Methods of Functional Analysis and Topology
Vol. 31 (2025), no. 2, pp. 136–152
doi.org/10.31392/MFAT-npu26_2.2025.04

THE PROJECTION SPECTRAL THEOREM, QUASI-FREE STATES AND POINT PROCESSES

EUGENE LYTVYNOV

Dedicated to the 100th anniversary of Yuri Makarovych Berezansky's birth

Abstract. In this review paper, we demonstrate that several classes of point processes in a locally compact Polish space X appear as the joint spectral measure of a rigorously defined particle density of a representation of the canonical anticommutation relations (CAR) or the canonical commutation relations (CCR) in a Fock space. For these representations of the CAR/CCR, the vacuum state on the corresponding $*$-algebra is quasi-free. The classes of point process that arise in such a way include determinantal and permanental point processes.

1. Introduction

The projection spectral theorem for a family of commuting self-adjoint (or more generally, normal) operators and the corresponding Fourier transform in generalized joint eigenvectors of this family play a fundamental role in functional analysis and its applications, in particular, to infinite dimensional analysis and mathematical physics. In the case where the operator family has additionally a cyclic vector, this result allows one to realize the operators from the family as multiplication operators in an L^2-space with respect to a probability measure. This theory was studied by many authors, including Yu.M. Berezansky, L. Gårding, I.M. Gelfand, A.G. Kostyuchechko, G. Kats, and K. Maurin. In the most general setting, this result was proved by Berezansky in [5], see also [6,7] and Chapter 3 in [9]. In particular, the theory of Berezansky allowed one to consider families of an uncountable number of operators

The projection spectral theorem below (Theorem 1.1) follows immediately from the results of [9, Chapter 3]. It is presented here in the form as in our review paper [16]. But first, let us remind the reader that we say that $\Phi \subset H \subset \Phi'$ is a standard triple if H is a separable Hilbert space, Φ is a nuclear space that is densely and continuously embedded into H, and Φ' is the dual space of Φ with respect to the center space H. The latter means that the dual pairing between $\omega \in \Phi'$ and $\varphi \in \Phi$, denoted by $\langle \omega, \varphi \rangle$, is determined by the inner product in H. Note that a standard triple can be real (in which case all three spaces in the triple are real) or complex (in which case all three spaces in the triple are complex, and one also typically assumes that the complex Hilbert space H is the complexification of a real Hilbert space $H_{\mathbb{R}}$).

Theorem 1.1. *Let $\Phi \subset H \subset \Phi'$ be a real standard triple, and let $\Psi \subset \mathcal{F} \subset \Psi'$ be a complex standard triple. Assume we are given a family $(A(\varphi))_{\varphi\in\Phi}$ of Hermitian operators in $\mathcal{F}$ such that*

1. *$D(A(\varphi)) = \Psi$, $\varphi \in \Phi$;*
2. *$A(\varphi)\Psi \subset \Psi$ for each $\varphi \in \Phi$, and furthermore $A(\varphi) : \Psi \to \Psi$ is continuous;*
3. *$A(\varphi_1)A(\varphi_2)f = A(\varphi_2)A(\varphi_1)f$ for each $\varphi_1, \varphi_2 \in \Phi$ and $f \in \Psi$;*
4. *for all $f, g \in \Psi$, the mapping $\Phi \ni \varphi \mapsto (A(\varphi)f, g)_{\mathcal{F}} \in \mathbb{C}$ is continuous;*

2020 *Mathematics Subject Classification.* 46L30, 47L60, 60G55.
Keywords. Projection spectral theorem, quasi-free state, determinantal point process, permanental point process, Poisson point process, hafnian point process.

(5) *there exists a vector Ω in Ψ which is cyclic for $(A(\varphi))_{\varphi\in\Phi}$, i.e., the linear span of the set*

$$\{\Omega,\ A(\varphi_1)\cdots A(\varphi_k)\Omega \mid \varphi_1,\dots,\varphi_k\in\Phi,\ k\in\mathbb{N}\}$$

is dense in $\mathcal{F}$;

(6) *for any $f\in\Psi$ and $\varphi\in\Phi$, the vector f is analytic for the operator $A(\varphi)$, i.e., for some $t>0$,*

$$\sum_{n=0}^{\infty} t^n\,\frac{\|(A(\varphi)^n f\|_{\mathcal{F}}}{n!}<\infty.$$

Then, each operator $A(\varphi)$, $\varphi\in\Phi$, is essentially self-adjoint in $\mathcal{F}$, and we denote its closure by $(\tilde A(\varphi), D(\tilde A(\varphi)))$. The operators $(\tilde A(\varphi))_{\varphi\in\Phi}$ commute in the sense of commutation of their projection-valued measures (resolutions of the identity). Furthermore, there exists a unique probability measure μ on $(\Phi',\mathcal{C}(\Phi'))$ such that the linear operator $I:\mathcal{F}\to L^2(\Phi',\mu)$ satisfying $I\Omega=1$ and

$$\begin{aligned} I(\tilde A(\varphi_1)\cdots\tilde A(\varphi_n)\Omega)&=I(A(\varphi_1)\cdots A(\varphi_n)\Omega)\\ &=\langle\cdot,\varphi_1\rangle\cdots\langle\cdot,\varphi_n\rangle\in L^2(\Phi',\mu)\end{aligned}$$

is unitary. Here, $\mathcal{C}(\Phi')$ denotes the cylinder σ-algebra on Φ'.

Under the action of I, each operator $(\tilde A(\varphi), D(\tilde A(\varphi)))$, $\varphi\in\Phi$, becomes the operator of multiplication by $\langle\cdot,\varphi\rangle$ in $L^2(\Phi',\mu)$, denoted by $M(\varphi)$, i.e.,

$$D(M(\varphi))=\big\{F\in L^2(\Phi',\mu)\mid \langle\cdot,\varphi\rangle F\in L^2(\Phi',\mu)\big\}$$

and for each $F\in D(M(\varphi))$,

$$(M(\varphi)F)(\omega)=\langle\omega,\varphi\rangle F(\omega).$$

The probability measure μ on Φ' from Theorem 1.1 is called the joint spectral measure of the family of (commuting self-adjoint) operators $(\tilde A(\varphi))_{\varphi\in\Phi}$ (with respect to the cyclic vector Ω).

Berezansky et al. [10] applied the projection spectral theorem to study of point processes and their correlation measures. The paper [10] was influenced by Kondratiev and Kuna's studies [13] of harmonic analysis on configuration spaces.

Let X be a locally compact Polish space, let $\Gamma(X)$ denote the set of simple configurations in X, and equip $\Gamma(X)$ with the cylinder σ-algebra $\mathcal{C}(\Gamma(X))$. Let $\Gamma_0(X)$ denote the (measurable) subset of $\Gamma(X)$ that consists of all finite configurations in X. A point process in X is a probability measure on $\Gamma(X)$. To each point process μ in X, there corresponds its correlation measure $\theta=\theta_\mu$, which is a (generally speaking infinite) measure on $\Gamma_0(X)$. The measure θ can be defined through the K-transform, which maps functions on $\Gamma_0(X)$ to functions on $\Gamma(X)$. In paper [13], the K-transform was studied as an operator $K:L^1(\Gamma_0(X),\theta)\to L^1(\Gamma(X),\mu)$ that satisfies[1]

$$\int_{\Gamma_0(X)} G(\eta)\,\theta(d\eta)=\int_{\Gamma(X)}(KG)(\gamma)\,\mu(d\gamma),\quad G\in L^1(\Gamma_0(X),\theta).\tag{1.1}$$

Furthermore, a convolution $\star$ of functions on $\Gamma_0(X)$ was explicitly constructed in [13], and this convolution satisfies $(K(G_1\star G_2))(\gamma)=(KG_1)(\gamma)(KG_2)(\gamma)$.

Let θ be the correlation measure of a point process μ. Define

$$\mathcal{R}_\theta=\{G\in L^1(\Gamma_0(X),\theta): KG\in L^2(\Gamma(X),\mu)\}.\tag{1.2}$$

The measure θ is $\star$-positive definite in the following sense:

$$\int_{\Gamma_0(X)}(G\star\overline{G})(\eta)\,\theta(d\eta)\ge 0\quad\text{for each } G\in\mathcal{R}_\theta.\tag{1.3}$$

[1] In fact, for a given point process μ, formula (1.1) uniquely determines its correlation measure θ.

Hence, $\mathcal{R}_\theta$ can be naturally interpreted as a complex Hilbert space with the inner product

$$(G_1, G_2)_{\mathcal{R}_\theta} = \int_{\Gamma_0(X)} (G_1 \star \overline{G_2})(\eta)\, \theta(d\eta), \quad G_1, G_2 \in \mathcal{R}_\theta. \tag{1.4}$$

Now assume that X is a Riemannian manifold and θ is a measure on $\Gamma_0(X)$ that is $\star$-positive definite, and satisfies some additional technical assumptions. We define a complex Hilbert space $\mathcal{R}_\theta$ of functions on $\Gamma_0(X)$, with the inner product given by (1.4). It is not difficult to construct a standard triple $\Psi \subset \mathcal{R}_\theta \subset \Psi'$.

Let $\mathcal{D}(X)$ denote the nuclear space of all smooth functions $\varphi : X \to \mathbb{R}$ with compact support. For each $\varphi \in \mathcal{D}(X)$, one can define a Hermitian operator $A(\varphi)$ in $\mathcal{R}_\theta$, with domain Ψ, that acts as follows: $A(\varphi)G = \varphi \star G$. In this formula, $\varphi \in \mathcal{D}(X)$ is naturally identified with a function on $\Gamma_0(X)$ (see Section 3 for details). It was proved in [10] that the family of operators $(A(\varphi))_{\varphi \in \mathcal{D}(X)}$ satisfies the conditions of Theorem 1.1. The corresponding spectral measure μ, defined initially on $\mathcal{D}'(X)$, was shown to be concentrated on the configuration space $\Gamma(X)$, and furthermore, μ is nothing but the point process for which θ is its correlation measure. Furthermore the unitary operator $I : \mathcal{R}_\theta \to L^2(\Gamma(X), \mu)$ from Theorem 1.1 is the restriction of the K-transform $K : L^1(\Gamma_0(X), \theta) \to L^1(\Gamma(X), \mu)$ to $\mathcal{R}_\theta \subset L^1(\Gamma_0(X), \theta)$.

