}, \quad j, k \in \mathbb{N}_0. \tag{3.17}$$

Let us choose a sequence $q = (q_n)_{n=0}^{\infty}$, $q_n \geq 1$, such that $\sum\limits_{n=0}^{\infty} s_{2n} q_n^{-1} < \infty$. Then from (3.14) and (3.17) it follows

$$\|f\|^2_S = \sum_{j,k=0}^{\infty} s_{j+k} f_k \bar{f}_j \leq \left(\sum_{j=0}^{\infty} \frac{s_{2j}}{q_j} \right) \|f\|^2_{l_2(q)}.$$

Thus, the embedding $l_2(q) \hookrightarrow S$ is topological. But if $\sum\limits_{n=0}^{\infty} q_n p_n^{-1} < \infty$, then the embedding $l_2(p) \hookrightarrow l_2(q)$ is quasi-nuclear. The composition of the topological and quasi-nuclear embeddings $l_2(p) \hookrightarrow S$ is quasi-nuclear. □

Now we can use the chain (3.16) to accommodate generalized eigenvectors, but the internal structure of $(l_2(p))_{-,S}$ is too complicated, since the structure of S is complicated. So, together with (3.16) we use the chain

$$l = (l_{fin})' \supset l_2(p^{-1}) \supset l_2 \supset l_2(p) \supset l_{fin}, \tag{3.18}$$

where $l_2(p^{-1})$ is the space l_2 with the weight $p^{-1} = (p_n^{-1})_{n=0}^{\infty}$ that is negative with respect to positive space $l_2(p)$ and the space l_2. The chains (3.16) and (3.18) have a common positive space $l_2(p)$. Thus, it is easy to understand the isometry between the spaces $(l_2(p))_{-,S}$ and $l_2(p^{-1})$. The following general lemma is used (see [15], Lemma 2.7.5).

Lemma 3.3. *Suppose that two equipments are given:*

$$\mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+, \quad \mathcal{F}_- \supset \mathcal{F} \supset \mathcal{F}_+ = \mathcal{H}_+ \tag{3.19}$$

with equal positive spaces. Then there exists an isometric operator $U : \mathcal{H}_- \to \mathcal{F}_-$, $U\mathcal{H}_- = \mathcal{F}_-$, *such that*

$$(U\xi, f)_{\mathcal{F}} = (\xi, f)_{\mathcal{H}}, \quad \xi \in \mathcal{H}_-, \quad f \in \mathcal{H}_+ = \mathcal{F}_+. \tag{3.20}$$

This operator is given by the expression: $U = \mathbb{I}_{\mathcal{F}}^{-1} \mathbb{I}_{\mathcal{H}}$, *where* $\mathbb{I}_{\mathcal{F}}$ *and* $\mathbb{I}_{\mathcal{H}}$ *are canonical isometric isomorphisms in chains according to* $\mathbb{I}_{\mathcal{F}} \mathcal{F}_- = \mathcal{F}_+$, $\mathbb{I}_{\mathcal{H}} \mathcal{H}_- = \mathcal{H}_+$.

Proof. For the sake of completeness, we will provide a full proof. Thus, the standard operators $\mathbb{I}_{\mathcal{H}}$: $\mathcal{H}_- \longmapsto \mathcal{H}_+$, $\mathbb{I}_{\mathcal{F}}$: $\mathcal{F}_- \longmapsto \mathcal{F}_+$ are isometric operators between the specified spaces i.e. the canonical isometric isomorphism of Berezanskiy. For such operators we have:

$$(\alpha, f)_{\mathcal{H}} = (\mathbb{I}_{\mathcal{H}} \alpha, f)_{\mathcal{H}_+} = (\alpha, \mathbb{I}_{\mathcal{H}}^{-1} f)_{\mathcal{H}_-}, \quad (\mathbb{I}_{\mathcal{H}} \alpha, \beta)_{\mathcal{H}} = (\alpha, \mathbb{I}_{\mathcal{H}} \beta)_{\mathcal{H}}, \ \forall \alpha, \beta \in \mathcal{H}_-, \ f \in \mathcal{H}_+.$$

