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CLARK-OCONE TYPE FORMULAS ON THE SPACES OF NONREGULAR GENERALIZED FUNCTIONS IN THE LÉVY WHITE NOISE ANALYSIS

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The paper is dedicated to Professor Yu. M. BEREZANSKY to his centennial

Abstract. In the classical Gaussian analysis the Clark-Ocone formula can be written in the form

$$F = \mathbf{E}F + \int \mathbf{E}\big(\partial_t F|_{\mathcal{F}_t}\big)dW_t,$$

where a function (a random variable) F is square integrable with respect to the Gaussian measure and differentiable by Hida; $\mathbf{E}$ denotes the expectation; $\mathbf{E}(\circ|_{\mathcal{F}_t})$— the conditional expectation with respect to the σ-algebra $\mathcal{F}_t$ that is generated by a Wiener process W up to the point of time t; $\partial_\cdot F$ is the Hida derivative of F; $\int \circ(t)dW_t$ denotes the Itô stochastic integral over a Wiener process. This formula has many applications, in particular, in the stochastic analysis and in the financial mathematics.

In this paper we generalize the Clark-Ocone formula to the spaces $(\mathcal{H}_{-\tau})_{-q}$ of nonregular generalized functions in the Lévy white noise analysis. More exactly, we prove that any element of $(\mathcal{H}_{-\tau})_{-q}$ can be represented as a sum of its expectation and a result of stochastic integration over a Lévy process of some generalized function, and construct Clark-Ocone type formulas on $(\mathcal{H}_{-\tau})_{-q}$ and on its subsets.

Introduction

Denote by $\mathcal{D}$ the Schwartz space of all real-valued infinite-differentiable functions on $\mathbb{R}_+ := [0, +\infty)$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [4], see also Subsection 1.6). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. Note that $\mathcal{D}$ and $\mathcal{D}'$ are the positive and negative spaces of a chain

$$\mathcal{D}' \supset L^2(\mathbb{R}_+) \supset \mathcal{D}, \tag{0.1}$$

where $L^2(\mathbb{R}_+)$ is the space of (classes of) real-valued functions on $\mathbb{R}_+$, square integrable over the Lebesgue measure (e.g., [4]).

Denote by $\langle\cdot,\cdot\rangle$ the dual pairing between elements of $\mathcal{D}'$ and $\mathcal{D}$, generated by the scalar product in $L^2(\mathbb{R}_+)$; by a lower index $\mathbb{C}$ the complexifications of linear topological spaces (for example, elements of $\mathcal{D}_\mathbb{C}$ are $a+bi$, $a,b\in\mathcal{D}$); and by $\mathcal{C}(\mathcal{D}')$ the cylindrical σ-algebra on $\mathcal{D}'$. Let γ be a standard Gaussian measure on $(\mathcal{D}',\mathcal{C}(\mathcal{D}'))$ ($\mathcal{C}(\mathcal{D}')$ is considered to be completed with respect to γ), i.e., a probability measure ($\gamma(\mathcal{D}')=1$) with the Laplace transform

$$l_\gamma(\lambda) := \int_{\mathcal{D}'} e^{\langle x,\lambda\rangle}\gamma(dx) = e^{\langle\lambda,\lambda\rangle/2}, \quad \lambda\in\mathcal{D}_\mathbb{C}.$$

As is well known (e.g., [6, 37, 32]), any square integrable with respect to γ and differentiable by Hida complex-valued function (a random variable) F on $\mathcal{D}'$ can be represented in the form

$$F = \mathbf{E}F + \int \mathbf{E}\big(\partial_t F|_{\mathcal{F}_t}\big)dW_t, \tag{0.2}$$

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where $\mathbf{E}$ denotes the (mathematical) expectation; $\mathbf{E}\big(\circ\,|_{\mathcal{F}_t}\big)$ is the conditional expectation with respect to a complete σ-algebra $\mathcal{F}_t$ generated by a Wiener process W up to the time t (i.e., $\mathcal{F}_t$ is the completion with respect to γ of the σ-algebra $\sigma(W_u : u \le t)$); $\partial.F$ is the Hida derivative of F; $\int \circ(t)dW_t$ denotes the Itô stochastic integral over the Wiener process. Formula (0.2) is called *the Clark-Ocone formula*. As we can see, this formula, in particular, gives a possibility to restore a version of the integrand (this integrand is not unique, generally speaking), if the result of stochastic integration is known.

As is known (e.g., [9, 43]), formula (0.2) holds true (up to clear modifications), if one considers a Poissonian measure instead of γ. We note also that one can easily avoid the restrictive assumption that a random variable F has to be differentiable by Hida: it is sufficient to generalize the Clark-Ocone formula to some spaces of generalized functions (in this case F can remain square integrable), see, e.g., [7, 8].

The Clark-Ocone formula and its generalizations have numerous applications, in particular, in stochastic analysis and in financial mathematics, see, e.g., [30, 1, 8, 39, 10, 38, 35, 2, 9, 43] and references therein. Currently, to match the needs of applications, a variety of Clark-Ocone type formulas have been built on different spaces, with use of different stochastic derivatives and with stochastic integrals over various random processes and measures, see, in particular, [31, 32, 1, 7, 3, 33, 8, 35, 43, 9, 18, 19, 20]. For example, in [32, 33] a Clark-Ocone type formula related to the Lévy process, containing stochastic integrals over a Wiener process and over a compensated Poissonian random measure, is obtained; in [8] uses an approach based on a so-called Nualart-Schoutens decomposition of square integrable random variables [36, 41], now the corresponding formula contains integrals over special random processes. It is worth noting that authors of [8] also generalize their results to certain spaces of generalized random variables.

In the author's papers [18, 19, 20] Clark-Ocone type formulas are constructed on the spaces of regular test, square integrable, and regular generalized functions of the Meixner white noise analysis [17]. This analysis is related to the generalized Meixner measure $\mathbf{m}$ [40] and to the corresponding Meixner random process, whose derivative (in the sense of generalized functions [13]) is the Meixner white noise (the measure of this noise as of a generalized random process [14] is $\mathbf{m}$). Note that the subclass of Meixner processes consisting of stationary random processes is a rather broad subclass of Lévy processes. Nevertheless, the constructions of [18, 19, 20] differ significantly from those of [32, 33] and [8]: we tried to keep as much as possible of the classical form of Clark-Ocone type formulas and therefore we used the Hida stochastic derivative and stochastic integration over the Meixner process only. In the author's paper [27] the results of [18, 19, 20] are transferred to the spaces of a so-called regular parametrized rigging of the space of square integrable random variables in the Lévy white noise analysis (see (1.21) below). The present paper is in a sense a continuation of [27], now our goal is to construct and study Clark-Ocone type formulas on the spaces of nonregular generalized functions in the Lévy white noise analysis (it is worth noting that properties of these spaces essentially differ from properties of the spaces of regular generalized functions). In particular, we show that, in contrast to the regular case, for any nonregular generalized function the Clark-Ocone type formula can be constructed using integration over a Lévy process only.

The paper is organized in the following manner. In the first section we consider a Lévy process L and recall the construction of a required probability triplet connected with L; afterwards we describe Lytvynov's generalization to the Lévy analysis of a so-called chaotic representation property, which is the base to construct spaces of regular and nonregular test and generalized functions, stochastic integrals and derivatives on these spaces, etc.; recall the construction of an extended stochastic integral and of a Hida stochastic derivative on the space of square integrable random variables (L^2); of regular and nonregular riggings of (L^2); of an extended stochastic integral and of a generalized Hida derivative on the spaces of nonregular generalized functions $(\mathcal{H}_{-\tau})_{-q}$. In the second section we prove that any element of $(\mathcal{H}_{-\tau})_{-q}$ can be represented as a sum of its expectation and a result of stochastic integration over a Lévy process of some generalized function; then we construct and study Clark-Ocone type formulas on $(\mathcal{H}_{-\tau})_{-q}$ and on its subsets.

1. Preliminaries

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the real (i.e., bilinear) scalar product in a space H; by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H; by $\mathcal{B}$ a Borel σ-algebra; by 1_Δ the indicator of a set Δ; and by $\widehat{\otimes}$ the symmetric tensor product. Also we use a notation "pr lim" (respectively, "ind lim") for a projective (respectively, inductive) limit of a family of spaces, this notation implies that the limit space is endowed with the projective (respectively, inductive) limit topology (see, e.g., [4] for a detailed description).

1.1. A Lévy process and its probability space.

Let $L=(L_t)_{t\in\mathbb{R}_+}$ be a real-valued locally square integrable Lévy process (a continuous in probability random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0=0$, see, e.g., [5] for a detailed description) without Gaussian part and drift. As is known (e.g., [8]), the characteristic function of L is

$$\mathbf{E}[e^{i\theta L_t}]=\exp\Big[t\int_{\mathbb{R}}(e^{i\theta x}-1-i\theta x)\nu(dx)\Big], \tag{1.3}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, $\mathbf{E}$, as above, denotes the expectation. We assume that *ν is a Radon measure whose support contains an infinite number of points, $\nu(\{0\})=0$, there exists $\varepsilon>0$ such that $\int_{\mathbb{R}}x^2e^{\varepsilon|x|}\nu(dx)<\infty$, and $\int_{\mathbb{R}}x^2\nu(dx)=1$.*

Now define a measure of the white noise of L.