The studies of the K-transform as the joint spectral measure of a family of commuting self-adjoint operators were advanced in [17]. That paper dealt with the more general case of a locally compact Polish space X, and it treated a family of commuting Hermitian operators $\rho(\Delta)$, indexed by all pre-compact Borel subsets Δ of X. For such a family of Hermitian operators, the notion of its correlation measure θ on $\Gamma_0(X)$ was given. Note that, unlike the case of a point process, the correlation measure of a family of Hermitian operators does not need to necessarily exist. However, if such a correlation measure θ does exist, then there also exists a point process μ in X for which θ is its correlation measure, while μ is the joint spectral measure of the (closures of the) operators $\rho(\Delta)$.

This result from [17] was applied to several families of Hermitian operators $\rho(\Delta)$ that arise as the particle density of a representation of the canonical anticommutation relations (CAR) or the canonical commutation relations (CCR). These representations are typically associated with a quasi-free state on the corresponding CAR/CCR $*$-algebra. The point processes that were discussed via this ansatz included Poisson point processes, determinantal and permanental point processes, and hafnian point processes. The aim of this paper is to briefly review these results.

The paper is organized as follows. In Section 2, we will recall some key definitions and results related to the K-transform. In Section 3, we will describe the main result of paper [10], while in Section 4 we will recall the above mentioned result from [17]. In Section 5, we will recall basic definitions and results related to quasi-free and gauge-invariant states on the CAR and CCR $*$-algebras. In Section 6, we will discuss symmetric and antisymmetric Fock spaces, creation and annihilation operators in them, and the differential second quantization. Finally, in Section 7, we will review several applications of the theorem from [17] to particle densities and point processes. Our key references in Section 7 are [1, 2, 15, 17].

2. Point process and its correlation measure

We mostly present in this section results from [13] (see also the references therein), sometimes in a slightly modified form.

Let X be a locally compact Polish space, let $\mathcal{B}(X)$ denote the Borel σ-algebra on X, and let $\mathcal{B}_0(X)$ denote the algebra of all pre-compact sets from $\mathcal{B}(X)$. Let σ be a measure on $(X, \mathcal{B}(X))$ which is non-atomic (i.e., $\sigma(\{x\}) = 0$ for all $x \in X$) and Radon (i.e., $\sigma(\Delta) < \infty$ for all $\Delta \in \mathcal{B}_0(X)$). For applications, the most important example is $X = \mathbb{R}^d$ and $\sigma(dx) = dx$ is the Lebesgue measure.

A (simple) configuration γ in X is a Radon measure on X of the form $\gamma = \sum_i \delta_{x_i}$, where δ_x denotes the Dirac measure with mass at x, and $x_i \neq x_j$ if $i \neq j$. By definition, the zero measure on X is a configuration. Note that, since γ is a Radon measure, it has a finite number of atoms x_i in each compact set in X, however the total mass of X can be ∞. Let $\Gamma(X)$ denote the set of all configurations γ in X. Let $\mathcal{C}(\Gamma(X))$ denote the minimal σ-algebra on $\Gamma(X)$ with respect to which the map $\Gamma(X) \ni \gamma \mapsto \gamma(\Delta)$ is measurable for each $\Delta \in \mathcal{B}_0(X)$. The σ-algebra $\mathcal{C}(\Gamma(X))$ coincides with the Borel σ-algebra of the vague topology on Γ_X, i.e., it is the minimal σ-algebra on Γ_X with respect to which each map of the form $\Gamma_X \ni \gamma \mapsto \langle \gamma, \varphi \rangle = \int_X \varphi \, d\gamma \in \mathbb{R}$ is continuous, where $\varphi : X \to \mathbb{R}$ is a continuous function with compact support.

A (simple) point process in X is a probability measure on $(\Gamma(X), \mathcal{C}(\Gamma(X)))$.

A measure on X^n is called symmetric if it remains invariant under the natural action of the symmetric group $\mathfrak{S}_n$ on X^n. For each $\gamma = \sum_i \delta_{x_i} \in \Gamma(X)$, the spatial falling factorial $(\gamma)_n$ is the symmetric measure on X^n of the form

$$(\gamma)_n = \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_n \neq i_1, \dots, i_n \neq i_{n-1}} \delta_{(x_{i_1}, x_{i_2}, \dots, x_{i_n})}.$$

The measures $(\gamma)_n$ ($n \in \mathbb{N}$) satisfy the recurrence formula

$$\begin{aligned}
&(\gamma)_1(\Delta) = \gamma(\Delta),\\
&(\gamma)_{n+1}(\Delta_1 \times \cdots \times \Delta_{n+1}) = \gamma(\Delta_{n+1})(\gamma)_n(\Delta_1 \times \cdots \times \Delta_n)\\
&\qquad - \sum_{i=1}^{n} (\gamma)_n(\Delta_1 \times \Delta_{i-1} \times (\Delta_i \cap \Delta_{n+1}) \times \Delta_{i+1} \times \cdots \times \Delta_n), \quad n \in \mathbb{N},
\end{aligned} \tag{2.5}$$

where $\Delta, \Delta_1, \dots, \Delta_{n+1} \in \mathcal{B}_0(X)$. For $n \geq 2$, denote

$$X^{(n)} = \{(x_1, \dots, x_n) \in X^n : x_i \neq x_j \text{ if } i \neq j\}.$$

Let also $X^{(1)} = X$. For $\gamma \in \Gamma(X)$ and $n \in \mathbb{N}$, the measure $(\gamma)_n$ is concentrated on $X^{(n)}$, i.e., $(\gamma)_n(X^n \setminus X^{(n)}) = 0$.

Let μ be a point process in X. For $n \in \mathbb{N}$, the nth correlation measure of μ is the symmetric measure $\theta^{(n)}$ on X^n defined by

$$\theta^{(n)}(dx_1 \cdots dx_n) = \frac{1}{n!} \int_{\Gamma(X)} (\gamma)_n(dx_1 \cdots dx_n) \, \mu(d\gamma).$$

The measure $\theta^{(n)}$ is concentrated on $X^{(n)}$. If each measure $\theta^{(n)}$ is absolutely continuous with respect to $\sigma^{\otimes n}$, then the symmetric functions $k^{(n)} : X^n \to [0, \infty)$ satisfying

$$d\theta^{(n)} = \frac{1}{n!} \, k^{(n)} d\sigma^{\otimes n} \tag{2.6}$$

are called the correlation functions of the point process μ. Under a very weak assumption, the correlation measures (or the correlations functions) uniquely identify a point process, see [14].

Let $\Gamma_0(X) = \{\gamma \in \Gamma(X) : \gamma(X) < \infty\}$. Then $\Gamma_0(X) = \bigcup_{n=0}^{\infty} \Gamma^{(n)}(X)$, where

$$\Gamma^{(n)}(X) = \{\gamma \in \Gamma_0(X) : \gamma(X) = n\}.$$

For each $n \in \mathbb{N}$, consider the map

$$X^{(n)} \ni (x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} \delta_{x_i} \in \Gamma^{(n)}(X). \tag{2.7}$$

Let $\mathcal{B}(\Gamma^{(n)}(X))$ denote the σ-algebra on $\Gamma^{(n)}(X)$ that is the image of the Borel σ-algebra $\mathcal{B}(X^{(n)})$ under the map in (2.7). (Equivalently, instead of $\mathcal{B}(X^{(n)})$, one can use the sub-σ-algebra $\mathcal{B}_{\mathrm{sym}}(X^{(n)})$ that consists of all symmetric sets from $\mathcal{B}(X^{(n)})$). Let $\mathcal{B}(\Gamma_0(X))$ denote the σ-algebra on $\Gamma_0(X)$ whose trace on each $\Gamma^{(n)}(X)$ ($n \in \mathbb{N}$) coincides with $\mathcal{B}(\Gamma^{(n)}(X))$. In fact, $\mathcal{B}(\Gamma_0(X))$ coincides with the trace of the σ-algebra $\mathcal{C}(\Gamma(X))$ on $\Gamma_0(X)$.

Let μ be a point process and let $(\theta^{(n)})_{n=1}^\infty$ be the sequence of its correlation measures. It is convenient to interpret the sequence $(\theta^{(n)})_{n=1}^\infty$ as a single measure θ on $(\Gamma_0(X), \mathcal{B}(\Gamma_0(X)))$. To this end, let us keep the notation $\theta^{(n)}$ for the push-forward of the measure $\theta^{(n)}$ on $X^{(n)}$ under the map in (2.7). Then, we define the measure θ on $(\Gamma_0(X), \mathcal{B}(\Gamma_0(X)))$ that coincides with $\theta^{(n)}$ on each $\Gamma^{(n)}(X)$ for $n \in \mathbb{N}$, and $\theta(\Gamma^{(0)}(X)) = 1$. The measure θ is called the correlation measure (on $\Gamma_0(X)$) of the point process μ.

The correlation measure θ can also be defined through the so-called K-transform. For a configuration $\gamma \in \Gamma(X)$ and a finite configuration $\eta \in \Gamma_0(X)$, we will write $\eta \prec \gamma$ if $\gamma - \eta \in \Gamma(X)$, i.e., the support of η is a subset of the support of γ. For a measurable function $G : \Gamma_0(X) \to [0, \infty)$, we define a measurable function $KG : \Gamma(X) \to [0, \infty]$ by

$$(KG)(\gamma) = \sum_{\eta \in \Gamma_0(X),\ \eta \prec \gamma} G(\eta), \quad \gamma \in \Gamma(X).$$

Then, for a point process μ, its correlation measure θ on $\Gamma_0(X)$ satisfies

$$\theta(\Lambda) = \int_{\Gamma(X)} (K\chi_\Lambda)(\gamma)\, \mu(d\gamma), \quad \Lambda \in \mathcal{B}(\Gamma_0(X)), \tag{2.8}$$

where χ_Λ is the indicator function of the set Λ. Formula (2.8) serves as an equivalent definition of the correlation measure θ.