and a similar equality for the second chain (3.19). Using these equalities, we obtain:

$$(U\xi, f)_{\mathcal{F}} = (\mathbb{I}_{\mathcal{F}}^{-1}\mathbb{I}_{\mathcal{H}}\xi, f)_{\mathcal{F}} = (\mathbb{I}_{\mathcal{H}}\xi, f)_{\mathcal{F}_+}$$
$$= (\mathbb{I}_{\mathcal{H}}\xi, f)_{\mathcal{H}_+} = (\xi, f)_{\mathcal{H}}, \quad \xi \in \mathcal{H}_-, \quad f \in \mathcal{H}_+ = \mathcal{F}_+.$$

□

Let us now apply lemma 3.3. As chains (3.19) we consider (3.16) and (3.18). Let $\xi_\lambda \in (l_2(p))_{-,S}$ be the generalized eigenvector of the operator J in terms of the chain (3.16). In this case, according to theorem 2.1 we have

$$(\xi_\lambda, Jf)_S = \lambda(\xi_\lambda, f)_S, \quad \lambda \in \mathbb{R},\ f \in l_{\text{fin}}. \tag{3.21}$$

Let us denote

$$P(\lambda) = U\xi_\lambda \in l_2(p^{-1}) \subset l,\ P(\lambda) = (P_n(\lambda))_{n=0}^{\infty}.$$

Using (3.20), the expression (3.21) is rewritten as

$$(P(\lambda), Jf)_{l_2} = \lambda(P(\lambda), f)_{l_2}, \quad \lambda \in \mathbb{R},\ f \in l_{\text{fin}}. \tag{3.22}$$

The corresponding Fourier transform has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(\lambda) = \hat{f}(\lambda) = (f, P(\lambda))_{l_2} \in L^2(\mathbb{R}, d\rho(\lambda)). \tag{3.23}$$

Let us calculate $P(\lambda)$. The operator A defined in (3.12) gives

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda P_n(\lambda)\bar{f}_n &= \lambda(P(\lambda), f)_{l_2} \\ &= (P(\lambda), Jf)_{l_2} = \sum_{n=0}^{\infty} P_{n+1}(\lambda)\bar{f}_n, \quad \forall f \in l_{\text{fin}}.\end{aligned} \tag{3.24}$$

Hence

$$\lambda P_n(\lambda) = P_{n+1}(\lambda), \quad n \in \mathbb{N}_0.$$

Without loss of generality, it is assumed $P_0(\lambda) = 1,\ \lambda \in \mathbb{R}$. Now the last two formulas give

$$P_n(\lambda) = \lambda^n,\ n \in \mathbb{N}_0. \tag{3.25}$$

Therefore, the Fourier transform (3.23) has the form

$$S \supset l_{\text{fin}} \ni f \to (Ff)(\lambda) = \hat{f}(\lambda) = \sum_{n=0}^{\infty} f_n \lambda^n \in L^2(\mathbb{R}, d\rho(\lambda)), \tag{3.26}$$

and Parseval's equality has the form

$$(f, g)_S = \int_{\mathbb{R}} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}\, d\rho(\lambda), \quad f, g \in l_{\text{fin}}. \tag{3.27}$$

To construct the Fourier transform (3.23) and justify the formulas (3.24)-(3.27), it is necessary to find out for the operator J the existence of a strong cyclic vector $\Omega = \delta_0 \in l_{\text{fin}}$ in the sense of the rigging (2.5).

This is indeed the case, since from (3.13) we have $J^p\delta_0 = \delta_p$.

The Parseval equality (3.27) directly leads to the representation (3.8): according to (3.25), (3.26) $\hat{\delta}_n = \lambda^n$ and $\hat{\delta}_0 = 1$, and from (3.14) it follows

$$s_n = (\delta_n, \delta_0)_S = (\hat{\delta}_n, \hat{\delta}_0)_{L_2(\mathbb{R}, d\rho(\lambda))} = \int_{\mathbb{R}} \lambda^n\, d\rho(\lambda), \quad n \in \mathbb{N}_0. \tag{3.28}$$

So, the first main part of the theorem is proven. To prove its second final part, it is necessary to verify that the condition (3.10) leads to the self-adjoint operator A in the space S.