Definition 1.1. A probability measure μ on $(\mathcal{D}',\mathcal{C}(\mathcal{D}'))$ with the Fourier transform

$$\int_{\mathcal{D}'}e^{i\langle\omega,\varphi\rangle}\mu(d\omega)=\exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}}(e^{i\varphi(t)x}-1-i\varphi(t)x)dt\nu(dx)\Big],\quad \varphi\in\mathcal{D}, \tag{1.4}$$

is called the measure of the Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [15]), see [34]. Below we assume that *the σ-algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to μ.*

Consider a probability space (probability triplet) $(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$. Denote by $(L^2):=L^2(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$ the space of square integrable random variables, i.e., the space of (classes of) complex-valued functions on $\mathcal{D}'$, square integrable with respect to μ. Let $f\in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k\in\mathcal{D})_{k\in\mathbb{N}}$ converge to f in $L^2(\mathbb{R}_+)$ as $k\to\infty$ (remind that $\mathcal{D}$ is a dense set in $L^2(\mathbb{R}_+)$). One can show [34, 8, 9, 22] that $\langle\circ,f\rangle:=(L^2)\text{-}\lim_{k\to\infty}\langle\circ,\varphi_k\rangle$ is a well-defined element of (L^2).

Put $1_{[0,0)}\equiv 0$. It follows from (1.3) and (1.4) that $\big(\langle\circ,1_{[0,t)}\rangle\big)_{t\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$, see, e.g., [8, 9]. So, for each $t\in\mathbb{R}_+$ we have $L_t=\langle\circ,1_{[0,t)}\rangle\in(L^2)$.

Note that the derivative in the sense of generalized functions of a Lévy process (the Lévy white noise) is $\dot{L}_\cdot(\omega)=\langle\omega,\delta_\cdot\rangle\equiv\omega(\cdot)$, where δ is the Dirac delta-function. Therefore $\dot{L}$ is a generalized random process in the sense of [13] with trajectories from $\mathcal{D}'$, and μ is the measure of $\dot{L}$ in the classical sense of this notion [14].

1.2. **Lytvynov's generalization of the chaotic representation property.** In what follows, we preserve the above-introduced notation $\langle\cdot,\cdot\rangle$ for the dual pairings in symmetric tensor powers of the complexification of chain (0.1) (actually, of more general chain (1.23)), i.e., for the dual pairings between elements of negative and positive spaces from chains (1.34)). Designate $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$. Let $\mathcal{P}$ be the set of complex-valued polynomials on $\mathcal{D}'$ that consists of zero and elements of the form

$$f(\omega)=\sum_{n=0}^{N_f}\langle\omega^{\otimes n},f^{(n)}\rangle,\quad \omega\in\mathcal{D}',\ f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n},\ N_f\in\mathbb{Z}_+,\ f^{(N_f)}\neq 0,$$

here N_f is called *the power of a polynomial* f; $\langle\omega^{\otimes 0},f^{(0)}\rangle := f^{(0)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}0}:=\mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (1.4) and properties of the measure ν, see also [34]), therefore $\mathcal{P}$ is a dense set in (L^2) [42]. Let $\mathcal{P}_n$, $n\in\mathbb{Z}_+$, be the set of polynomials of power smaller than or equal to n, by $\overline{\mathcal{P}}_n$ we denote the closure of $\mathcal{P}_n$ in (L^2). Let for $n\in\mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n\ominus\overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)); put $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2)=\mathop{\oplus}_{n=0}^{\infty}\mathbf{P}_n. \tag{1.5}$$

Let $f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n\in\mathbb{Z}_+$. Denote by $:\!\langle\circ^{\otimes n},f^{(n)}\rangle\!:\ \in(L^2)$ the orthogonal projection of a monomial $\langle\circ^{\otimes n},f^{(n)}\rangle$ onto $\mathbf{P}_n$. We define real (bilinear) scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n\in\mathbb{Z}_+$, by setting for $f^{(n)},g^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$

$$(f^{(n)},g^{(n)})_{ext}:=\frac{1}{n!}\int_{\mathcal{D}'}:\!\langle\omega^{\otimes n},f^{(n)}\rangle\!::\!\langle\omega^{\otimes n},g^{(n)}\rangle\!:\mu(d\omega). \tag{1.6}$$

The well-posedness of this definition is proved (up to obvious modifications) in [34].

Let $\mathcal{H}_{ext}^{(n)}$, $n\in\mathbb{Z}_+$, be the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to the norms, generated by scalar products (1.6). For each $F^{(n)}\in\mathcal{H}_{ext}^{(n)}$ we define *a Wick monomial* $:\!\langle\circ^{\otimes n},F^{(n)}\rangle\!:\ \stackrel{\mathrm{def}}{=} (L^2)\text{-}\lim_{k\to\infty}:\!\langle\circ^{\otimes n},f_k^{(n)}\rangle\!:$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}\ni f_k^{(n)}\to F^{(n)}$ as $k\to\infty$ in $\mathcal{H}_{ext}^{(n)}$. It is easy to prove by the method of "mixed sequences" that this definition is well-posed; and it is not difficult to show that $:\!\langle\circ^{\otimes 0},F^{(0)}\rangle\!:\ =\langle\circ^{\otimes 0},F^{(0)}\rangle=F^{(0)}$ and $:\!\langle\circ,F^{(1)}\rangle\!:\ =\langle\circ,F^{(1)}\rangle$ (cf. [34]).

In the next statement, which follows from (1.5) and the fact that for each $n\in\mathbb{Z}_+$ the set $\{:\!\langle\circ^{\otimes n},f^{(n)}\rangle\!:\,|f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}\}$ is dense in $\mathbf{P}_n$ (and therefore $\mathbf{P}_n=\{:\!\langle\circ^{\otimes n},F^{(n)}\rangle\!:\,|F^{(n)}\in\mathcal{H}_{ext}^{(n)}\}$), Lytvynov's generalization of the chaotic representation property is described.

Theorem 1.2. (cf. [34]) *A random variable $F\in(L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)}\in\mathcal{H}_{ext}^{(n)}$, $n\in\mathbb{Z}_+$, such that*

$$F=\sum_{n=0}^{\infty}:\!\langle\circ^{\otimes n},F^{(n)}\rangle\!: \tag{1.7}$$

(the series converges in (L^2)) and

$$\|F\|_{(L^2)}^2=\int_{\mathcal{D}'}|F(\omega)|^2\mu(d\omega)=\mathbf{E}|F|^2=\sum_{n=0}^{\infty}n!|F^{(n)}|_{\mathcal{H}_{ext}^{(n)}}^2<\infty. \tag{1.8}$$

Remark 1.3. Decomposition (1.7) is an analog of the decomposition of square integrable random variable by the Hermite polynomials in the Gaussian analysis (which is equivalent to the decomposition by the repeated Itô stochastic integrals). At the same time Wick monomials from (1.7) are polynomials if and only if the corresponding Lévy process is a Meixner process, see [34] for details.

It is worth noting that in this paper we do not use directly an explicit (calculation-friendly) formula for scalar products in the spaces $\mathcal{H}_{ext}^{(n)}$ and therefore, we prefer not to write it down. The interested reader can find this formula in [34]; in another record form it is given in, e.g., [22, 21, 12, 11, 26].

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$. It follows from the explicit formula for the scalar products in $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and for $n \in \mathbb{N}\backslash\{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(t_1, \dots, t_n) = 0$ if there exist $k, j \in \{1, \dots, n\}$: $k \neq j$, but $t_k = t_j$). In this sense *the space* $\mathcal{H}_{ext}^{(n)}$ *is an extension of* $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, this explains why we use the subscript "*ext*" in our designations. Also we note that for each $n \in \mathbb{N}\backslash\{1\}$ the space $\mathcal{H}_{ext}^{(n)}$ is the symmetric subspace of the space of (classes of) complex-valued functions on $\mathbb{R}_+^n$, square integrable with respect to a certain Radon measure.