The K-transform can be extended to act on a class of measurable functions $G : \Gamma_0(X) \to \mathbb{R}$. Indeed, let $G^+(\eta) = \max\{G(\eta), 0\}$, $G^-(\eta) = \max\{-G(\eta), 0\}$, so that $G = G^+ - G^-$. For $\eta \in \Gamma_0(X)$, we define $(KG)(\eta) = (KG^+)(\eta) - (KG^-)(\eta)$, provided both terms $(KG^+)(\eta)$ and $(KG^-)(\eta)$ are finite. Similarly, one defines the K-transform of a function $G : \Gamma_0(X) \to \mathbb{C}$.

If μ is a point process and θ is its correlation measure on $\Gamma_0(X)$, then, for each function $G \in L^1(\Gamma_0(X), \theta)$, the function KG is well-defined μ-a.e., $KG \in L^1(\Gamma(X), \mu)$, and furthermore formula (1.1) holds.

The K-transform leads to a natural convolution of functions on $\Gamma_0(X)$. For any measurable functions $G_1, G_2 : \Gamma_0(X) \to \mathbb{C}$, we define a function $G_1 \star G_2 : \Gamma_0(X) \to \mathbb{C}$ by

$$(G_1 \star G_2)(\eta) = \sum_{\substack{\eta_1, \eta_2 \in \Gamma_0(X) \\ \eta_1 \prec \eta,\ \eta_2 \prec \eta}} G_1(\eta_1) G_2(\eta_2), \quad \eta \in \Gamma_0(X).$$

Then,

$$(K(G_1 \star G_2))(\gamma) = (KG_1)(\gamma)(KG_2)(\gamma) \tag{2.9}$$

for each $\gamma \in \Gamma(X)$ such that both $(KG_1)(\gamma)$ and $(KG_2)(\gamma)$ are well defined and finite.

In view of formula (2.9), formula (1.3) holds, where $\mathcal{R}_\theta$ is defined by (1.2).

3. Spectral representation of the correlation measure

In this section, we will briefly explain the main result of Berezansky et al. [10].

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. Let σ be the volume measure on X.

We will say that a function $G : \Gamma_0(X) \to \mathbb{C}$ has a bounded support if there exist a compact set $\Lambda \subset X$ and $N \in \mathbb{N}$ such that the function G vanishes outside the set $\bigcup_{n=0}^N \Gamma^{(n)}(\Lambda)$, where $\Gamma^{(n)}(\Lambda) = \{\eta \in \Gamma^{(n)}(X) : \eta(\Lambda^c) = 0\}$. Let us denote by $\mathcal{F}(\Gamma_0(X))$ the set of all measurable bounded functions $G : \Gamma_0(X) \to \mathbb{C}$ with bounded support. It is easy to see that, for each $G \in \mathcal{F}(\Gamma_0(X))$, $(KG)(\gamma)$ is well-defined for all $\gamma \in \Gamma(X)$, and KG is a bounded function on $\Gamma(X)$.

Let θ be a measure on $\Gamma_0(X)$. We make the following assumptions about θ:

(A1): *Normalization*: $\theta(\Gamma^{(0)}(X)) = 1$.

(A2): *Local finiteness*: For each $n \in \mathbb{N}$ and each compact set $\Lambda \subset X$, we have $\theta(\Gamma_\Lambda^{(n)}) < \infty$.

(A3): $\star$*-positive definiteness*: For each $G \in \mathcal{F}(\Gamma_0(X))$, $\int_{\Gamma_0(X)} (G \star \overline{G})(\eta)\, \theta(d\eta) \geq 0$.

We define an inner product on $\mathcal{F}(\Gamma_0(X))$ by

$$(G_1, G_2)_{\mathcal{R}_\theta} = \int_{\Gamma_0(X)} (G_1 \star \overline{G_2})(\eta)\, \theta(d\eta).$$

Let $\mathcal{R}_\theta$ denote the complex Hilbert space obtained as the completion of $\mathcal{F}(\Gamma_0(X))$ in the norm induced by the inner product $(\cdot,\cdot)_{\mathcal{R}_\theta}$. (As usual, one identifies in $\mathcal{R}_\theta$ any two functions $G_1, G_2 \in \mathcal{F}(\Gamma_0(X))$ for which $\|G_1 - G_2\|_{\mathcal{R}_\theta} = 0$.)

We denote by $\mathcal{D}(X)$ the nuclear space of all real-valued smooth functions on X with compact support. Let $\mathcal{D}'(X)$ denote the dual space of $\mathcal{D}(X)$ with respect to the center space $L^2(X \to \mathbb{R}, \sigma)$. We denote by $\mathcal{D}_\mathbb{C}(X)$ the complexification of $\mathcal{D}(X)$, i.e., the nuclear space of all complex-valued smooth functions on X with compact support. For $n \geq 2$, we denote by $\mathcal{D}_\mathbb{C}^{\odot n}(X)$ the nth symmetric tensor power of $\mathcal{D}_\mathbb{C}(X)$. This is a nuclear space that consists of all complex-valued symmetric smooth functions on X^n with compact support. We also denote $\mathcal{D}_\mathbb{C}^{\odot 1}(X) = \mathcal{D}_\mathbb{C}(X)$ and $\mathcal{D}_\mathbb{C}^{\odot 0}(X) = \mathbb{C}$.

Each function $G^{(n)} \in \mathcal{D}_\mathbb{C}^{\odot n}(X)$ can be naturally identified with a function (still denoted by) $G^{(n)}$ on $\Gamma^{(n)}(X)$ by using the formula

$$G^{(n)}(x_1, \dots, x_n) = G^{(n)}(\delta_{x_1} + \cdots + \delta_{x_n}), \quad (x_1, \dots, x_n) \in X^{(n)}.$$

Note that, in view of the continuity of the function $G^{(n)} : X^n \to \mathbb{C}$, the restriction of the function $G^{(n)}$ to $X^{(n)}$ completely determines $G^{(n)}$ on X^n. With this identification, we will interpret each $\mathcal{D}_\mathbb{C}^{\odot n}(X)$ as the nuclear space of functions on $\Gamma^{(n)}(X)$. We extend these functions to the whole space $\Gamma_0(X)$ by setting them to be equal to zero on $\Gamma^{(m)}(X)$ for $m \neq n$.

Let Ψ be the topological direct sum of the nuclear spaces $\mathcal{D}_\mathbb{C}^{\odot n}(X)$ with $n \in \mathbb{N}_0$. Then Ψ is a nuclear space, and it can naturally be interpreted as the space of functions on $\Gamma_0(X)$. Note that $\Psi \subset \mathcal{F}(\Gamma_0(X))$. In fact, the nuclear space Ψ is densely and continuously embedded into the Hilbert space $\mathcal{R}_\theta$. Hence, we get a standard (complex) triple $\Psi \subset \mathcal{R}_\theta \subset \Psi'$.

The above considerations imply that the real nuclear space $\mathcal{D}(X)$ is identified with the real nuclear space of functions on $\Gamma^{(1)}(X)$, extended by zero to the whole $\Gamma_0(X)$.

For each $\varphi \in \mathcal{D}(X)$, we define a Hermitian operator $A(\varphi)$ in the Hilbert space $\mathcal{R}_\theta$ with domain Ψ by $A(\varphi)G = \varphi \star G$ for $G \in \Psi$. Note that $A(\varphi)G \in \Psi$. We now strengthen condition (A2) by demanding the following:

(A2$'$): For every compact $\Lambda \subset X$, there exists a constant $C_\Lambda > 0$ such that $\theta(\Gamma_\Lambda^{(n)}) \leq C_\Lambda^n$ for all $n \in \mathbb{N}$.

Under the assumptions (A1), (A2$'$) and (A3), Theorem 1.1 can be applied to the Hermitian operators $(A(\varphi))_{\varphi \in \mathcal{D}(X)}$, the cyclic vector $\Omega = \chi_0$, the indicator function of the zero measure, and the standard triples $\mathcal{D}(X) \subset L^2(X \to \mathbb{R}, \sigma) \subset \mathcal{D}'(X)$ and $\Psi \subset \mathcal{R}_\theta \subset \Psi'$.

However, Theorem 1.1 yields the spectral measure μ as a probability measure on $(\mathcal{D}'(X), \mathcal{C}(\mathcal{D}'(X)))$, whereas we are interested in a point process.

Using ideas from [13], it was shown in [10] that the spectral measure μ is, in fact, concentrated on $\Gamma(X)$, provided the following additional assumption is satisfied:

(A4): Every compact $\Lambda\subset X$ can be covered by a finite union of open sets $\Lambda_1,\dots,\Lambda_k$, $k\in\mathbb N$, which have compact closures and satisfy the estimate

$$\theta(\Gamma^{(n)}(\Lambda_i))\le(2+\epsilon)^{-n}\quad\text{for all } i=1,\dots,k \text{ and } n\in\mathbb N,$$

where $\epsilon=\epsilon(\Lambda)>0$.

For the obtained point process μ, the measure θ is its correlation measure on $\Gamma_0(X)$. Furthermore the obtained unitary operator $I:\mathcal R_\theta\to L^2(\Gamma(X),\mu)$ is just the restriction of the K-transform $K:L^1(\Gamma_0(X),\theta)\to L^1(\Gamma(X),\mu)$ to $\mathcal R_\theta\subset L^1(\Gamma_0(X),\theta)$. These results imply in particular, the following

Theorem 3.1. *Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. Let θ be a measure on $(\Gamma_0(X),\mathcal B(\Gamma_0(X)))$ that satisfies assumptions (A1), (A2′), (A3), and (A4). Then there exists a unique point process μ in X whose correlation measure on $\Gamma_0(X)$ is θ.*

It should be noted that, of the assumptions of Theorem 3.1, the really difficult one to check is (A3). We will discuss below a class of examples where it is indeed possible to check condition (A3).