To do this, consider a Hermitian operator defined on a linear set invariant with respect to its action

$$\mathcal{D} = l_{\text{fin}} = \operatorname{span}\{\delta_n\}, \quad n \in \mathbb{N}_0$$

by the expression $J\delta_n = \delta_{n+1}$, and for $p \geq 1$ we have $J^p\delta_n = \delta_{n+p}$.

According to (3.14), the space S has the norm $\|f\|_S = \sqrt{(f,f)_S}$. Therefore, for each $\delta_n \in \mathcal{D}$ we have $\|J^p\delta_n\|_S^2 = \|\delta_{n+p}\|_S^2 = s_{2n+2p}$. Since

$$\sum_{p=1}^{\infty} \frac{1}{\sqrt[p]{\|J^p\delta_n\|_S}} = \sum_{p=1}^{\infty} \frac{1}{\sqrt[2p]{s_{2n+2p}}},$$

then one can claim that the quasi-analyticity of the class $\mathrm{C}\{\|J^p\delta_n\|\}$ is equivalent to the quasi-analyticity of the class $\mathrm{C}\{\sqrt{s_{2n+2p}}\}$ and according to the properties of quasi-analytic classes this class is equivalent to $\mathrm{C}\{\sqrt{s_{2p}}\}$. □

4. Example by Koshmanenko V. D.

4.1. **Example 1.** Let $\mathcal{H}$ be a (separable) Hilbert space with scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$, and $A = A^* > 1$ be an unbounded self-adjoint operator with a dense in $\mathcal{H}$ domain $\mathfrak{D}(A)$.

Let us construct an equipment of the space $\mathcal{H}$ using the operators A since $\mathfrak{D}(A)$ is dense in $\mathcal{H}$. Let $\mathcal{H}_+ = \mathfrak{D}(A)$ be the space with scalar product $(\cdot,\cdot)_+ := (A\cdot, A\cdot)$ and the norm $\|\cdot\|_+ := \sqrt{(\cdot,\cdot)_+}$, and $\mathcal{H}_-$ is the completion of $\mathcal{H}$ by the norm $\|\cdot\|_- = \sqrt{(A^{-1}\cdot, A^{-1}\cdot)}$.

For the operator A, we choose a densely defined symmetric restriction of $\dot{A}$ in $\mathcal{H}$, $\mathfrak{D}(\dot{A})$, with defect indices $(1,1)$. The density fact of the corresponding embedding is denoted by $\mathcal{H} \sqsupset \mathfrak{D}(\dot{A})$. Let $M_+ := \mathfrak{D}(\dot{A})$ denotes a set in $\mathcal{H}_+$. Then the orthogonal complement in $\mathcal{H}_+$ to M_+ is denoted by $\{\eta_+\} = \mathcal{H}_+ \ominus M_+$. Their images in $\mathcal{H}$ are denoted by $M_0 := AM_+$, $\eta_0 := A\eta_+$, respectively. So, we have $\mathcal{H} = M_0 \oplus \{\eta_0\}$. In particular, $AM_+ = \dot{A}M_+ = M_0$.

From the book [9], Chapter 6, Theorems 6.1.1, 6.1.4 and next theorems there, $\mathcal{H} \sqsupset M_+$ if and only if $\eta_0 \in \mathcal{H} \setminus \mathcal{H}_+$, and also $\mathcal{H}_- \sqsupset M_0$.