1.3. **An extended stochastic integral on** (L^2)**.** Decomposition (1.7) defines an isometric isomorphism (a generalized Wiener-Itô-Segal isomorphism)

$$\mathbf{I} : (L^2) \to \mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$$

between the space of square integrable random variables (L^2) and the weighted extended symmetric Fock space $\mathop{\oplus}\limits_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$: for $G \in (L^2)$ with decomposition (1.7)

$$\mathbf{I}G = (G^{(0)}, G^{(1)}, \dots) \in \mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}.$$

Let $\mathbf{1} : \mathcal{H}_{\mathbb{C}} \to \mathcal{H}_{\mathbb{C}}$ be the identity operator. Then the operator

$$\mathbf{I} \otimes \mathbf{1} : (L^2) \otimes \mathcal{H}_{\mathbb{C}} \to \big(\mathop{\oplus}_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)} \big) \otimes \mathcal{H}_{\mathbb{C}} \cong \mathop{\oplus}_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})$$

is an isometric isomorphism between the spaces $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$ and $\mathop{\oplus}\limits_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})$. It is clear that for arbitrary $m \in \mathbb{Z}_+$ and $F_{\cdot}^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_{\mathbb{C}}$ a vector $(\underbrace{0, \dots, 0}_{m}, F_{\cdot}^{(m)}, 0, \dots)$ belongs to $\mathop{\oplus}\limits_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})$. *We define a Wick monomial*

$$:\langle \circ^{\otimes m}, F_{\cdot}^{(m)} \rangle: \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1} (\underbrace{0, \dots, 0}_{m}, F_{\cdot}^{(m)}, 0, \dots) \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}. \tag{1.9}$$

By construction elements $:\langle \circ^{\otimes n}, F_{\cdot}^{(n)} \rangle:$, $n \in \mathbb{Z}_+$, $F_{\cdot}^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, form an orthogonal basis in the space $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$ in the sense that $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ if and only if F can be uniquely represented as

$$F \equiv F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F_{\cdot}^{(n)} \rangle: \tag{1.10}$$

(the series converges in $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$), with

$$\|F\|_{(L^2) \otimes \mathcal{H}_{\mathbb{C}}}^2 = \|(\mathbf{I} \otimes \mathbf{1})F\|^2_{\mathop{\oplus}\limits_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}})} = \sum_{n=0}^{\infty} n! |F_{\cdot}^{(n)}|^2_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}} < \infty. \tag{1.11}$$

Now we describe the construction of an extended stochastic integral over a Lévy process L, that is based on decomposition (1.10) (a detailed presentation is given in [22]). Let $F_{\cdot}^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}_{\cdot}^{(n)} \in F_{\cdot}^{(n)}$ such that

$$\dot{f}_t^{(n)}(t_1, \dots, t_n) = 0 \text{ if for some } k \in \{1, \dots, n\}\ t = t_k. \tag{1.12}$$

Let $\hat{f}^{(n)}$ be the symmetrization of a function $\dot{f}^{(n)}_{\cdot}$ by all variables. Define $\widehat{F}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ as the equivalence class in $\mathcal{H}_{ext}^{(n+1)}$ generated by $\hat{f}^{(n)}$ (i.e., $\hat{f}^{(n)} \in \widehat{F}^{(n)}$). It is proved in [22, 21] that this definition is well-posed (in particular, $\widehat{F}^{(n)}$ does not depend on a choice of a representative $\dot{f}^{(n)}_{\cdot} \in F^{(n)}_{\cdot}$ satisfying (1.12)) and $|\widehat{F}^{(n)}|_{\mathcal{H}_{ext}^{(n+1)}} \leq |F^{(n)}_{\cdot}|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}}$.

Definition 1.4. For $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ we define the extended stochastic integral over a Lévy process $\int F(t)\widehat{d}L_t \in (L^2)$ by setting

$$\int F(t)\widehat{d}L_t := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{F}^{(n)} \rangle :, \tag{1.13}$$

where $\widehat{F}^{(0)} := F^{(0)}_{\cdot} \in \mathcal{H}_{\mathbb{C}} = \mathcal{H}_{ext}^{(1)}$, and $\widehat{F}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{N}$, are constructed by the kernels $F^{(n)}_{\cdot} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (1.10) for F, if the series in the right hand side of (1.13) converges in (L^2).

The domain of this integral, i.e., of the operator

$$\int \circ(t)\widehat{d}L_t : (L^2) \otimes \mathcal{H}_{\mathbb{C}} \to (L^2), \tag{1.14}$$

consists of elements $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ such that (see (1.8))

$$\Big\| \int F(t)\widehat{d}L_t \Big\|^2_{(L^2)} = \sum_{n=0}^{\infty} (n+1)! |\widehat{F}^{(n)}|^2_{\mathcal{H}_{ext}^{(n+1)}} < \infty. \tag{1.15}$$

Note that integral (1.14) is called an *extended* stochastic integral because it is a generalization of the Itô stochastic integral, see [22].

Remark 1.5. In this paper we do not need stochastic integrals with respect to measurable sets, which differ from $\mathbb{R}_+$; but such integrals often arise in applications. The definition of the mentioned integrals can be given by the classical way: for arbitrary $\Delta \in \mathcal{B}(\mathbb{R}_+)$ set

$$\int_\Delta \circ(t)\widehat{d}L_t := \int \circ(t) 1_\Delta(t)\widehat{d}L_t$$

(in particular, $\int_{\mathbb{R}_+} \circ(t)\widehat{d}L_t = \int \circ(t)\widehat{d}L_t$). The interested reader can find a detailed information about such integrals in [22, 26].

1.4. **A Hida stochastic derivative on** (L^2)**.** We describe the construction of a Hida stochastic derivative that is based on decomposition (1.7). Let $G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in G^{(n)}$ be a representative of $G^{(n)}$. We consider $\dot{g}^{(n)}(\cdot)$, i.e., separate a one argument of $\dot{g}^{(n)}$, and define $G^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ as the equivalence class in $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ generated by $\dot{g}^{(n)}(\cdot)$ (i.e., $\dot{g}^{(n)}(\cdot) \in G^{(n)}(\cdot)$). By analogy with [22] one can prove that for each $G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, the element $G^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ is well-defined (in particular, $G^{(n)}(\cdot)$ does not depend on a choice of a representative $\dot{g}^{(n)} \in G^{(n)}$) and

$$|G^{(n)}(\cdot)|_{\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}} \leq |G^{(n)}|_{\mathcal{H}_{ext}^{(n)}}. \tag{1.16}$$

Remark 1.6. Note that, in spite of estimate (1.16), the space $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N} \backslash \{1\}$, cannot be considered as a subspace of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ because different elements of $\mathcal{H}_{ext}^{(n)}$ can coincide as elements of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ (i.e., representatives of different equivalence classes in $\mathcal{H}_{ext}^{(n)}$ can fall into the same equivalence class in $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$).

Definition 1.7. For $G \in (L^2)$ we define the Hida stochastic derivative $\partial_{\cdot} G \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ by setting

$$\partial_{\cdot} G := \sum_{n=0}^{\infty} (n+1) :\langle \circ^{\otimes n}, G^{(n+1)}(\cdot) \rangle :, \tag{1.17}$$

where $G^{(n+1)}(\cdot) \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ are constructed by the kernels $G^{(n+1)} \in \mathcal{H}_{ext}^{(n+1)}$ from decomposition (1.7) for G, if the series in the right hand side of (1.17) converges in $(L^2) \otimes \mathcal{H}_{\mathbb{C}}$.

The domain of this derivative, i.e., of the operator

$$\partial_{\cdot} : (L^2) \to (L^2) \otimes \mathcal{H}_{\mathbb{C}}, \tag{1.18}$$

consists of elements $G \in (L^2)$ such that (see (1.11))

$$\|\partial_{\cdot} G\|^2_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}} = \sum_{n=0}^{\infty} (n+1)!(n+1)|G^{(n+1)}(\cdot)|^2_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} < \infty. \tag{1.19}$$

As in the classical Gaussian analysis, we have the following statement.

Proposition 1.8. ([22]) *Extended stochastic integral* (1.14) *and Hida stochastic derivative* (1.18) *are mutually adjoint operators, i.e., for arbitrary* $F \in (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ *and* $G \in (L^2)$ *satisfying conditions* (1.15) *and* (1.19), *respectively,*

$$\Big(\int F(t) \widehat{d} L_t, G \Big)_{(L^2)} = (F(\cdot), \partial_{\cdot} G)_{(L^2)\otimes\mathcal{H}_{\mathbb{C}}}.$$

In particular, integral (1.14) *and derivative* (1.18) *are closed operators.*

Remark 1.9. Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$. Note that the operator adjoint to the extended stochastic integral $\int_\Delta \circ(t)\widehat{d}L_t$ (see Remark 1.5) is $1_\Delta(\cdot)\partial_{\cdot}$. The interested reader can find a more detailed information in [22, 26].

1.5. **A regular rigging of** (L^2)**.** Let $\beta \in [0,1]$, $q \in \mathbb{Z}$ in the case $\beta \in (0,1]$ and $q \in \mathbb{Z}_+$ if $\beta = 0$. Denote by $(L^2)_q^\beta$ the subspace of (L^2) that consists of $f \in (L^2)$ such that

$$\|f\|^2_{(L^2)_q^\beta} := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |f^{(n)}|^2_{\mathcal{H}_{ext}^{(n)}} < \infty, \tag{1.20}$$

where $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decomposition (1.7) for f. Set $(L^2)^\beta := \operatorname{pr}\lim_{q\to+\infty} (L^2)_q^\beta$. It is easy to see that $f \in (L^2)^\beta$ if and only if f can be uniquely represented in form (1.7) with convergent series (1.20) for each $q \in \mathbb{Z}_+$.

It is clear that each $(L^2)_q^\beta$ (as well as $(L^2)^\beta$) densely and continuously embedded into (L^2) (cf. (1.20) and (1.8)), therefore we can consider a chain

$$(L^2)^{-\beta} \supset (L^2)_{-q}^{-\beta} \supseteq (L^2) = (L^2)_0^0 \supseteq (L^2)_q^\beta \supset (L^2)^\beta, \tag{1.21}$$

where $(L^2)_{-q}^{-\beta}$ and $(L^2)^{-\beta} = \operatorname{ind}\lim_{q'\to+\infty} (L^2)_{-q'}^{-\beta}$ are the spaces dual of $(L^2)_q^\beta$ and $(L^2)^\beta$, respectively.