4. Correlation measure of a family of Hermitian operators

In this section, we will briefly discuss a result from [17], which was a further development of the studies in [10]. Our presentation of this result follows [1] and [2].

Let X be a locally compact Polish space, and let σ be a non-atomic Radon measure on $(X,\mathcal B(X))$.

Let $\mathcal F$ be a separable Hilbert space and let $\mathcal S$ be a dense subspace of $\mathcal F$. For each $\Delta\in\mathcal B_0(X)$, let $\rho(\Delta):\mathcal S\to\mathcal S$ be a linear Hermitian operator in $\mathcal F$. We further assume:

- for any $\Delta_1,\Delta_2\in\mathcal B_0(X)$ with $\Delta_1\cap\Delta_2=\varnothing$, we have $\rho(\Delta_1\cup\Delta_2)=\rho(\Delta_1)+\rho(\Delta_2)$;
- the operators $\rho(\Delta)$ commute, i.e., $[\rho(\Delta_1),\rho(\Delta_2)]=0$ for any $\Delta_1,\Delta_2\in\mathcal B_0(X)$.

Let $\mathcal A$ denote the (commutative) $*$-algebra generated by $(\rho(\Delta))_{\Delta\in\mathcal B_0(X)}$. Let Ω be a fixed vector in $\mathcal S$ with $\|\Omega\|_{\mathcal F}=1$, and let a state $\tau:\mathcal A\to\mathbb C$ be defined by $\tau(a)=(a\Omega,\Omega)_{\mathcal F}$ for $a\in\mathcal A$.

In view of formula (2.5), we define Wick polynomials in $\mathcal A$ by the following recurrence formula:

$$\begin{aligned}{:}\rho(\Delta){:}&=\rho(\Delta),\\ {:}\rho(\Delta_1)\cdots\rho(\Delta_{n+1}){:}&=\rho(\Delta_{n+1})\,{:}\rho(\Delta_1)\cdots\rho(\Delta_n){:}\\ &\quad-\sum_{i=1}^n {:}\rho(\Delta_1)\cdots\rho(\Delta_{i-1})\rho(\Delta_i\cap\Delta_{n+1})\rho(\Delta_{i+1})\cdots\rho(\Delta_n){:}\,,\end{aligned}\tag{4.10}$$

where $\Delta,\Delta_1,\dots,\Delta_{n+1}\in\mathcal B_0(X)$ and $n\in\mathbb N$. It is easy to see that, for each permutation $\pi\in\mathfrak S_n$,

$${:}\rho(\Delta_1)\cdots\rho(\Delta_n){:}={:}\rho(\Delta_{\pi(1)})\cdots\rho(\Delta_{\pi(n)}){:}\,.$$

We assume that, for each $n\in\mathbb N$, there exists a symmetric measure $\theta^{(n)}$ on X^n that is concentrated on $X^{(n)}$ and such that

$$\theta^{(n)}\big(\Delta_1\times\dots\times\Delta_n\big)=\frac{1}{n!}\,\tau\big({:}\rho(\Delta_1)\cdots\rho(\Delta_n){:}\big),\quad \Delta_1,\dots,\Delta_n\in\mathcal B_0(X).\tag{4.11}$$

Note that, if the measure $\theta^{(n)}$ exists, then it is unique. The $\theta^{(n)}$ is called the nth correlation measure of the operators $\rho(\Delta)$. If $\theta^{(n)}$ has a density $k^{(n)}$ with respect to $\frac{1}{n!}\sigma^{\otimes n}$, then $k^{(n)}$ is called the nth correlation function of the operators $\rho(\Delta)$.

Theorem 4.1 ([17]). *Let $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ be a family of Hermitian operators in $\mathcal{F}$ as above. In particular, these operators have correlation measures $(\theta^{(n)})_{n=1}^\infty$ respective the state τ. Furthermore, we assume that the following two conditions are satisfied.*

(B1) *For each $\Delta\in\mathcal{B}_0(X)$, there exists a constant $C_\Delta>0$ such that $\theta^{(n)}(\Delta^n)\le C_\Delta^n$ for all $n\in\mathbb{N}$.*

(B2) *For each $x\in X$ and any sequence $\{\Delta_l\}_{l\in\mathbb{N}}\subset\mathcal{B}_0(X)$ such that $\Delta_l\downarrow\{x\}$ (i.e., $\Delta_1\supset\Delta_2\supset\Delta_3\supset\cdots$ and $\bigcap_{l=1}^\infty\Delta_l=\{x\}$), we have $C_{\Delta_l}\to 0$ as $l\to\infty$.*

Then the following statements hold.

(i) *Let $\mathfrak{D}=\{a\Omega\mid a\in\mathcal{A}\}$ and let $\mathfrak{F}$ denote the closure of $\mathfrak{D}$ in $\mathcal{F}$. Each operator $(\rho(\Delta),\mathfrak{D})$ is essentially self-adjoint in $\mathfrak{F}$, i.e., the closure of $\rho(\Delta)$, denoted by $\widetilde{\rho}(\Delta)$, is a self-adjoint operator in $\mathfrak{F}$.*

(ii) *For any $\Delta_1,\Delta_2\in\mathcal{B}_0(X)$, the operator-valued measures (resolutions of the identity) of the operators $\widetilde{\rho}(\Delta_1)$ and $\widetilde{\rho}(\Delta_2)$ commute.*

(iii) *There exist a unique point process μ in X and a unique unitary operator $U:\mathfrak{F}\to L^2(\Gamma_X,\mu)$ satisfying $U\Omega=1$ and*

$$\big(U\rho(\Delta_1)\cdots\rho(\Delta_n)\Omega\big)(\gamma)=\gamma(\Delta_1)\cdots\gamma(\Delta_n)$$

for any $\Delta_1,\dots,\Delta_n\in\mathcal{B}_0(X)$ ($n\in\mathbb{N}$). In particular,

$$\tau\big(\rho(\Delta_1)\cdots\rho(\Delta_n)\big)=\int_{\Gamma_X}\gamma(\Delta_1)\cdots\gamma(\Delta_n)\,\mu(d\gamma).$$

(iv) *The correlations measures of the point process μ are $(\theta^{(n)})_{n=1}^\infty$.*

Let us stress that, in Theorem 4.1, the correlation measure θ on $\Gamma_0(X)$ automatically satisfies condition (A3). On the other hand, the most difficult condition of Theorem 4.1 is the very existence of the correlation measures $(\theta^{(n)})_{n=1}^\infty$ for a given family of Hermitian operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$.

Below we will discuss several examples of application of Theorem 4.1, but first we will recall some definitions related to quasi-free states.

5. Quasi-free states on the CAR and CCR algebras

In this section, we follow [12, Section 5.2.3], see also [1].

We start with the case of the CAR algebra. Let $\mathcal{H}$ and $\mathcal{F}$ be separable complex Hilbert spaces. Let $a^+(\varphi)$ and $a^-(\varphi)$ ($\varphi\in\mathcal{H}$) be bounded linear operators in $\mathcal{F}$ such that $a^+(\varphi)$ linearly depends on φ and $a^-(\varphi)=(a^+(\varphi))^*$. Let $a^+(\varphi)$ and $a^-(\varphi)$ satisfy the CAR:

$$\{a^+(\varphi),a^+(\psi)\}=\{a^-(\varphi),a^-(\psi)\}=0,\qquad \{a^-(\varphi),a^+(\psi)\}=(\psi,\varphi)_{\mathcal{H}}.$$

Here $\{A,B\}=AB+BA$ is the anticommutator. Let $\mathbb{A}$ be the unital $*$-algebra generated by these operators, called the CAR algebra. We define field operators

$$b(\varphi)=a^+(\varphi)+a^-(\varphi),\quad \varphi\in\mathcal{H}.$$

As easily seen, these operators also generate $\mathbb{A}$.

Let $\tau:\mathbb{A}\to\mathbb{C}$ be a state on $\mathbb{A}$, i.e., τ is a linear functional on the vector space $\mathbb{A}$, $\tau(1)=1$, and $\tau(a^*a)\ge 0$ for each $a\in\mathbb{A}$. The state τ is completely determined by the functionals $T^{(n)}:\mathcal{H}^n\to\mathbb{C}$ ($n\in\mathbb{N}$) defined by

$$T^{(n)}(\varphi_1,\dots,\varphi_n)=\tau\big(b(\varphi_1)\cdots b(\varphi_n)\big). \tag{5.12}$$

The state τ is called quasi-free if

$$T^{(2n-1)}=0, \tag{5.13}$$

$$T^{(2n)}(\varphi_1,\dots,\varphi_{2n})=\sum(-1)^{\mathrm{Cross}(\nu)}\,T^{(2)}(\varphi_{i_1},\varphi_{j_1})\cdots T^{(2)}(\varphi_{i_n},\varphi_{j_n}),\quad n\in\mathbb{N}, \tag{5.14}$$

where the summation in (5.14) is over all partitions $\nu = \big\{\{i_1, j_1\}, \dots, \{i_n, j_n\}\big\}$ of $\{1, \dots, 2n\}$ with $i_k < j_k$ $(k = 1, \dots, n)$ and $\mathrm{Cross}(\nu)$ denotes the number of all crossings in ν, i.e., the number of all choices of $\{i_k, j_k\}, \{i_l, j_l\} \in \nu$ such that $i_k < i_l < j_k < j_l$.

The state τ is called gauge-invariant if, for each $q \in \mathbb{C}$ with $|q| = 1$, we have $T^{(n)}(q\varphi_1, \dots, q\varphi_n) = T^{(n)}(\varphi_1, \dots, \varphi_n)$ for all $\varphi_1, \dots, \varphi_n \in \mathcal{H}$, $n \in \mathbb{N}$.