The operator A can be extended to $\mathcal{H}_-$ as an operator with domain $\mathcal{H}$ that is dense in $\mathcal{H}_-$ with respect to its norm. In fact, this operator coincides with the canonical isometric isomorphism of Berezansky from $\mathcal{H}$ into $\mathcal{H}_-$. Thus we constructed the diagram.

$$\begin{array}{ccccccc}
\mathcal{H}_- & \supset & \mathcal{H} & \supset & \mathcal{H}_+ & = & \mathfrak{D}(A) \\
 & & \shortparallel & & \shortparallel & & \\
 & \sqsupset & M_0 & \sqsupset & M_+ & = & \mathfrak{D}(\dot{A}) \\
 & & \oplus & & \oplus & & \\
 & & \{\eta_0\} & & \{\eta_+\} & &
\end{array}$$

Let us next choose another self-adjoint extension $\tilde{A} = \tilde{A}^*$, $\mathfrak{D}(\tilde{A})$ in $\mathcal{H}$ of the operator $\dot{A}$, different from A, but such that, for simplicity, $\tilde{A} > 1$.

Let us construct an equipment of the space $\mathcal{H}$ using the operator $\tilde{A}$. Let us now set $\tilde{\mathcal{H}}_+ = \mathfrak{D}(\tilde{A})$ with the scalar product $\widetilde{(\cdot,\cdot)}_+ := (\tilde{A}\cdot, \tilde{A}\cdot)$ and the norm $\widetilde{\|\cdot\|}_+ := \sqrt{\widetilde{(\cdot,\cdot)}_+}$, and $\tilde{\mathcal{H}}_-$ is the completion of $\mathcal{H}$ by the norm $\widetilde{\|\cdot\|}_- = \sqrt{(\tilde{A}^{-1}\cdot, \tilde{A}^{-1}\cdot)}$.

For the operator $\tilde{A}$, the symmetric restriction densely defined in $\mathcal{H}$ is the same $\dot{A}$, $\mathfrak{D}(\dot{A})$, with defect indices $(1,1)$; we also denote $\mathcal{H} \sqsupset \mathfrak{D}(\dot{A})$. Let us denote the same $M_+ := \mathfrak{D}(\dot{A})$ as a set but in $\tilde{\mathcal{H}}_+$. Then the orthogonal complement in $\tilde{\mathcal{H}}_+$ to M_+ is denoted by $\{\tilde\eta_+\} = \tilde{\mathcal{H}}_+ \ominus M_+$. Their images in $\mathcal{H}$ will be the same $M_0 := AM_+$, $\eta_0 := A\eta_+$. So, we also have $\mathcal{H} = M_0 \oplus \{\eta_0\}$. In particular, $\tilde{A}M_+ = \dot{A}M_+ = M_0$.

Similarly, from the book [9] Chapter 6, Theorems 6.1.1, 6.1.4 and next theorems there, $\mathcal{H} \sqsupset M_+$ if and only if $\eta_0 \in \mathcal{H} \setminus \tilde{\mathcal{H}}_+$, and also $\tilde{\mathcal{H}}_- \sqsupset M_0$.

The operator $\tilde{A}$ can also be extended by continuity onto $\tilde{\mathcal{H}}_-$, as an operator with domain $\mathcal{H}$.

Thus we constructed the second diagram.

$$\begin{array}{ccccccc}
\tilde{\mathcal{H}}_- & \supset & \mathcal{H} & \supset & \tilde{\mathcal{H}}_+ & = & \mathfrak{D}(\tilde{A})\\
 & & \| & & \| & & \\
 & \supset & M_0 & \supset & M_+ & = & \mathfrak{D}(\dot{A})\\
 & & \oplus & & \oplus & & \\
 & & \{\eta_0\} & & \{\tilde\eta_+\} & &
\end{array}$$

The operators A and $\tilde{A}$, as different self-adjoint extensions of the common symmetric operator $\dot{A}$, are related by the M. Krein formula

$$\tilde{A}^{-1} = A^{-1} + b(\cdot, \eta_0)\eta_0,$$

where b is some constat.

Now it is obvious that the norms of $\tilde{\mathcal{H}}_-$ and $\mathcal{H}_-$ are different and at the same time they are the same on M_0, because $(M_0, \eta_0) = 0$, and M_0 is dense in both $\tilde{\mathcal{H}}_-$ and $\mathcal{H}_-$.