Definition 1.10. Chain (1.21) is called a parametrized regular rigging of (L^2). The spaces $(L^2)_q^\beta$, $(L^2)^\beta$ are called (parametrized Kondratiev-type) spaces of regular test functions, and the spaces $(L^2)_{-q}^{-\beta}$, $(L^2)^{-\beta}$ are called (parametrized Kondratiev-type) spaces of regular generalized functions.

The next statement follows from the general duality theory.

Proposition 1.11. *1) Any regular generalized function* $F \in (L^2)_{-q}^{-\beta}$ *can be uniquely represented as formal series* (1.7) *(with kernels* $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$*) that converges in* $(L^2)_{-q}^{-\beta}$*, and*

$$\|F\|^2_{(L^2)_{-q}^{-\beta}} = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|^2_{\mathcal{H}_{ext}^{(n)}} < \infty. \tag{1.22}$$

Vice versa, any formal series (1.7) *such that series* (1.22) *converges, is a regular generalized function from* $(L^2)_{-q}^{-\beta}$ *(i.e., now series* (1.7) *converges in* $(L^2)_{-q}^{-\beta}$*).*

2) The dual pairing between $F \in (L^2)_{-q}^{-\beta}$ *and* $f \in (L^2)_q^{\beta}$ *that is generated by the scalar product in* (L^2)*, has a form* $\langle\!\langle F, f\rangle\!\rangle_{(L^2)} = \sum_{n=0}^{\infty} n!(F^{(n)}, f^{(n)})_{\mathcal{H}_{ext}^{(n)}}$*, where* $F^{(n)}, f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ *are the kernels from decompositions* (1.7) *for* F *and* f*, respectively.*

3) $F \in (L^2)^{-\beta}$ *if and only if* F *can be uniquely represented in form* (1.7) *and norm* (1.22) *is finite for some* $q \in \mathbb{Z}_+$*.*

Note that the term "*regular* test and generalized functions" is connected with the fact that the kernels from decompositions (1.7) for elements of all spaces of chain (1.21) belong to the same spaces $\mathcal{H}_{ext}^{(n)}$.

Remark 1.12. One can consider more general spaces of regular test and generalized functions, using in formula (1.20) $q \in \mathbb{R}$ in the case $\beta \in (0,1]$ and $q \in \mathbb{R}_+$ if $\beta = 0$, as well as K^{qn} with any $K > 1$ instead of 2^{qn}. But such a generalization is not essential for the rang of problems considered in this paper.

Comparing formulas (1.20) and (1.22), it is easy to conclude that one can denote the spaces $(L^2)_q^{\beta}$, $(L^2) = (L^2)_0^0$ and $(L^2)_{-q}^{-\beta}$ from chain (1.21) by $(L^2)_q^{\beta}$, $\beta \in [-1,1]$, $q \in \mathbb{Z}$. The norms in these spaces are given, obviously, by formula (1.20).

The extended stochastic integral on the spaces of regular test and generalized functions can be defined by formula (1.13) as a linear continuous operator acting from $(L^2)_q^{\beta} \otimes \mathcal{H}_{\mathbb{C}}$ into $(L^2)_{q-1}^{\beta}$ (from $(L^2)_q^{-1} \otimes \mathcal{H}_{\mathbb{C}}$ into $(L^2)_q^{-1}$, from $(L^2)^{\beta} \otimes \mathcal{H}_{\mathbb{C}} := \operatorname{pr}\lim_{q\to+\infty} (L^2)_q^{\beta} \otimes \mathcal{H}_{\mathbb{C}}$ into $(L^2)^{\beta}$ and from $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} := \operatorname{ind}\lim_{q\to+\infty} (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ into $(L^2)^{-\beta}$, here $\beta \in [0,1]$).

The Hida stochastic derivative, in turn, can be defined by formula (1.17) as a linear continuous operator acting from $(L^2)_{q+1}^{\beta}$ into $(L^2)_q^{\beta} \otimes \mathcal{H}_{\mathbb{C}}$ (from $(L^2)_q^1$ into $(L^2)_q^1 \otimes \mathcal{H}_{\mathbb{C}}$, from $(L^2)^{\beta}$ into $(L^2)^{\beta} \otimes \mathcal{H}_{\mathbb{C}}$ and from $(L^2)^{-\beta}$ into $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$, here $\beta \in [0,1]$). The statement of Proposition 1.8 holds true up to obvious modification.

The interested reader can find a more detailed information about the spaces of regular test and generalized functions, as well as about stochastic integrals and derivatives on these spaces in [21, 26].

1.6. **A nonregular rigging of** (L^2)**.** Let T be the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Denote by $\mathcal{H}_\tau$ the real Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is the completion of $\mathcal{D}$ with respect to the norm generated by the scalar product

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ, respectively. As is known (e.g., [4]), $\mathcal{D} = \operatorname{pr}\lim_{\tau\in T} \mathcal{H}_\tau$ (moreover, for any $n \in \mathbb{N}$ $\mathcal{D}^{\widehat{\otimes} n} = \operatorname{pr}\lim_{\tau\in T} \mathcal{H}_\tau^{\widehat{\otimes} n}$), and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$. Therefore, one can consider a chain (cf. (0.1))

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D}, \tag{1.23}$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem [4] $\mathcal{D}' = \operatorname{ind}\lim_{\tau\in T} \mathcal{H}_{-\tau}$ (in what follows we'll consider $\mathcal{D}'$ as a topological space with the inductive limit topology).

It follows from results of [21] that one can modify T (it is necessary to remove from T some “bad" indexes; and we further assume that T is modified) in order to obtain the following statement.

Proposition 1.13. *1) For each $\tau \in T$ the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\tau}$, i.e., $\mu(\mathcal{H}_{-\tau}) = 1$.*

2) For each $\tau \in T$ and each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$

$$|f^{(n)}|^2_{\mathcal{H}_{ext}^{(n)}} \leq n! c(\tau)^n |f^{(n)}|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}. \tag{1.24}$$

Let $\tau \in T$ and $q \in \mathbb{Z}_+$. Denote

$$\mathcal{P}_W := \Big\{ f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle : \mid f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \subset \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \subset \mathcal{H}_{ext}^{(n)}, N_f \in \mathbb{Z}_+ \Big\} \subset (L^2).$$

Define real (bilinear) scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \; g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle : \in \mathcal{P}_W$$

$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}. \tag{1.25}$$

The well-posedness of this definition is proved in [25].

Remark 1.14. (cf. Remark 1.12) One can introduce more general scalar products on $\mathcal{P}_W$, using in formula (1.25) $q \in \mathbb{R}_+$ and K^{qn} with any $K > 1$ instead of 2^{qn}. But, as in the regular case, such a generalization is not essential for our considerations.

Let $(\mathcal{H}_\tau)_q$ be the completions of $\mathcal{P}_W$ with respect to the norms, generated by scalar products (1.25). Set $(\mathcal{H}_\tau) := \operatorname*{pr\,lim}_{q\to+\infty} (\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \operatorname*{pr\,lim}_{\tau\in T, q\to+\infty} (\mathcal{H}_\tau)_q$. As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be uniquely represented in the form (cf. (1.7))

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \; f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \subset \mathcal{H}_{ext}^{(n)} \tag{1.26}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|^2_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}} < \infty. \tag{1.27}$$

Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be uniquely presented in form (1.26) and norm (1.27) is finite for each $q \in \mathbb{Z}_+$ (for each $\tau \in T$ and each $q \in \mathbb{Z}_+$).

Proposition 1.15. ([21, 25]) *For each $\tau \in T$ there exists $q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0(\tau)} := \{q_0(\tau), q_0(\tau)+1, \ldots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2).*

In view of this proposition one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \;\; \tau \in T, \;\; q \in \mathbb{N}_{q_0(\tau)}, \tag{1.28}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \operatorname*{ind\,lim}_{q'\to+\infty} (\mathcal{H}_{-\tau})_{-q'}$ and $(\mathcal{D}') = \operatorname*{ind\,lim}_{\widetilde{\tau}\in T, q'\to+\infty} (\mathcal{H}_{-\widetilde{\tau}})_{-q'}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$, respectively.

In what follows we accept on default $\tau \in T$, $q \in \mathbb{N}_{q_0(\tau)}$.

Definition 1.16. Chain (1.28) is called a nonregular rigging of the space (L^2). The spaces $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ are called (Kondratiev-type) spaces of nonregular test functions, and the spaces $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$ and $(\mathcal{D}')$ are called (Kondratiev-type) spaces of nonregular generalized functions.