The state τ can also be uniquely characterized by the n-point functions

$$S^{(m,n)} : \mathcal{H}^{m+n} \to \mathbb{C}, \quad m, n \in \mathbb{N}_0,\ m + n \geq 1,$$

defined by

$$\begin{aligned} &S^{(m,n)}(\varphi_m, \dots, \varphi_2, \varphi_1, \psi_1, \psi_2, \dots, \psi_n) \\ &\quad = \tau\big(a^+(\varphi_m) \cdots a^+(\varphi_2) a^+(\varphi_1) a^-(\psi_1) a^-(\psi_2) \cdots a^-(\psi_n)\big). \end{aligned} \tag{5.15}$$

In fact, the state τ is gauge-invariant quasi-free if and only if

$$S^{(m,n)}(\varphi_m \dots, \varphi_1, \psi_1, \dots, \psi_n) = \delta_{m,n} \det \Big[S^{(1,1)}(\varphi_i, \psi_j)\Big]_{i,j=1,\dots,n}. \tag{5.16}$$

There exists a one-to-one correspondence between the set of gauge-invariant quasi-free states τ and the set of all bounded linear operators K in $\mathcal{H}$ that satisfy $0 \leq K \leq 1$. More exactly, a state τ on the CAR algebra $\mathbb{A}$ is gauge-invariant quasi-free if and only if, for some operator K in $\mathcal{H}$ satisfying $0 \leq K \leq 1$, formula (5.16) holds with $S^{(1,1)}(\varphi, \psi) = (K\varphi, \psi)_{\mathcal{H}}$ for $\varphi, \psi \in \mathcal{H}$.

Next, let us briefly discuss the CCR case. As before, let $\mathcal{H}$ and $\mathcal{F}$ be separable complex Hilbert spaces. Let $\mathcal{V}$ be a dense subspace of $\mathcal{H}$, let D be a dense subspace of $\mathcal{F}$, and let $a^+(\varphi) : D \to D$ and $a^-(\varphi) : D \to D$ $(\varphi \in \mathcal{V})$ be linear operators such that $a^+(\varphi)$ linearly depends on $\varphi \in \mathcal{V}$ and $a^-(\varphi) = (a^+(\varphi))^* \restriction_D$. Assume $a^+(\varphi)$ and $a^-(\varphi)$ satisfy the CCR:

$$[a^+(\varphi), a^+(\psi)] = [a^-(\varphi), a^-(\psi)] = 0, \qquad [a^-(\varphi), a^+(\psi)] = (\psi, \varphi)_{\mathcal{H}}.$$

Here $[A, B] = AB - BA$ is the commutator. Let $\mathbb{A}$ be the unital $*$-algebra generated by these operators, called the CCR algebra. We define field operators $b(\varphi) = a^+(\varphi) + a^-(\varphi)$ $(\varphi \in \mathcal{V})$; these operators also generate $\mathbb{A}$.

Let $\tau : \mathbb{A} \to \mathbb{C}$ be a state on $\mathbb{A}$. The state τ is completely determined by the functionals $T^{(n)} : \mathcal{V}^n \to \mathbb{C}$ $(n \in \mathbb{N})$ defined by (5.12). The state τ is called quasi-free if formula (5.13) holds, and the following counterpart of formula (5.14) holds:

$$T^{(2n)}(\varphi_1, \dots, \varphi_{2n}) = \sum T^{(2)}(\varphi_{i_1}, \varphi_{j_1}) \cdots T^{(2)}(\varphi_{i_n}, \varphi_{j_n}), \quad n \in \mathbb{N}.$$

Remark 5.1. Let $\varkappa : \mathcal{V} \to \mathbb{C}$ be a linear functional. Assume that the operators $a^+(\varphi)$, $a^-(\varphi)$ $(\varphi \in \mathcal{V})$ satisfy the CCR. Then the operators $A^+(\varphi) = a^+(\varphi) + \varkappa(\varphi)$, $A^-(\varphi) = a^-(\varphi) + \overline{\varkappa(\varphi)}$ $(\varphi \in \mathcal{V})$ also satisfy the CCR. Note that the unital $*$-algebra $\mathbb{A}$ generated by the operators $a^+(\varphi)$, $a^-(\varphi)$ coincides with the unital $*$-algebra generated by the operators $A^+(\varphi)$, $A^-(\varphi)$. Assume that τ is a quasi-free state on $\mathbb{A}$ relative the operators $a^+(\varphi)$, $a^-(\varphi)$. But then τ is not a quasi-free state on $\mathbb{A}$ relative the operators $A^+(\varphi)$, $A^-(\varphi)$. Indeed, in the latter case, $T^{(2n-1)} \neq 0$. Nevertheless, one can easily generalize the definition of a quasi-free state in such a way that, if τ is a quasi-free state relative the operators $a^+(\varphi)$, $a^-(\varphi)$, then τ is also a quasi-free state relative the operators $A^+(\varphi)$, $A^-(\varphi)$.

Similarly to the CAR case, we define a gauge-invariant state τ on $\mathbb{A}$ and n-point functions

$$S^{(m,n)} : \mathcal{V}^{m+n} \to \mathbb{C}, \quad m, n \in \mathbb{N}_0,\ m + n \geq 1,$$

by formula (5.15).

A state τ is gauge-invariant quasi-free if and only if the formula (5.16) holds in which the determinant det is replaced with the permanent per. In fact, a state τ on the CCR algebra $\mathbb{A}$ is gauge-invariant quasi-free if and only if, for a bounded linear operator K in $\mathcal{H}$ satisfying $K \geq 0$, we have

$$S^{(m,n)}(\varphi_m \dots, \varphi_1, \psi_1, \dots, \psi_n) = \delta_{m,n} \operatorname{per}\left[(K\varphi_i, \psi_j)_{\mathcal{H}}\right]_{i,j=1,\dots,n} .$$

Following [1] (see also the references therein), let us now present a formal observation. Choose $\mathcal{H} = L^2(X, \sigma)$, and in the CCR case, assume that $\mathcal{V}$ contains all bounded functions with compact support. Let $a^+(\varphi)$ and $a^-(\varphi)$ satisfy either the CAR or the CCR, and let τ be a state on the corresponding $*$-algebra $\mathbb{A}$. For $x \in X$, define (formal) operators $a^+(x)$, $a^-(x)$ that satisfy

$$a^+(\chi_\Delta) = \int_\Delta a^+(x)\,\sigma(dx), \quad a^-(\chi_\Delta) = \int_\Delta a^-(x)\,\sigma(dx) \quad \text{for each } \Delta \in \mathcal{B}_0(X).$$

(Here χ_Δ denotes the indicator of Δ.) We now define particle density $\rho(x) = a^+(x)a^-(x)$ ($x \in X$), and in the smeared form

$$\rho(\Delta) = \int_\Delta \rho(x)\,\sigma(dx) = \int_\Delta a^+(x)a^-(x)\,\sigma(dx), \quad \Delta \in \mathcal{B}_0(X).$$

It follows from the CAR/CAR that the Hermitian operators $\rho(\Delta)$ ($\Delta \in \mathcal{B}_0(X)$) commute, and furthermore, formula (4.10) implies, for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}_0(X)$,

$$:\!\rho(\Delta_1)\cdots\rho(\Delta_n)\!: = \int_{\Delta_1\times\cdots\times\Delta_n} a^+(x_n)\cdots a^+(x_1)a^-(x_1)\cdots a^-(x_n)\,\sigma^{\otimes n}(dx_1\cdots dx_n). \tag{5.17}$$

Thus, the Wick polynomials correspond to the Wick (normal) ordering, in which all the operators $a^+(x_i)$ are to the left of all the operators $a^-(x_j)$. Hence, by (4.11) and (5.17), we formally obtain

$$\begin{aligned}&\theta^{(n)}\big(\Delta_1 \times \cdots \times \Delta_n\big)\\ &\quad = \frac{1}{n!}\int_{\Delta_1\times\cdots\times\Delta_n} \tau\big(a^+(x_n)\cdots a^+(x_1)a^-(x_1)\cdots a^-(x_n)\big)\sigma^{\otimes n}(dx_1\cdots dx_n).\end{aligned}$$

Therefore, by (2.6), the family of the operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ has correlation functions

$$k^{(n)}(x_1, \dots, x_n) = \tau\big(a^+(x_n)\cdots a^+(x_1)a^-(x_1)\cdots a^-(x_n)\big).$$

Below we will present several examples where a rigorous representation of the CAR/CCR leads, through an informal particle density $\rho(x)$ ($x \in X$), to a rigorously defined family of Hermitian operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ that satisfy the conditions of Theorem 4.1, and hence a point process μ exists and is the spectral measure of the family of commuting self-adjoint operators $(\tilde{\rho}(\Delta))_{\Delta\in\mathcal{B}_0(X)}$. But first we will recall several basic facts about antisymmetric and symmetric Fock spaces.

6. Fock spaces

The material of this section is based on [9, Chapter 2, Section 2.2] and [18, Section 19]

Let $\mathcal{G}$ denote a separable complex Hilbert space. Let $\mathcal{AF}(\mathcal{G}) = \bigoplus_{n=0}^\infty \mathcal{G}^{\wedge n} n!$ denote the antisymmetric Fock space over $\mathcal{G}$. Here $\wedge$ denotes the antisymmetric tensor product and elements of the Hilbert space $\mathcal{AF}(\mathcal{G})$ are sequences $g = (g^{(n)})_{n=0}^\infty$ with $g^{(n)} \in \mathcal{G}^{\wedge n}$ ($\mathcal{G}^{\wedge 0} = \mathbb{C}$) and $\|g\|^2_{\mathcal{AF}(\mathcal{G})} = \sum_{n=0}^\infty \|g^{(n)}\|^2_{\mathcal{G}^{\wedge n}}\, n! < \infty$. The vector $\Omega = (1, 0, 0, \dots)$ is called the vacuum.