5. Examples by Nizhnik L. P.

5.1. **Example 1.** Consider the space $L_2 := L_2[0,\pi]$ of square-integrable functions on the interval $[0,\pi]$. Let $(\cdot,\cdot)$ be its scalar product. As is well known, the set of functions $\mathfrak{D} = \{\sin nx\}$, $n \in \mathbb{N}$ is dense in L_2 and $f(x) \equiv 1$ belongs to L_2. It is also known that "1" has a Fourier series expansion $1 = \sum\limits_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin(2k+1)x$.

Consider another new space $\tilde{L_2} := \tilde{L_2}[0,\pi]$, formed as a completion on continuous functions of the form $\tilde f = f(x) \dotplus f(0)\cdot 1$ with respect to the norm, given by the scalar product

$$\begin{aligned}(\tilde f, \tilde g)^{\sim} &= (f(x) \dotplus f(0)\cdot 1, g(x) \dotplus g(0)\cdot 1)^{\sim}\\ &= (f,g) + f(0)(1,g) + g(0)(f,1) + f(0)g(0)(1,1),\end{aligned}$$

where $f, g \in C[0,\pi]$ – the space of continuous functions on the interval $[0,\pi]$, $\dotplus$ denotes a direct sum. It is clear that the norm in $\tilde{L_2}$ is given by the formula

$$\begin{aligned}\|\tilde f\|^{\sim 2} &= (\tilde f, \tilde f)^{\sim} = (f(x) \dotplus f(0)\cdot 1, f(x) \dotplus f(0)\cdot 1)^{\sim}\\ &= (f(x), f(x)) + 2f(0)(f(x),1) + f(0)^2(1,1).\end{aligned}$$

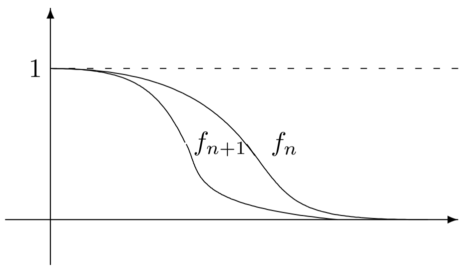
It is not difficult to verify that the axioms of the scalar product and the norm hold. It is obvious that $(\cdot,\cdot) \restriction \mathfrak{D} = (\cdot,\cdot)^{\sim} \restriction \mathfrak{D}$, because $\sin n0 = 0$.

Thus, the spaces L_2 and $\tilde{L_2}$, which coincide on the dense set $\mathfrak{D}$ and have the same norm on this set, are constructed. Let us show that these norms are different outside their common set.

If we take $E = \begin{cases} 1, & x = 0; \\ 0, & x \in (0,\pi] \end{cases}$, then, it is obviously $\|E\|_{L_2} = 0$, but

$$\|E\|^{\sim} = (E,E)^{\sim} = (E \dotplus 1\cdot 1, E \dotplus 1 \cdot 1)^{\sim} = (E,E) + 2\cdot 1\cdot (E,1) + 1^2(1,1) = 1.$$

Let us show that the norms $\|\cdot\|$ $\|\cdot\|^{\sim}$ are not equivalent. Let us choose a sequence of the form



that converges to E pointwise. Obviously $f_n \xrightarrow{L_2} 0$ and $f_n(0) = 1$. But at the same time $\|f_n(x)\|\tilde{}^2 \xrightarrow{\tilde{L_2}} \frac{1}{2}$. Really,

$$\begin{aligned} \|f_n(x) - \frac{1}{2}\|\tilde{}^2 &= (f_n(x) - \frac{1}{2}, f_n(x) - \frac{1}{2})\tilde{} \\ &= ([f_n(x) - \frac{1}{2}]\dot{+}(f_n(x) - \frac{1}{2})|_{x=0} \cdot 1, [f_n(x) - \frac{1}{2}]\dot{+}(f_n(x) - \frac{1}{2})|_{x=0} \cdot 1) \\ &= (([f_n(x) - \frac{1}{2}]\dot{+}(1 - \frac{1}{2})), ([f_n(x) - \frac{1}{2}]\dot{+}(1 - \frac{1}{2}))) = (f_n(x), f_n(x)) \\ &= \|f_n(x)\|^2 \xrightarrow{L_2} 0, \quad n \to 0. \end{aligned}$$