Remark 1.17. Let $\tau \in T$, $q \in \mathbb{Z}_+$ and $\beta \in [0,1]$. By analogy with the regular case one can introduce on $\mathcal{P}_W$ scalar products $(\cdot,\cdot)_{\tau,q,\beta}$ by setting for $f,g \in \mathcal{P}_W$

$$(f,g)_{\tau,q,\beta} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^{1+\beta} 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}}$$

(cf. (1.25)), and define "parametrized spaces of nonregular test functions" $(\mathcal{H}_\tau)_q^\beta$ as completions of $\mathcal{P}_W$ with respect to the norms generated by these scalar products. It is possible to study properties of the spaces $(\mathcal{H}_\tau)_q^\beta$ and its projective limits, to introduce and to study different operators and operations on them; such considerations are interesting by itself and can be useful for applications. But $(\mathcal{H}_\tau)_q^\beta \not\subset (L^2)$ if $\beta < 1$ (except for the Gaussian and Poissonian special cases, which we do not consider in this paper), so, we cannot consider $(\mathcal{H}_\tau)_q^\beta$ with $\beta < 1$ as spaces of test functions in the framework of the Lévy white noise analysis.

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}'^{(n)}_{\mathbb{C}} \supset \mathcal{H}^{(n)}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{(n)}_{ext} \supset \mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes}n}_{\mathbb{C}}, \tag{1.29}$$

$n \in \mathbb{N}$, where $\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ and $\mathcal{D}'^{(n)}_{\mathbb{C}} = \operatorname{ind\,lim}_{\widetilde{\tau}\in T} \mathcal{H}^{(n)}_{-\widetilde{\tau},\mathbb{C}}$ are the spaces dual of $\mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}$ and $\mathcal{D}^{\widehat{\otimes}n}_{\mathbb{C}}$ with respect to $\mathcal{H}^{(n)}_{ext}$. Set $\mathcal{D}^{\widehat{\otimes}0}_{\mathbb{C}} = \mathcal{H}^{\widehat{\otimes}0}_{\tau,\mathbb{C}} = \mathcal{H}^{(0)}_{ext} = \mathcal{H}^{(0)}_{-\tau,\mathbb{C}} = \mathcal{D}'^{(0)}_{\mathbb{C}} := \mathbb{C}$. In what follows we denote by $\langle\cdot,\cdot\rangle_{ext}$ the real (bilinear) dual pairings between elements of negative and positive spaces from chains (1.29), these pairings are generated by the scalar products in $\mathcal{H}^{(n)}_{ext}$.

The next statement follows from the general duality theory (cf. [17, 21]).

Proposition 1.18. (cf. Proposition 1.11) *There exists a system of generalized functions*

$$\left\{ :\langle\circ^{\otimes n}, F^{(n)}_{ext}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \mid F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}},\ n \in \mathbb{Z}_+ \right\}$$

such that

1) for $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{ext} \subset \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ the generalized function $:\langle\circ^{\otimes n}, F^{(n)}_{ext}\rangle:$ is a Wick monomial that is defined in Subsection 1.2;

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ can be uniquely represented as a series

$$F = \sum_{n=0}^{\infty} :\langle\circ^{\otimes n}, F^{(n)}_{ext}\rangle:,\ \ F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}, \tag{1.30}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, i.e.,

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{n=0}^{\infty} 2^{-qn} |F^{(n)}_{ext}|^2_{\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}} < \infty; \tag{1.31}$$

and, vice versa, any series (1.30) with finite norm (1.31) is a generalized function from $(\mathcal{H}_{-\tau})_{-q}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q}$);

3) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_q$ that is generated by the scalar product in (L^2), has a form $\langle\langle F, f\rangle\rangle_{(L^2)} = \sum_{n=0}^{\infty} n! \langle F^{(n)}_{ext}, f^{(n)}\rangle_{ext}$, where $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ and $f^{(n)} \in \mathcal{H}^{\widehat{\otimes}n}_{\tau,\mathbb{C}}$ are the kernels from decompositions (1.30) and (1.26) for F and f, respectively;

4) $F \in (\mathcal{H}_{-\tau})$ (respectively, $F \in (\mathcal{D}')$) if and only if F can be uniquely represented in form (1.30) and norm (1.31) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (respectively, for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

1.7. **An extended stochastic integral on $(\mathcal{H}_{-\tau})_{-q}$.** In Subsection 1.3 we described a generalized Wiener-Itô-Segal isomorphism $\mathbf{I}$ between the space of square integrable random variables (L^2) and the weighted extended symmetric Fock space. It is clear that the restrictions of $\mathbf{I}$ to the spaces $(\mathcal{H}_\tau)_q \subset (L^2)$ are isometric isomorphisms between $(\mathcal{H}_\tau)_q$ and the weighted symmetric Fock spaces $\bigoplus_{n=0}^{\infty}(n!)^2 2^{qn}\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$ (cf. [29]). Therefore, for arbitrary $n \in \mathbb{Z}_+$ and $f_\cdot^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{\tau,\mathbb{C}} \subset \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ the Wick monomial $:\langle \circ^{\otimes n}, f_\cdot^{(n)}\rangle:$ (see (1.9)) belongs to $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}} \subset (L^2) \otimes \mathcal{H}_{\mathbb{C}}$ and, moreover, such monomials form orthogonal bases in the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ in the sense that $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ if and only if f can be uniquely represented as (cf. (1.10))

$$f \equiv f(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_\cdot^{(n)}\rangle:$$

(the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$), with

$$\|f\|^2_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}} = \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f_\cdot^{(n)}|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \otimes \mathcal{H}_{\tau,\mathbb{C}}} < \infty.$$

So, as in the case of the spaces $(\mathcal{H}_{-\tau})_{-q}$, it follows from the general duality theory that in each space $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$, $\tau \in T$, $q \in \mathbb{N}_{q_0(\tau)}$ (see Proposition 1.15), there exists a system of generalized functions

$$\big\{:\langle \circ^{\otimes n}, F_{ext,\cdot}^{(n)}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}} \mid F_{ext,\cdot}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}},\ n \in \mathbb{Z}_+\big\}$$

such that for $F_{ext,\cdot}^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ the generalized function $:\langle \circ^{\otimes n}, F_{ext,\cdot}^{(n)}\rangle:$ is a Wick monomial that is defined by formula (1.9); any generalized function $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ can be uniquely represented as a series

$$F \equiv F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F_{ext,\cdot}^{(n)}\rangle:,\ \ F_{ext,\cdot}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}, \tag{1.32}$$

that converges in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$, i.e.,

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} = \sum_{n=0}^{\infty} 2^{-qn}|F_{ext,\cdot}^{(n)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}} < \infty; \tag{1.33}$$

and, vice versa, any series (1.32) with finite norm (1.33) is a generalized function from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$). Moreover, it is clear that $F \in (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau,\mathbb{C}} := \operatorname{ind\,lim}_{q' \to +\infty}(\mathcal{H}_{-\tau})_{-q'} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ (respectively, $F \in (\mathcal{D}') \otimes \mathcal{D}'_{\mathbb{C}} := \operatorname{ind\,lim}_{\widetilde{\tau} \in T, q' \to +\infty} (\mathcal{H}_{-\widetilde{\tau}})_{-q'} \otimes \mathcal{H}_{-\widetilde{\tau},\mathbb{C}}$) if and only if F can be uniquely represented in form (1.32) and norm (1.33) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (respectively, for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

Let us describe the construction of an extended stochastic integral over a Lévy process L, that is based on decomposition (1.32) (a detailed presentation is given in [21], see also [24]). We need a small preparation. Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes}n}_{\mathbb{C}} \supset \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n} \supset \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n} \equiv L^2(\mathbb{R}_+)_{\mathbb{C}}^{\widehat{\otimes}n} \supset \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \supset \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}, \tag{1.34}$$

$n \in \mathbb{N}$ (as is known (cf. [4]), $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}$ and $\mathcal{D}'^{\widehat{\otimes}n}_{\mathbb{C}} = \operatorname{ind\,lim}_{\widetilde{\tau} \in T} \mathcal{H}_{-\widetilde{\tau},\mathbb{C}}^{\widehat{\otimes}n}$ are the spaces dual of $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$). Set $\mathcal{D}'^{\widehat{\otimes}0}_{\mathbb{C}} = \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}0} = \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}0} = \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}0} = \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}0} := \mathbb{C}$. Since the spaces of test functions in chains (1.34) and (1.29) coincide, there exists a family of natural isomorphisms

$$U_n : \mathcal{D}'_{\mathbb{C}}{}^{(n)} \to \mathcal{D}'_{\mathbb{C}}{}^{\widehat{\otimes} n},\ n \in \mathbb{Z}_+,$$

such that for all $F_{ext}^{(n)} \in \mathcal{D}'_{\mathbb{C}}{}^{(n)}$ and $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$\langle F_{ext}^{(n)}, f^{(n)} \rangle_{ext} = \langle U_n F_{ext}^{(n)}, f^{(n)} \rangle.$$

It is easy to see that the restrictions of U_n to the spaces $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ are *isometric* isomorphisms between the spaces $\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ and $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n}$.

Remark 1.19. Recall that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, therefore in the case $n = 1$ chains (1.34) and (1.29) coincide. Hence U_1 is the identity operator on $\mathcal{D}'_{\mathbb{C}}{}^{(1)} = \mathcal{D}'_{\mathbb{C}}$. In the case $n = 0$ U_0 is, obviously, the identity operator on $\mathbb{C}$.