For $\varphi \in \mathcal{G}$, we define a creation operator $a^+(\varphi)$ as a bounded linear operator in $\mathcal{AF}(\mathcal{G})$ that satisfies $a^+(\varphi)g^{(n)} = \varphi \wedge g^{(n)}$ for $g^{(n)} \in \mathcal{G}^{\wedge n}$. For each $\varphi \in \mathcal{G}$, we define an annihilation operator $a^-(\varphi) = a^+(\varphi)^*$. Then,

$$a^-(\varphi) g_1 \wedge \cdots \wedge g_n = \sum_{i=1}^n (-1)^{i+1} (g_i, \varphi)_{\mathcal{G}}\, g_1 \wedge \cdots \wedge g_{i-1} \wedge g_{i+1} \wedge \cdots \wedge g_n$$

for all $g_1,\dots,g_n\in\mathcal G$. Note that the norm of the operators $a^+(\varphi)$, $a^-(\varphi)$ in $\mathcal{AF}(\mathcal G)$ is equal to $\|\varphi\|_{\mathcal G}$. The operators $a^+(\varphi)$, $a^-(\varphi)$ satisfy the CAR over $\mathcal G$.

We will also need below the differential second quantization. We denote by $\mathcal{AF}_{\mathrm{fin}}(\mathcal G)$ the subspace of $\mathcal{AF}(\mathcal G)$ that consists of all $(g^{(n)})_{n=0}^\infty\in\mathcal{AF}(\mathcal G)$ such that, for some $N\in\mathbb N$, $g^{(n)}=0$ for all $n>N$. For a bounded linear operator A in $\mathcal G$, its differential second quantization $d\Gamma(A)$ is the linear operator in $\mathcal{AF}(\mathcal G)$, with domain $\mathcal{AF}_{\mathrm{sym}}(\mathcal G)$, that maps each subspace $\mathcal G^{\wedge n}$ continuously into itself and satisfies $d\Gamma(A)\Omega=0$ and

$$d\Gamma(A)g_1\wedge\dots\wedge g_n=\sum_{i=1}^n g_1\wedge\dots\wedge g_{i-1}\wedge(Ag_i)\wedge g_{i+1}\dots\wedge g_n$$

for all $g_1,\dots,g_n\in\mathcal G$.

Next, let $\mathcal{SF}(\mathcal G)=\bigoplus_{n=0}^\infty \mathcal G^{\odot n}n!$ denote the symmetric Fock space of $\mathcal G$. Here $\odot$ denotes the symmetric tensor product and elements of the Hilbert space $\mathcal{SF}(\mathcal G)$ are sequences $g=(g^{(n)})_{n=0}^\infty$ with $g^{(n)}\in\mathcal G^{\odot n}$ ($\mathcal G^{\odot 0}=\mathbb C$) and $\|g\|^2_{\mathcal{SF}(\mathcal G)}=\sum_{n=0}^\infty\|g^{(n)}\|^2_{\mathcal G^{\odot n}}\,n!<\infty$. The vector $\Omega=(1,0,0,\dots)$ is again called the vacuum. Let also $\mathcal{SF}_{\mathrm{fin}}(\mathcal G)$ be the subspace of $\mathcal{SF}(\mathcal G)$ defined similarly to $\mathcal{AF}_{\mathrm{fin}}(\mathcal G)$.

For $\varphi\in\mathcal G$, we define a creation operator $a^+(\varphi)$ as a linear operator in $\mathcal{SF}(\mathcal G)$ with domain $\mathcal{SF}_{\mathrm{fin}}(\mathcal G)$ that satisfies $a^+(\varphi)g^{(n)}=\varphi\odot g^{(n)}$ for $g^{(n)}\in\mathcal G^{\odot n}$. For each $\varphi\in\mathcal G$, we define an annihilation operator $a^-(\varphi)$ as the adjoint of the operator $a^+(\varphi)$ restricted to $\mathcal{SF}_{\mathrm{fin}}(\mathcal G)$. Then,

$$a^-(\varphi)g_1\odot\dots\odot g_n=\sum_{i=1}^n(g_i,\varphi)_{\mathcal G}\,g_1\odot\dots\odot g_{i-1}\odot g_{i+1}\odot\dots\odot g_n$$

for all $g_1,\dots,g_n\in\mathcal G$. The operators $a^+(\varphi)$, $a^-(\varphi)$ satisfy the CAR over $\mathcal G$.

Next, we define a differential second quantization in $\mathcal{SF}(\mathcal G)$. For a bounded linear operator A in $\mathcal G$, its differential second quantization $d\Gamma(A)$ is a linear operator in $\mathcal{SF}(\mathcal G)$, with domain $\mathcal{SF}_{\mathrm{sym}}(\mathcal G)$, that maps each subspace $\mathcal G^{\odot n}$ continuously into itself and satisfies $d\Gamma(A)\Omega=0$ and

$$d\Gamma(A)g_1\odot\dots\odot g_n=\sum_{i=1}^n g_1\odot\dots\odot g_{i-1}\odot(Ag_i)\odot g_{i+1}\odot\dots\odot g_n$$

for all $g_1,\dots,g_n\in\mathcal G$.

7. Examples of application of Theorem 4.1

Everywhere below X is a locally compact Polish space and σ is a non-atomic Radon measure on X.

7.1. **Determinantal point process with a Hermitian kernel.** We discuss here a result from [17], see also [15].

First, let us briefly recall Araki–Wyss' explicit construction of the gauge-invariant quasi-free states on the CAR algebra [4]. Let $\mathcal H$ be a separable complex Hilbert space with an antiunitary involution $\mathcal I$, i.e., $\mathcal I$ is an antilinear operator in $\mathcal H$ that satisfies $(\mathcal I\varphi,\mathcal I\psi)_{\mathcal H}=(\psi,\varphi)_{\mathcal H}$ for $\varphi,\psi\in\mathcal H$ and $\mathcal I^2=\mathbf 1$. Let $\mathcal G=\mathcal H\oplus\mathcal H$, and for $\varphi\in\mathcal H$ and $\sharp\in\{+,-\}$, we define operators $a_1^\sharp(\varphi)=a^\sharp(\varphi,0)$, $a_2^\sharp(\varphi)=a^\sharp(0,\varphi)$ in $\mathcal{AS}(\mathcal G)$.

We fix a bounded linear operator K in $\mathcal H$ that satisfies $0\le K\le 1$, and define operators $K_1=\sqrt K$ and $K_2=\sqrt{1-K}$. For each $\varphi\in\mathcal H$, we define the following bounded linear operators in $\mathcal{AF}(\mathcal G)$:

$$A^+(\varphi)=a_2^+(K_2\varphi)+a_1^-(\mathcal IK_1\varphi),\quad A^-(\varphi)=a_2^-(K_2\varphi)+a_1^+(\mathcal IK_1\varphi).\tag{7.18}$$

The operators $A^+(\varphi)$, $A^-(\varphi)$ ($\varphi\in\mathcal H$) satisfy the CAR.

Let $\mathbb{A}$ denote the corresponding CAR $*$-algebra. The vacuum state on $\mathbb{A}$ is defined by $\tau(a) = (a\Omega, \Omega)_{\mathcal{AF}(\mathcal{G})}$ $(a \in \mathbb{A})$. This state is gauge-invariant quasi-free, with

$$S^{(1,1)}(\varphi, \psi) = (K\varphi, \psi)_{\mathcal{H}}, \quad \varphi, \psi \in \mathcal{H}. \tag{7.19}$$

Now, let $\mathcal{H} = L^2(X, \sigma)$ with $\mathcal{I}$ being the complex conjugation in $\mathcal{H}$. For each $\Delta \in \mathcal{B}_0(X)$, we denote by P_Δ the orthogonal projection of $L^2(X, \sigma)$ onto its subspace $L^2(\Delta, \sigma)$, i.e., for $f \in L^2(X, \sigma)$, $(P_\Delta f)(x) = \chi_\Delta(x) f(x)$, where χ_Δ is the indicator function of the set Δ.

Let K be a bounded linear operator in $\mathcal{H}$ that satisfies $0 \le K \le 1$, and additionally we assume that the operator K is locally trace-class, i.e., for each $\Delta \in \mathcal{B}_0(X)$, the operator $P_\Delta K P_\Delta$ is trace-class. Under these assumptions, K is an integral operator, and furthermore its integral kernel $K(x, y)$ can be chosen so that $\operatorname{Tr}(P_\Delta K P_\Delta) = \int_\Delta K(x, x)\, \sigma(dx)$ for each $\Delta \in \mathcal{B}_0(X)$.

Under these assumptions, the particle density corresponding to the gauge-invariant quasi-free (vacuum) state τ on $\mathbb{A}$ is the family of (unbounded) Hermitian operators $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ that are defined as follows.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ denote two copies of $\mathcal{H}$, so that

$$\mathcal{G}^{\otimes 2} = (\mathcal{H}_1 \oplus \mathcal{H}_2)^{\otimes 2} = \bigoplus_{i,j \in \{1,2\}} \mathcal{H}_i \otimes \mathcal{H}_j.$$

Let $\mathcal{E}_{i,j} : \mathcal{H}^{\otimes 2} \to \mathcal{G}^{\otimes 2}$ denote the operator of embedding of $\mathcal{H}^{\otimes 2}$ into $\mathcal{G}^{\otimes 2}$ under which and element $f^{(2)} \in \mathcal{H}^{\otimes 2}$ is mapped to $f^{(2)} \in \mathcal{H}_i \otimes \mathcal{H}_j \subset \mathcal{G}^{\otimes 2}$. We denote $(f^{(2)})_{i,j} = \mathcal{E}_{i,j} f^{(2)}$.

Next, for $g^{(2)} \in \mathcal{G}^{\otimes 2}$ we define a creation operator $a^+(g^{(2)})$ which satisfies

$$a^+(g^{(2)})\varphi^{(n)} = \operatorname{ASym}_{n+2}(g^{(2)} \otimes \varphi^{(n)}), \quad \varphi^{(n)} \in \mathcal{G}^{\wedge n},\ n \in \mathbb{N}_0, \tag{7.20}$$

where $\operatorname{ASym}_{n+2}$ is the operator of antisymmetrization that projects $\mathcal{G}^{\otimes(n+2)}$ onto $\mathcal{G}^{\wedge(n+2)}$. The operator $a^+(g^{(2)})$ is bounded in $\mathcal{AF}(\mathcal{G})$ and we denote by $a^-(g^{(2)})$ its adjoint operator. In particular, for each $f^{(2)} \in \mathcal{H}^{\otimes 2}$ and $i, j \in \{1, 2\}$, we have defined bounded linear operators $a^+((f^{(2)})_{i,j})$ and $a^-((f^{(2)})_{i,j})$ in $\mathcal{AF}(\mathcal{G})$.