5.2. **Example 2.** Consider the space $L_2 := L_2[1, \infty]$ of square-integrable functions on the semi-axis $[1, \infty]$. Let $(\cdot, \cdot)$ be its scalar product, $\mathfrak{D} := C_0^\infty$ – infinitely differentiable functions with compact support. As is known, $\mathfrak{D}$ is dense in L_2 and $E := E(x) = \frac{1}{x}$ belongs to L_2 since $\int_1^\infty \frac{1}{x^2}\, dx = 1$, therefore E can be approximated by elements of of $\mathfrak{D}$.

Consider another new space $\tilde{L_2} := \tilde{L_2}[1, \infty]$, formed as a completion on continuous functions of the form $\tilde{f} = f(x)\dot{+}f_\infty \cdot E$ with respect to the norm, given by the scalar product

$$\begin{aligned} (\tilde{f}, \tilde{g})\tilde{} &= (f(x)\dot{+}f_\infty \cdot E, g(x)\dot{+}g_\infty \cdot E)\tilde{} \\ &= (f, g) + f_\infty(E, g) + g_\infty(f, E) + f_\infty g_\infty(E, E), \end{aligned}$$

where $f, g \in C[0, \infty]$ – the space of continuous functions on the interval $[0, \infty]$ and such that $f_\infty := \lim\limits_{x\to\infty} f(x)x < \infty$; $\dot{+}$ denotes a direct sum. It is clear that the norm in $\tilde{L_2}$ is given by the formula

$$\begin{aligned} \|\tilde{f}\|\tilde{}^2 &= (\tilde{f}, \tilde{f})\tilde{} = (f(x)\dot{+}f_\infty \cdot E, f(x)\dot{+}f_\infty \cdot E)\tilde{} \\ &= (f(x), f(x)) + 2f_\infty(f(x), E) + f_\infty^2(E, E). \end{aligned}$$

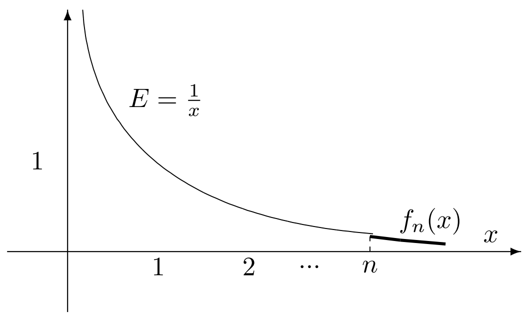
It is not difficult to verify that the axioms of the scalar product and the norm hold. It is obvious that $(\cdot, \cdot) \restriction \mathfrak{D} = (\cdot, \cdot)\tilde{} \restriction \mathfrak{D}$, because $f_\infty := \lim\limits_{x\to\infty} f(x)x = 0$, for $f \in \mathfrak{D}$.

Thus, the spaces L_2 and $\tilde{L_2}$, which coincide on the dense set $\mathfrak{D}$ and have the same norm on this set, are constructed. Let us show that these norms are different outside their common set.

Let us take E, then obviously $\|E\|_{L_2} = 1$, but

$$\|E\|\tilde{} = (E, E)\tilde{} = (E + E, E + E) = 4(E, E) = 4 \neq 1.$$

Let us show that the norms are not equivalent. Let us choose a sequence of the form:



namely $f_n(x) = \begin{cases} 0, & x \in [1, n); \\ \frac{1}{x}, & x \in [n, \infty] \end{cases}$