Definition 1.20. Let $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$. We define an extended stochastic integral over a Lévy process $\int F(t)\widehat{d}L_t \in (\mathcal{H}_{-\tau})_{-q}$ by setting

$$\int F(t)\widehat{d}L_t := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \widehat{F}_{ext}^{(n)} \rangle: \tag{1.35}$$

(cf. (1.13)), where

$$\widehat{F}_{ext}^{(n)} := U_{n+1}^{-1}\{Pr[(U_n \otimes \mathbf{1})F_{ext,\cdot}^{(n)}]\} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+1)}, \tag{1.36}$$

Pr is the symmetrization operator (more exactly, the orthoprojector acting for each $n \in \mathbb{Z}_+$ from $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ to $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$), $F_{ext,\cdot}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ are the kernels from decomposition (1.32) for F.

By analogy with [21, 16, 26] it is easy to verify that

$$\Big\| \int F(t)\widehat{d}L_t \Big\|_{(\mathcal{H}_{-\tau})_{-q}} \le 2^{-q/2} \|F\|_{(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}},$$

hence this definition is well-posed and, moreover, the extended stochastic integral

$$\int \circ(t)\widehat{d}L_t : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}} \to (\mathcal{H}_{-\tau})_{-q} \tag{1.37}$$

is a linear *bounded* and, therefore, *continuous* operator.

The following assertion is a trivial modification of the corresponding statement from [21].

Proposition 1.21. *Extended stochastic integral* (1.37) *is an extension of integral* (1.14) *and therefore is an extension of the Itô stochastic integral over a Lévy process.*

Note that, as is easily seen, an extended stochastic integral can be defined by (1.35), (1.36) as a linear continuous operator acting from $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ to $(\mathcal{H}_{-\tau})$, or from $(\mathcal{D}') \otimes \mathcal{D}'_{\mathbb{C}}$ to $(\mathcal{D}')$.

Remark 1.22. Integral (1.37) cannot be redefined as an integral with respect to a measurable set $\Delta \neq \mathbb{R}_+$, because it is impossible to multiply elements of the space $\mathcal{H}_{-\tau,\mathbb{C}}$ by the indicator 1_Δ, generally speaking. However, one can easily pass over this problem, introducing a stochastic integral on the space $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{\mathbb{C}}$, see [21, 16, 26] for details.

Finally, we note that the operator adjoint to extended stochastic integral (1.37) is a Hida stochastic derivative $\partial_\cdot : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ that is the restriction to the space $(\mathcal{H}_\tau)_q$ of operator (1.18).

1.8. **A generalized Hida derivative on $(\mathcal{H}_{-\tau})_{-q}$.** In contrast to the regular case, the extended stochastic integral cannot be naturally restricted to the spaces of nonregular test functions [23, 24]. The Hida stochastic derivative, in turn, has no a natural extension to the spaces of nonregular generalized functions [24]. Nevertheless, one can define its natural analog—a generalized Hida derivative on the mentioned spaces. Now we describe the construction of this derivative, the interested reader can find more details in [23, 24, 28].

Definition 1.23. We define a generalized Hida derivative

$$\widetilde{\partial}_{\cdot} : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau,\mathbb{C}} \tag{1.38}$$

by setting for each $F \in (\mathcal{H}_{-\tau})_{-q}$

$$\widetilde{\partial}_{\cdot} F := \sum_{n=0}^{\infty} (n+1) {:}\langle \circ^{\otimes n}, F_{ext}^{(n+1)}(\cdot)\rangle{:}, \tag{1.39}$$

(cf. (1.17)), where

$$F_{ext}^{(n+1)}(\cdot) := (U_n \otimes \mathbf{1})^{-1}(U_{n+1}F_{ext}^{(n+1)})(\cdot) \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}, \tag{1.40}$$

$F_{ext}^{(n+1)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+1)}$ are the kernels from decomposition (1.30) for F.

The well-posedness of this definition, as well as the fact that operator (1.38) is linear and *bounded* (and, therefore, *continuous*) from the results of [24] follow. It's also clear that the generalized Hida derivative can be introduced by formulas (1.39), (1.40) as a linear continuous operator acting from $(\mathcal{H}_{-\tau})$ to $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau,\mathbb{C}}$, or from $(\mathcal{D}')$ to $(\mathcal{D}') \otimes \mathcal{D}'_{\mathbb{C}}$.

Finally, we note that the operator adjoint to the generalized Hida derivative (1.38) is a so-called *generalized stochastic integral* on the space $(\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ [23]—a natural analog of the extended stochastic integral on the mentioned space.

2. Clark-Ocone type formulas and related topics

Let $F \in (\mathcal{H}_{-\tau})_{-q}$ (remind that we accepted on default $\tau \in T$, $q \in \mathbb{N}_{q_0(\tau)}$, see Proposition 1.15). We define an expectation

$$\mathbf{E}F := \langle\!\langle F, 1 \rangle\!\rangle_{(L^2)} = F_{ext}^{(0)} \in \mathbb{C}, \tag{2.41}$$

where $F_{ext}^{(0)}$ is a kernel from decomposition (1.30) for F. It is clear that if $F \in (L^2) \subset (\mathcal{H}_{-\tau})_{-q}$ then $\mathbf{E}F$ is the usual (classical) expectation.

2.1. **Belonging of a random variable to the range of values of the extended stochastic integral.** It is clear that in order to talk about Clark-Ocone type formulas for nonregular generalized functions, at first it is necessary to find out the conditions under which $F \in (\mathcal{H}_{-\tau})_{-q}$ can be represented in the form

$$F = \mathbf{E}F + \int G(t)\widehat{d}L_t \tag{2.42}$$

with some integrable G. It turns out, unlike the regular case [27], now no additional restrictions for F are required. More exactly, we have the following statement.

Theorem 2.1. *Any $F \in (\mathcal{H}_{-\tau})_{-q}$ can be represented in form (2.42) with some $G \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$.*

Proof. Let $F_{ext}^{(n+1)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+1)}$, $n \in \mathbb{Z}_+$, be the kernels from decomposition (1.30) for F. We put

$$G_{ext,\cdot}^{(n)} := F_{ext}^{(n+1)}(\cdot) = (U_n \otimes \mathbf{1})^{-1}(U_{n+1}F_{ext}^{(n+1)})(\cdot) \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}} \tag{2.43}$$

(see (1.40)),

$$G(\cdot):=\sum_{n=0}^{\infty}:\langle\circ^{\otimes n},G_{ext,\cdot}^{(n)}\rangle:.$$

Then by (1.33), (2.43), isometricity of the operators U_n, $\mathbf{1}$, U_{n+1}, and (1.31)

$$\|G\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau,\mathbb{C}}}=\sum_{n=0}^{\infty}2^{-qn}|G_{ext,\cdot}^{(n)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}\otimes\mathcal{H}_{-\tau,\mathbb{C}}}$$

$$=\sum_{n=0}^{\infty}2^{-qn}|(U_{n+1}F_{ext}^{(n+1)})(\cdot)|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}\otimes\mathcal{H}_{-\tau,\mathbb{C}}}=\sum_{n=0}^{\infty}2^{-qn}|U_{n+1}F_{ext}^{(n+1)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n+1}}$$

$$=2^q\sum_{n=0}^{\infty}2^{-q(n+1)}|F_{ext}^{(n+1)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n+1)}}\le 2^q\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}}<\infty,$$

therefore $G\in(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$. Futher, by (1.35), (1.36) and (2.43) we have

$$\int G(t)\widehat{d}L_t=\sum_{n=0}^{\infty}:\langle\circ^{\otimes n+1},U_{n+1}^{-1}\{Pr[(U_n\otimes\mathbf{1})G_{ext,\cdot}^{(n)}]\}\rangle:$$

$$=\sum_{n=0}^{\infty}:\langle\circ^{\otimes n+1},U_{n+1}^{-1}\{Pr[(U_n\otimes\mathbf{1})(U_n\otimes\mathbf{1})^{-1}(U_{n+1}F_{ext}^{(n+1)})(\cdot)]\}\rangle:$$

$$=\sum_{n=0}^{\infty}:\langle\circ^{\otimes n+1},U_{n+1}^{-1}\{Pr[(U_{n+1}F_{ext}^{(n+1)})(\cdot)]\}\rangle:=\sum_{n=0}^{\infty}:\langle\circ^{\otimes n+1},U_{n+1}^{-1}U_{n+1}F_{ext}^{(n+1)}\rangle:$$

$$=\sum_{n=0}^{\infty}:\langle\circ^{\otimes n+1},F_{ext}^{(n+1)}\rangle:$$

($Pr[(U_{n+1}F_{ext}^{(n+1)})(\cdot)]=U_{n+1}F_{ext}^{(n+1)}$ because $U_{n+1}F_{ext}^{(n+1)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n+1}$), whence equality (2.42) follows. □

Remark 2.2. It is worth noting that an integrand G in representation (2.42) is not unique for a given F, generally speaking.

2.2. **A Clark-Ocone type formula on** $(\mathcal{H}_{-\tau})_{-q}$**.** As previously noted, the role of the Hida stochastic derivative on the spaces $(\mathcal{H}_{-\tau})_{-q}$ is played by the generalized Hida derivative $\widetilde{\partial}_\cdot$, see Subsection 1.8. Therefore, now it is natural to construct Clark-Ocone type formulas using exactly the operator $\widetilde{\partial}_\cdot$.