For each $\Delta \in \mathcal{B}_0(X)$, $K_2 P_\Delta K_1$ is a Hilbert–Schmidt operator in $\mathcal{H}$, hence it can be standardly identified with an element of $\mathcal{H}^{\otimes 2}$. Then, for each $\Delta \in \mathcal{B}_0(X)$, the particle density $\rho(\Delta)$ is the linear Hermitian operator in $\mathcal{AF}(\mathcal{G})$ with domain $\mathcal{AF}_{\mathrm{fin}}(\mathcal{G})$ given by

$$\begin{aligned}\rho(\Delta) &= a^+((K_2 P_\Delta K_1)_{2,1}) + a^-((K_2 P_\Delta K_1)_{2,1}) \\ &\quad + d\Gamma\big((-\mathcal{I} K_1 P_\Delta K_1 \mathcal{I}) \oplus (K_2 P_\Delta K_2)\big) + \operatorname{Tr}(P_\Delta K P_\Delta).\end{aligned} \tag{7.21}$$

The family of the Hermitian operators $(\rho(\Delta))_{\Delta \in \mathcal{B}_0(X)}$ satisfies the conditions of Theorem 4.1. The corresponding point process μ has correlation functions

$$k^{(n)}(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1,\dots,n}\,.$$

Thus, μ is the determinantal point process whose (Hermitian) correlation kernel is $K(x, y)$.

7.2. **Determinantal point process with a J-Hermitian kernel.** We discuss here a result from [2], see also the references therein.

Assume that the underlying space X is split into two disjoint measurable parts, X_1 and X_2, of positive measure σ. We denote by P_i the orthogonal projection of $\mathcal{H} = L^2(X, \sigma)$ onto $\mathcal{H}_i = L^2(X_i, \sigma)$, and we define $J = P_1 - P_2$.

According to the orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, each bounded linear operator A in $\mathcal{H}$ can be represented in the block form,

$$A = \begin{bmatrix} A^{11} & A^{21} \\ A^{12} & A^{22} \end{bmatrix},$$

where $A^{ij} = P_i A P_j$.

One says that a bounded linear operator A in $\mathcal{H}$ is J-self-adjoint if $(A^{ii})^* = A^{ii}$ ($i = 1, 2$) and $(A^{21})^* = -A^{12}$. If a J-self-adjoint operator A is an integral operator, then its integral kernel $A(x, y)$ is called J-Hermitian. A J-Hermitian kernel $A(x, y)$ satisfies $A(x, y) = \overline{A(y, x)}$ if both x and y are from the same part X_i, and $A(x, y) = -\overline{A(y, x)}$ if x and y are from different parts X_i and X_j ($i \neq j$).

For a bounded linear operator A in $\mathcal{H}$, we denote $\widehat{A} = AP_1 + (1 - A)P_2$. As easily seen, the operation $A \mapsto \widehat{A}$ is an involution in the space of bounded linear operators in $\mathcal{H}$. Furthermore, if an operator A is self-adjoint, then $\widehat{A}$ is J-self-adjoint, and if A is J-self-adjoint, then $\widehat{A}$ is self-adjoint.

Below we will use some of the notations of Section 7.1. Let K be a bounded linear operator in $\mathcal{H}$ with $0 \leq K \leq 1$, and let $K_1 = \sqrt{K}$, $K_2 = \sqrt{1-K}$. For each $\varphi \in \mathcal{H}$, consider the following bounded linear operators in $\mathcal{AF}(\mathcal{G})$:

$$A^+(\varphi) = a^+(\mathcal{I}K_1\mathcal{I}P_2\varphi,\, K_2P_1\varphi) + a^-(\mathcal{I}K_1P_1\varphi,\, \mathcal{I}K_2P_2\varphi),$$
$$A^-(\varphi) = a^-(\mathcal{I}K_1\mathcal{I}P_2\varphi,\, K_2P_1\varphi) + a^+(\mathcal{I}K_1P_1\varphi, \mathcal{I}K_2P_2\varphi).$$

These operators satisfy the CAR, and let $\mathbb{A}$ be the corresponding CAR $*$-algebra. The vacuum state τ on $\mathbb{A}$ is not gauge-invariant but it is still quasi-free. In fact, formulas (5.13), (5.14) hold in this case with

$$T^{(2)}(\varphi, \psi) = 2i\Im(K\mathbb{J}\varphi, \mathbb{J}\psi)_{\mathcal{H}} + (\mathbb{J}\psi, \mathbb{J}\varphi)_{\mathcal{H}}, \quad \varphi, \psi \in \mathcal{H},$$

where $\mathbb{J}\varphi = P_1\varphi + P_2\mathcal{I}\varphi$.

Let us now additionally assume that both operators P_1KP_1 and $P_2(1 - K)P_2$ are locally trace-class. Then the corresponding particle density can be rigorously defined as follows. For each $\Delta \in \mathcal{B}_0(X)$, the particle density $\rho(\Delta)$ is the linear Hermitian operator in $\mathcal{AF}(\mathcal{G})$ with domain $\mathcal{AF}_{\mathrm{fin}}(\mathcal{G})$ given by

$$\begin{aligned}\rho(\Delta) =& a^+\big((K_2J_\Delta K_1)_{2,1}\big) + a^-\big((K_2J_\Delta K_1)_{2,1}\big) + d\Gamma\big((-\mathcal{I}K_1J_\Delta K_1\mathcal{I}) \oplus (K_2J_\Delta K_2)\big)\\ &+ \mathrm{Tr}(P_{\Delta\cap X_1}KP_{\Delta\cap X_1}) + \mathrm{Tr}(P_{\Delta\cap X_2}(1 - K)P_{\Delta\cap X_2}),\end{aligned}$$

where $J_\Delta = P_{\Delta\cap X_1} - P_{\Delta\cap X_2}$. Here $(K_2J_\Delta K_1)_{2,1} \in \mathcal{G}^{\otimes 2}$ is defined similarly to $(K_2P_\Delta K_1)_{2,1}$ in Section 7.1.

The family of the Hermitian operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ satisfies the conditions of Theorem 4.1. The corresponding point process μ has correlation functions

$$k^{(n)}(x_1, \ldots, x_n) = \det[\mathbb{K}(x_i, x_j)]_{i,j=1,\ldots,n}.$$

Here $\mathbb{K} = \widehat{K}$ is a J-self-adjoint operator in $\mathcal{H}$, and $\mathbb{K}(x, y)$ is its integral J-Hermitian kernel. Note that $\mathbb{K}(x, y)$ indeed exists due to our assumptions, and furthermore $\mathbb{K}(x, y)$ is chosen so that $\mathrm{Tr}(P_\Delta KP_\Delta) = \int_\Delta \mathbb{K}(x, x)\,\sigma(dx)$ for each $\Delta \in \mathcal{B}_0(X)$ with $\Delta \subset X_1$ and $\mathrm{Tr}(P_\Delta(1 - K)P_\Delta) = \int_\Delta \mathbb{K}(x, x)\,\sigma(dx)$ for each $\Delta \in \mathcal{B}_0(X)$ with $\Delta \subset X_2$.

Thus, μ is a determinantal point process with a J-Hermitian correlation kernel $\mathbb{K}(x, y)$.

7.3. **Poisson point process.** We will discuss here a result from [1], compare with [8].

Let $\mathcal{H} = L^2(X, \sigma)$ and let $\mathcal{V} = L^2(X, \sigma) \cap L^1(X, \sigma)$. For each $\varphi \in \mathcal{V}$, we define the following linear operators in $\mathcal{SF}(\mathcal{H})$ with domain $\mathcal{SF}_{\mathrm{fin}}(\mathcal{H})$:

$$A^+(\varphi) = a^+(\varphi) + \int_X \varphi(x)\,\sigma(dx), \quad A^-(\varphi) = a^-(\varphi) + \int_X \varphi(x)\,\sigma(dx).$$

These operators satisfy the CCR. The vacuum state on the corresponding CCR algebra $\mathbb{A}$ is quasi-free in the sense as explained in Remark 5.1.

The corresponding particle density has the form

$$\rho(\Delta) = a^+(\chi_\Delta) + a^-(\chi_\Delta) + d\Gamma(P_\Delta) + \sigma(\Delta),$$

where P_Δ is the orthogonal projection of $\mathcal{H}$ onto $L^2(\Delta, \sigma)$.

The family of the Hermitian operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ satisfies the conditions of Theorem 4.1. The corresponding point process μ has correlation functions $k^{(n)}(x_1,\dots,x_n)=1$ for all $n\in\mathbb{N}$. Thus, μ is the Poisson point process with intensity measure σ.

7.4. **Permanental point process.** We discuss here a result from [17], see also [15].

The results of this section are pretty similar to those of Section 7.1, so we will only outline the necessary changes.

First, let us briefly recall Araki–Woods' explicit construction of the gauge-invariant quasi-free states on the CCR algebra [3].

We fix a bounded linear operator K in $\mathcal{H}$ that satisfies $K\ge 0$, and define operators $K_1=\sqrt{K}$ and $K_2=\sqrt{1+K}$. For each $\varphi\in\mathcal{H}$, we define the linear operators in $\mathcal{SF}(\mathcal{G})$ with the domain $\mathcal{SF}_{\mathrm{fin}}(\mathcal{G})$ through formula (7.18), with the corresponding operators acting in $\mathcal{SF}(\mathcal{G})$ rather than $\mathcal{AF}(\mathcal{G})$. These operators satisfy the CCR, and the vacuum state on the corresponding CCR algebra is gauge-invariant quasi-free, and (7.19) holds.