Obviously $f_n(x) \xrightarrow{L_2} 0$. But together with that $f_n(x) \xrightarrow{\tilde{L_2}} \frac{1}{2}E$. Indeed

$$\begin{aligned}\|\frac{1}{2}E - f_n\|^{\sim 2} &= (\frac{1}{2}E - f_n, \frac{1}{2}E - f_n)^{\sim} \\ &= ((\frac{1}{2}E - f_n)\dot{+}\{\frac{1}{2}E_\infty - f_{n,\infty}\}E, (\frac{1}{2}E - f_n)\dot{+}\{\frac{1}{2}E_\infty - f_{n,\infty}\}E) \\ &= \int_1^\infty (\frac{1}{2}E - f_n + \frac{1}{2}E - E)(\frac{1}{2}E - f_n + \frac{1}{2}E - E)dx \\ &= \int_1^\infty f_n^2 dx \longrightarrow 0, \quad n \to 0.\end{aligned}$$

5.3. **Example 3.** Now it is not difficult to construct a discrete analogue of Example 2. Namely, consider the space l_2 – summable with the square of the sequences $f = (f_n)$. Let $(\cdot, \cdot)$ be its scalar product. As is known, the set of finite sequences $\mathfrak{D}$ is dense in l_2 and we put $E := \{1, \frac{1}{2}, ..., \frac{1}{n}, ...\}$ belongs to l_2 since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. It is also known that E can be approximated by sequences from $\mathfrak{D}$.

Consider the new space $\tilde{l_2}$ formed as a completion on continuous functions of the form $\tilde{f} = f(x)\dot{+}f_\infty \cdot E$ with respect to the norm, given by the scalar product

$$\begin{aligned}(\tilde{f}, \tilde{g})^{\sim} &= (f(x)\dot{+}f_\infty \cdot E, g(x)\dot{+}g_\infty \cdot E)^{\sim} \\ &= (f, g) + f_\infty(E, g) + g_\infty(f, E) + f_\infty g_\infty(E, E),\end{aligned}$$

where $f, g \in l$ – the space of infinite sequences and such that $f_\infty := \lim\limits_{n\to\infty} f_n n < \infty$; $\dot{+}$ denotes a direct sum. It is clear that the norm in $\tilde{l_2}$ is given by the formula

$$\begin{aligned}\|\tilde{f}\|^{\sim 2} &= (\tilde{f}, \tilde{f})^{\sim} = (f\dot{+}f_\infty \cdot E, f\dot{+}f_\infty \cdot E)^{\sim} \\ &= (f, f) + 2f_\infty(f, E) + f_\infty^2(E, E).\end{aligned}$$

It is not difficult to verify that the axioms of the scalar product and the norm hold. It is obvious that $(\cdot, \cdot) \upharpoonright \mathfrak{D} = (\cdot, \cdot)^{\sim} \upharpoonright \mathfrak{D}$, since $f_\infty := \lim\limits_{n\to\infty} f_n n = 0$ for $f \in \mathfrak{D}$.

Thus, the spaces l_2 and $\tilde{l_2}$, which coincide on the dense set $\mathfrak{D}$ and have the same norm on this set, are constructed. Let us show that these norms are different outside their common set.

Let us take E, then as it is known $\|E\|_{l_2} = \frac{\pi}{\sqrt{6}}$, but

$$\|E\|\tilde{} = (E,E)\tilde{} = (E+E, E+E) = 4(E,E) = \frac{2\pi^2}{3}.$$

The non-equivalence of norms is now obvious for the sequence

$$E_n(x) = \begin{cases} 0, & n \in [0, n-1]; \\ \frac{1}{n}, & n \in [n, \infty] \end{cases},$$

namely

$$E_n = (0,0,...,0,\frac{1}{n},\frac{1}{n+1},...) \xrightarrow{l_2} 0, \quad E_n = (0,0,...,0,\frac{1}{n},\frac{1}{n+1},...) \xrightarrow{\tilde{l}_2} \frac{1}{2}E.$$

The last convergences are proved similarly as in Example 2.

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Mykola Dudkin: `dudkin@imath.kiev.ua`
National Technical University of Ukraine “Igor Sikorsky Kyiv Polytechnic Institute”, Kyiv, Ukraine