Definition 2.3. We define an operator $B:(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau,\mathbb{C}}\to(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$ by setting

$$(BG)(\cdot):=\sum_{n=0}^{\infty}\frac{1}{n+1}:\langle\circ^{\otimes n},G_{ext,\cdot}^{(n)}\rangle:,\tag{2.44}$$

where $G_{ext,\cdot}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$ are the kernels from decomposition (1.32) for $G\in(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$.

It is clear that B is a linear continuous operator.

Theorem 2.4. *Let $F\in(\mathcal{H}_{-\tau})_{-q}$. Then we have a representation (a Clark-Ocone type formula)*

$$F=\mathbf{E}F+\int B\widetilde{\partial}_tF\widehat{d}L_t,\tag{2.45}$$

where $B\widetilde{\partial}_\cdot F\in(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau,\mathbb{C}}$, $\widetilde{\partial}_\cdot F$ is the generalized Hida derivative of F (cf. (0.2), (2.42)).

Proof. It follows from (1.39), (2.44) and the properties of the operators $\widetilde{\partial}.$ and B that for $F \in (\mathcal{H}_{-\tau})_{-q}$

$$B\widetilde{\partial}.F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F_{ext}^{(n+1)}(\cdot)\rangle: \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau,\mathbb{C}},$$

where $F_{ext}^{(n+1)}(\cdot) \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ are constructed by the kernels $F_{ext}^{(n+1)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+1)}$ from decomposition (1.30) for F, with using formula (1.40). Using (2.43), one can conclude from the proof of Theorem 2.1 that actually $B\widetilde{\partial}.F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$ and equality (2.45) is fulfilled. □

Remark 2.5. At the moment we do not know how to construct a natural conditional expectation on the spaces of nonregular generalized functions, and therefore we cannot construct a direct analog of Clark-Ocone formula (0.2) in a general case. However, we'll consider some particular cases in the next subsection.

2.3. **Clark-Ocone type formulas on subsets of** $(\mathcal{H}_{-\tau})_{-q}$**.** As is known, in contrast to the Gaussian and Poissonian analysis, in the Lévy analysis the spaces of nonregular test functions are not embedded into the corresponding spaces of regular test functions, generally speaking. However, we have the following statement.

Proposition 2.6. *Let $q, q' \in \mathbb{Z}_+$ be such that $q \geq q' + \log_2 c(\tau)$, where $c(\tau) > 0$ is introduced in Proposition 1.13. Then $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into $(L^2)_{q'}^0$.*

Proof. Using (1.20), (1.24) and (1.27), for $f \in (\mathcal{H}_\tau)_q$ we obtain

$$\begin{aligned}\|f\|_{(L^2)_{q'}^0}^2 &= \sum_{n=0}^{\infty} n!2^{q'n}|f^{(n)}|_{\mathcal{H}_{ext}^{(n)}}^2 \leq \sum_{n=0}^{\infty}(n!)^2 2^{q'n}c(\tau)^n|f^{(n)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}}^2 \\ &\leq \sum_{n=0}^{\infty}(n!)^2 2^{qn}|f^{(n)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}}^2 = \|f\|_{(\mathcal{H}_\tau)_q}^2 < \infty,\end{aligned} \tag{2.46}$$

where $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \subset \mathcal{H}_{ext}^{(n)}$ are the kernels from decomposition (1.26) for f. Further, if for some $f \in (\mathcal{H}_\tau)_q$ we have $\|f\|_{(L^2)_{q'}^0} = 0$ then by (1.20) for each $n \in \mathbb{Z}_+$ $|f^{(n)}|_{\mathcal{H}_{ext}^{(n)}} = 0$, therefore $|f^{(n)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}} = 0$ (because $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n} \subset \mathcal{H}_{ext}^{(n)}$) and consequently $\|f\|_{(\mathcal{H}_\tau)_q} = 0$. Hence $(\mathcal{H}_\tau)_q \subset (L^2)_{q'}^0$. The density of this embedding is obvious, the continuity from calculation (2.46) follows. □

Taking into account the proved result, one can conclude that *for $q \in \mathbb{N}_{q_0(\tau)}$ (see Proposition 1.15) and $q' \in \mathbb{Z}_+$ such that $q \geq q' + \log_2 c(\tau)$ the space of regular generalized functions $(L^2)_{-q'}^0$ is densely and continuously embedded into the space of nonregular generalized functions $(\mathcal{H}_{-\tau})_{-q}$*. Actually, we have a chain

$$(\mathcal{H}_{-\tau})_{-q} \supset (L^2)_{-q'}^0 \supseteq (L^2) \supseteq (L^2)_{q'}^0 \supset (\mathcal{H}_\tau)_q.$$

Recall that elements of the spaces of regular test and generalized functions $(L^2)_q^\beta$ (for such spaces we accept on default $\beta \in [-1, 1]$, $q \in \mathbb{Z}$, see Subsection 1.5) and, in particular, elements of $(L^2)_{-q'}^0$, in general, cannot be represented in form (2.42) with a regular integrand G. More exactly, we have the following statement.

Theorem 2.7. ([27]) *Let $F \in (L^2)_q^\beta$. The following statements are equivalent:*

(1) *F can be represented in form (2.42) with $G \in (L^2)_q^\beta \otimes \mathcal{H}_\mathbb{C}$ if $\beta \geq 0$, and $G \in (L^2)_{q-1}^\beta \otimes \mathcal{H}_\mathbb{C}$ if $\beta < 0$;*

(2) *for each* $n \in \mathbb{N}\setminus\{1\}$ *the kernel* $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ *from decomposition* (1.7) *for* F *contains a representative* $f^{(n)} \in F^{(n)}$ *such that* $f^{(n)}(t_1, \dots, t_n) = 0$ *if for each* $i \in \{1, \dots, n\}$ *there exists* $j \in \{1, \dots, n\}\setminus\{i\}$ *such that* $t_i = t_j$.

For $n \in \mathbb{N}\setminus\{1\}$ and $t_1, \dots, t_n \in \mathbb{R}_+$ we define

$$\hbar_n(t_1, \dots, t_n) := nPr1_{\{t_1\neq t_n, t_2\neq t_n, \dots, t_{n-1}\neq t_n\}}$$
$$= 1_{\{t_1\neq t_n, t_2\neq t_n, \dots, t_{n-1}\neq t_n\}} + 1_{\{t_n\neq t_{n-1}, t_1\neq t_{n-1}, \dots, t_{n-2}\neq t_{n-1}\}} + \dots + 1_{\{t_2\neq t_1, t_3\neq t_1, \dots, t_n\neq t_1\}}$$

(here, as usual, Pr is the symmetrization operator), and set $\hbar_1 \equiv 1$. Further, for $G_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{Z}_+$, we set

$$\widetilde{G}_\cdot^{(n)}(\cdot_1, \dots, \cdot_n) := \begin{cases} \frac{G_\cdot^{(n)}(\cdot_1,\dots,\cdot_n)}{\hbar_{n+1}(\cdot_1,\dots,\cdot_n,\cdot)}, & \text{if } \hbar_{n+1}(\cdot_1, \dots, \cdot_n, \cdot) \neq 0 \\ 0, & \text{if } \hbar_{n+1}(\cdot_1, \dots, \cdot_n, \cdot) = 0 \end{cases} .$$

It is clear that $\widetilde{G}_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ and $|\widetilde{G}_\cdot^{(n)}|_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \le |G_\cdot^{(n)}|_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}$.

For $G \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ we define

$$(AG)(\cdot) := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, \widetilde{G}_\cdot^{(n)}\rangle:,$$

where the kernels $\widetilde{G}_\cdot^{(n)}$ are constructed by the kernels $G_\cdot^{(n)}$ from decomposition (1.10) for G. It is clear that A is a linear continuous operator in $(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$.

Theorem 2.8. *Let* $F \in (L^2)_{-q'}^0 \subset (\mathcal{H}_{-\tau})_{-q}$ $(q \in \mathbb{N}_{q_0(\tau)},\ q' \in \mathbb{Z}_+,\ q \ge q' + \log_2 c(\tau))$ *can be represented in form* (2.42) *(see Theorem 2.7). Then we have a representation (a Clark-Ocone type formula)*

$$F = \mathbf{E}F + \int A\partial_t F \widehat{d}L_t, \tag{2.47}$$

where $A\partial_\cdot F \in (L^2)_{-q'}^0 \otimes \mathcal{H}_{\mathbb{C}} \subset (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$, $\partial_\cdot F$ *is the Hida derivative of* F *(cf.* (0.2), (2.45)*).*

Proof. The result easily follows from Theorem 2.3.2 and Proposition 2.3.1 in [27]. □

It is worth noting that formula (2.47) (as well as formula (2.45)) is not a direct analog of Clark-Ocone formula (0.2), see [27] for details. So, it is natural to consider a Clark-Ocone type formula that is a straight analog of (0.2) in the Lévy analysis.