Let now $\mathcal{H}=L^2(X,\sigma)$ and assume that the operator $K\ge 0$ is locally trace-class. For $g^{(2)}\in\mathcal{G}^{\otimes 2}$, we define operators $a^+(g^{(2)})$, $a^-(g^{(2)})$ similarly to Section 7.1. In particular, to define $a^+(g^{(2)})$, we use formula (7.20) in which we replace $\mathcal{G}^{\wedge n}$ with $\mathcal{G}^{\odot n}$, and the antysimmetrization operator ASym_{n+2} with the symmetrization operator Sym_{n+2}. Next, we define the corresponding particle density $\rho(\Delta)$ similarly to formula (7.21). The family of the Hermitian operators $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ satisfies the conditions of Theorem 4.1. The corresponding point process μ has correlation functions

$$k^{(n)}(x_1,\dots,x_n)=\mathrm{per}[K(x_i,x_j)]_{i,j=1,\dots,n}.$$

Thus, μ is the permanental point process whose correlation kernel is $K(x,y)$.

7.5. **Hafnian point process.** In this section we will briefly discuss a result from [1], see also the references therein.

Let, as before, $\mathcal{H}=L^2(X,\sigma)$, and let $\mathcal{V}$ be the (dense) subspace of $\mathcal{H}$ that consists of all bounded measurable functions with compact support. Let $\mathcal{G}$ be a separable complex Hilbert space with an antiunitary involution $\mathcal{I}$.

Let $X\ni x\mapsto (L_1(x),L_2(x))\in\mathcal{G}^2$ be a measurable mapping. We assume that

$$\begin{aligned}
&(L_1(x),\mathcal{I}L_2(y))_{\mathcal{G}}=(L_1(y),\mathcal{I}L_2(x))_{\mathcal{G}},\\
&(L_1(x),L_1(y))_{\mathcal{G}}=(L_2(x),L_2(y))_{\mathcal{G}}\quad\text{for all } x,y\in X,\\
&\int_\Delta \|L_1(x)\|^2_{\mathcal{G}}\,\sigma(dx)=\int_\Delta \|L_2(x)\|^2_{\mathcal{G}}\,\sigma(dx)<\infty\quad\text{for each } \Delta\in\mathcal{B}_0(X).
\end{aligned}\tag{7.22}$$

By using e.g. [11, Chapter 10, Theorem 3.1], for each $h\in\mathcal{V}$, we define

$$\int_X hL_i\,d\sigma,\ \int_X h\mathcal{I}L_i\,d\sigma\in\mathcal{G},\quad i=1,2,$$

as Bochner integrals.

Denote $\mathcal{E}=\mathcal{H}\oplus\mathcal{G}$. We consider the following linear operators in $\mathcal{SF}(\mathcal{E})$ with domain $\mathcal{SF}_{\mathrm{fin}}(\mathcal{E})$:

$$\begin{aligned}
A^+(h)=&a^+\left(h,\int_X h\mathcal{I}L_2\,d\sigma\right)+a^-\left(0,\int_X \overline{h}\,L_1\,d\sigma\right),\\
A^-(h)=&a^-\left(h,\int_X h\mathcal{I}L_2\,d\sigma\right)+a^+\left(0,\int_X \overline{h}\,L_1\,d\sigma\right),\quad h\in\mathcal{V}.
\end{aligned}\tag{7.23}$$

These operators satisfy the CCR, and the vacuum state τ on the corresponding CCR algebra $\mathbb{A}$ is quasi-free with

$$T^{(2)}(f,h)=\int_X \overline{f(x)}\,h(x)\sigma(dx)$$

$$+2\int_{X^2}\Re\left(f(x)h(y)\overline{\mathcal{K}_2(x,y)}+\overline{f(x)}\,h(y)\mathcal{K}_1(x,y)\right)\sigma^{\otimes 2}(dx\,dy).$$

Here

$$\mathcal{K}_1(x,y)=(L_1(x),L_1(y))_{\mathcal{G}},$$
$$\mathcal{K}_2(x,y)=(L_1(x),\mathcal{I}L_2(y))_{\mathcal{G}},\quad x,y\in X.$$

The vacuum state τ is, however, not gauge-invariant.

It follows from (7.23) that the operators $A^+(x)$, $A^-(x)$ $(x\in X)$ are formally given by

$$A^+(x)=a^+(x,0)+a^+(0,\mathcal{I}L_2(x))+a^-(0,L_1(x)),$$
$$A^-(x)=a^-(x,0)+a^-(0,\mathcal{I}L_2(x))+a^+(0,L_1(x)). \tag{7.24}$$

Note that, in formula (7.24), $a^\sharp(0,\mathcal{I}L_2(x))$ and $a^\sharp(0,L_1(x))$ ($\sharp\in\{+,-\}$) are rigorously defined operators in $\mathcal{SF}(\mathcal{E})$ with domain $\mathcal{SF}_{\mathrm{fin}}(\mathcal{E})$. On the other hand, $a^\sharp(x,0)$ are formal operators that satisfy

$$\int_\Delta a^\sharp(x,0)\,\sigma(dx)=a^\sharp(\chi_\Delta,0),\quad \Delta\in\mathcal{B}_0(X),\ \sharp\in\{+,-\}.$$

In [1, Proposition 6.4], it is shown that the particle density $\rho(\Delta)=\int_\Delta A^+(x)A^-(x)\,\sigma(dx)$ $(\Delta\in\mathcal{B}_0(X))$ can be easily given a rigorous meaning as a Hermitian operator in $\mathcal{SF}(\mathcal{E})$ with domain $\mathcal{SF}_{\mathrm{fin}}(\mathcal{E})$.

The family $(\rho(\Delta))_{\Delta\in\mathcal{B}_0(X)}$ satisfies the conditions of Theorem 4.1. To write down the correlation functions of the corresponding point process μ, let us first recall that, for a symmetric $2n\times 2n$-matrix $C=[c_{ij}]_{i,j=1,\dots,2n}$, the hafnian of C is defined by

$$\operatorname{haf}(C)=\frac{1}{n!\,2^n}\sum_{\pi\in\mathfrak{S}_{2n}}\prod_{i=1}^n c_{\pi(2i-1)\pi(i)}.$$

(Note the the value of the hafnian of C does not depend on the diagonal elements of the matrix C.) The hafnian can also be written as

$$\operatorname{haf}(C)=\sum c_{i_1j_1}\cdots c_{i_nj_n},$$

where the summation is over all partitions $\nu=\big\{\{i_1,j_1\},\dots,\{i_n,j_n\}\big\}$ of $\{1,\dots,2n\}$. The point process μ has correlation functions

$$k^{(n)}(x_1,\dots,x_n)=\operatorname{haf}\left[\mathbb{K}(x_i,x_j)\right]_{i,j=1,\dots,n},$$

where

$$\mathbb{K}(x,y)=\begin{bmatrix}\mathcal{K}_2(x,y) & \mathcal{K}_1(x,y)\\ \overline{\mathcal{K}_1(x,y)} & \overline{\mathcal{K}_2(x,y)}\end{bmatrix}.$$

Hence, it is natural to call μ a hafnian point process.

In fact, μ is a Cox process. More exactly, μ is the Poisson point process with random intensity measure $|G(x)|^2\sigma(dx)$. Here $(G(x))_{x\in X}$ is a complex-valued Gaussian random field on X with covariance $\mathbb{E}(G(x)\overline{G(y)})=\mathcal{K}_1(x,y)$ and pseudo-covariance $\mathbb{E}(G(x)G(y))=\mathcal{K}_2(x,y)$.

Let us now consider a special case of the above construction. Let $K\geq 0$ be a locally trace-class operator in $\mathcal{H}$. Let $K_1=\sqrt{K}$. Then, for each $\Delta\in\mathcal{B}_0(X)$, $P_\Delta K_1$ is a Hilbert–Schmidt operator in $\mathcal{H}$. Hence, K_1 is an integral operator and its integral kernel $K_1(x,y)$ satisfies $\int_{\Delta\times X}|K_1(x,y)|^2\sigma(dx)\sigma(dy)<\infty$ for each $\Delta\in\mathcal{B}_0(X)$. Thus, we may assume that, for each $x\in X$, $K_1(x,\cdot)\in\mathcal{H}$.

Now, assume additionally that the integral kernel $K_1(x,y)$ is real-valued, i.e., the operator K_1 (equivalently the operator K) maps the space of real-valued functions from $\mathcal{H}$ into itself. Set $\mathcal{G}=\mathcal{H}$, let $\mathcal{I}$ be the complex conjugation in $\mathcal{H}$, and define

$L_1(x) = L_2(x) = K_1(x,\cdot) \in \mathcal{H}$ for $x \in X$. Then assumptions (7.22) are trivially satisfied. In this case,

$$\mathcal{K}_1(x,y) = \mathcal{K}_2(x,y) = (K_1(x,\cdot), K_1(y,\cdot))_{\mathcal{H}} = K(x,y) = \overline{K(x,y)},$$

where $K(x,y)$ is the integral kernel of the operator K. The correlation functions of the corresponding point process μ can be written in the form

$$k^{(n)}(x_1,\dots,x_n) = \det{}_2[K(x_i,x_j)]_{i,j=1,\dots,n}\,,$$

where $\det_2$ the the 2-determinant: for a matrix $B = [b_{ij}]_{i,j=1,\dots,n}$,

$$\det{}_2(B) = \sum_{\pi\in\mathfrak{S}_n}\prod_{i=1}^{n} 2^{n-C(\pi)}\, b_{i\,\pi(i)}.$$

Here $C(\pi)$ denotes the number of cycles in the permutation π. Such a point process μ is called 2-permanental.

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Eugene Lytvynov: e.lytvynov@swansea.ac.uk
Department of Mathematics, Swansea University, Bay Campus, Swansea SA1 8EN, U.K.

Received 27/08/2025; Revised 12/12/2025