For $n \in \mathbb{N}$ and $t_1, \dots, t_n, t \in \mathbb{R}_+$ define

$$\chi_{n,t}(t_1, \dots, t_n) := \begin{cases} 0, & \text{if } \exists i \in \{1, \dots, n\} : t_i \ge t \text{ and } \forall j \in \{1, \dots, n\}\setminus\{i\}\ t_i \neq t_j \\ 1, & \text{in other cases} \end{cases} ,$$

i.e., $\chi_{n,t}(t_1, \dots, t_n) = 0$, if there exists t_i of multiplicity 1 (meaning t_i is not equal to any other t_j, $j \neq i$) that is greater than t or equal to t. Let also $\chi_{0,\cdot} \equiv 1$. Further, for $G \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ we define

$$(\mathbf{E}_\cdot G)(\cdot) := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, G_\cdot^{(n)}\chi_{n,\cdot}\rangle: \equiv G_\cdot^{(0)} + \sum_{n=1}^{\infty} :\langle \circ^{\otimes n}, G_\cdot^{(n)}\chi_{n,\cdot}\rangle: \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}},$$

where $G_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.10) for G. It is clear that $G_\cdot^{(n)}\chi_{n,\cdot} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$, $|G_\cdot^{(n)}\chi_{n,\cdot}|_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}} \le |G_\cdot^{(n)}|_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}_{\mathbb{C}}}$, and therefore $\mathbf{E}_\cdot$ is a linear continuous operator in $(L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$.

Theorem 2.9. (cf. Theorem 2.8) *Let* $F \in (L^2)^0_{-q'} \subset (\mathcal{H}_{-\tau})_{-q}$ *(*$q \in \mathbb{N}_{q_0(\tau)}$*,* $q' \in \mathbb{Z}_+$*,* $q \geq q' + \log_2 c(\tau)$*) can be represented in form* (2.42) *(see Theorem 2.7). Then we have a representation (a Clark-Ocone type formula)*

$$F = \mathbf{E}F + \int \mathbf{E}_t \partial_t F \widehat{d}L_t, \tag{2.48}$$

where $\mathbf{E}_{\cdot}\partial_{\cdot} F \in (L^2)^0_{-q'} \otimes \mathcal{H}_{\mathbb{C}} \subset (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$*,* $\partial_{\cdot} F$ *is the Hida derivative of* F *(cf.* (0.2)*,* (2.45)*,* (2.47)*).*

Proof. The result easily follows from Theorem 2.4.3 and Proposition 2.4.1 in [27]. □

Let $\mathcal{F}_t$, $t \in \mathbb{R}_+$, be the completion with respect to the Lévy white noise measure of the σ-algebra $\sigma(L_u : u \leq t)$ that is generated by the Lévy process L up to time t. For $F \in (L^2)^\beta_q$, $\beta \in [-1, 0)$ and $q \in \mathbb{Z}$, or $\beta = 0$ and $-q \in \mathbb{N}$ (i.e., for generalized random variables), we define an expectation $\mathbf{E}F$ by formula (2.41), and a conditional expectation $\mathbf{E}\big(F|_{\mathcal{F}_t}\big)$ by setting

$$\mathbf{E}\big(F|_{\mathcal{F}_t}\big) := F^{(0)} + \sum_{n=1}^{\infty} :\langle \circ^{\otimes n}, F^{(n)} 1_{[0,t)^n} \rangle: \in (L^2)^\beta_q,$$

where $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decomposition (1.7) for F. For $F \in (L^2) \subset (L^2)^\beta_q$, as is easy to see, $\mathbf{E}F$ is the usual expectation of F; and one can show by analogy with the proof of Theorem 4.2 in [29] that $\mathbf{E}\big(F|_{\mathcal{F}_t}\big)$ is the conditional expectation of F with respect to $\mathcal{F}_t$.

Based on the definition above, for $G \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ it is natural to set

$$\mathbf{E}\big(G(\cdot)|_{\mathcal{F}_\cdot}\big) := G^{(0)}_\cdot + \sum_{n=1}^{\infty} :\langle \circ^{\otimes n}, G^{(n)}_\cdot 1_{[0,\cdot)^n} \rangle: \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}},$$

where $G^{(n)}_\cdot \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ are the kernels from decomposition (1.10) for G. It is clear that $G^{(n)}_\cdot 1_{[0,\cdot)^n} \in \mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}$ and $|G^{(n)}_\cdot 1_{[0,\cdot)^n}|_{\mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}} \leq |G^{(n)}_\cdot|_{\mathcal{H}^{(n)}_{ext} \otimes \mathcal{H}_{\mathbb{C}}}$, therefore $\mathbf{E}\big(\circ(\cdot)|_{\mathcal{F}_\cdot}\big)$ is a linear continuous operator in $(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$.

Theorem 2.10. ([27], cf. Theorem 2.7) *Let* $F \in (L^2)^\beta_q$*. The following statements are equivalent:*

(1) *F can be represented in the form*

$$F = \mathbf{E}F + \int \mathbf{E}\big(\partial_t F|_{\mathcal{F}_t}\big) \widehat{d}L_t \tag{2.49}$$

with $\mathbf{E}\big(\partial_\cdot F|_{\mathcal{F}_\cdot}\big) \in (L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ *if* $\beta \geq 0$*, and* $\mathbf{E}\big(\partial_\cdot F|_{\mathcal{F}_\cdot}\big) \in (L^2)^\beta_{q-1} \otimes \mathcal{H}_{\mathbb{C}}$ *if* $\beta < 0$*;*

(2) *for each* $n \in \mathbb{N}\backslash\{1\}$ *the kernel* $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$ *from decomposition* (1.7) *for* F *contains a representative* $f^{(n)} \in F^{(n)}$ *such that* $f^{(n)}(t_1, \dots, t_n) = 0$ *if there exist* $i, j \in \{1, \dots, n\}$*,* $i \neq j$*, such that* $\max\{t_1, \dots, t_n\} = t_i = t_j$ *(i.e., if the multiplicity of the greatest* $t_\cdot \in \{t_1, \dots, t_n\}$ *is greater than one).*

Corollary 2.11. *Let* $F \in (L^2)^0_{-q'} \subset (\mathcal{H}_{-\tau})_{-q}$ *(*$q \in \mathbb{N}_{q_0(\tau)}$*,* $q' \in \mathbb{Z}_+$*,* $q \geq q' + \log_2 c(\tau)$*) satisfy assumption* (2) *of Theorem 2.10. Then F can be represented in form* (2.49)*, where* $\mathbf{E}\big(\partial_\cdot F|_{\mathcal{F}_\cdot}\big) \in (L^2)^0_{-q'} \otimes \mathcal{H}_{\mathbb{C}} \subset (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$*.*

It is worth noting that, as is proved in [27], if $F \in (L^2)^\beta_q$ can be represented in form (2.49) then

$$\mathbf{E}\big(\partial_\cdot F|_{\mathcal{F}_\cdot}\big) = \mathbf{E}_\cdot \partial_\cdot F \tag{2.50}$$

in $(L^2)^\beta_q \otimes \mathcal{H}_{\mathbb{C}}$ if $\beta \geq 0$, and in $(L^2)^\beta_{q-1} \otimes \mathcal{H}_{\mathbb{C}}$ if $\beta < 0$ (cf. (2.49) and (2.48)). In particular, equality (2.50) is true for $F \in (L^2)^0_{-q'} \subset (\mathcal{H}_{-\tau})_{-q}$ $(q \in \mathbb{N}_{q_0(\tau)}$, $q' \in \mathbb{Z}_+$, $q \geq q' + \log_2 c(\tau))$

that can be represented in form (2.49). So, formula (2.48) is a direct analog of Clark-Ocone formula (0.2) in the Lévy analysis.

Remark 2.12. Let $F \in (L^2)^\beta_{q'} \cap (\mathcal{H}_{-\tau})_{-q}$ with some $\beta \in [-1,1]$, $q' \in \mathbb{Z}$, $\tau \in T$ and $q \in \mathbb{N}_{q_0(\tau)}$. Now if F can be represented in form (2.42) (see Theorem 2.7) then representations (2.47), (2.48) hold true and the integrands belong to the space $(L^2)^\beta_{q'} \otimes \mathcal{H}_{\mathbb{C}}$ for $\beta \geq 0$, and to the space $(L^2)^\beta_{q'-1} \otimes \mathcal{H}_{\mathbb{C}}$ for $\beta < 0$ (actually this statement is true for all $F \in (L^2)^\beta_{q'}$ that can be represented in form (2.42)) [27]; but there is no evidence to suggest that these integrands belong to the space $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau,\mathbb{C}}$, generally speaking.

Remark 2.13. If $F \in (L^2)^\beta_q$, but assumption (2) (or assumption (1), which is the same) of Theorem 2.7 is not fulfilled, one still can construct the integrands $A\partial.F, \mathbf{E}.\partial.F \in (L^2)^\beta_{q-1} \otimes \mathcal{H}_{\mathbb{C}}$ and the corresponding stochastic integrals. But, of course, representations (2.47) and (2.48) will not be fair.

Finally, we note that all results of this paper hold true (up to obvious modifications), if one considers the "boundary" spaces $(\mathcal{H}_{-\tau})$, $(\mathcal{D}')$ instead of $(\mathcal{H}_{-\tau})_{-q}$.

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