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INEQUALITY OF RIEMANN-STIELTJES-Δ-INTEGRAL FOR HILBERT SPACES ON TIME SCALES

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Abstract. In this paper, the inequality of Riemann-Stieltjes integral is defined for functions of Hilbert space. The concept of time scales is introduced to unify both discrete and continuous problems. Also, the definition and properties of Riemann-Stieltjes integral are used in the application of self-adjoint and unitary operators in Hilbert spaces. Thees results are obtained on time scales.

1. Introduction and Preliminaries

The theory of Riemann integration finds application in almost every aspect of mathematical analysis. Lebesgue integral seems to be more difficult to handle because of its measure theory. The definition and properties of Riemann integral are more easier than that of Lebesgue integral, and therefore used in this paper. Riemann-Stieltjes integral is widely used everywhere especially in the cases when the integrand and the integrator have no common discontinuous points. For more details on Riemann and Stieltjes integrals, see ([2], [3], [4], [5], [7], [8] and [9]). The theory of time scale is introduced to unify discrete and continuous problems. Bohner and Guseinov [3], worked on Riemann and Lebesgue integration which gives advances in dynamic equations on time scales. See ([1], [3], [8], [10] and [11]) for more researchers that worked on calculus of time scales. The aim of this paper is to apply the properties of the Riemann–Stieltjes-Δ-integral to the study of selfadjoint and unitary operators in Hilbert spaces on time scales.

Let $I = [a, b]$ be a real closed interval. A partition of I is any finite ordered subset $P = t_0, t_1, ..., t_n \subset [a, b]$, where $a = t_0 < t_1 < ... < t_n = b$. Each partition $P = t_0, t_1, ..., t_n$ of I decomposes I into subintervals $I_{\triangle_j} = [t_{j-1}, t_j],\ j = 1, 2, ..., n$ such that $I_{\triangle_j} \cap I_{\triangle_k} = \emptyset$ for any $k \neq j$. By $\triangle t_j = t_j - t_{j-1}$, we denote the length of the j^{th} subinterval in the partition P; by $P(I)$ the set of all partitions of I (see [8]).

Let $P_m, P_n \in P(I)$. If $P_m \subset P_n$, we call P_n a refinement of P_m. If P_m, P_n are independently chosen, then the partition $P_m \cup P_n$ is a common refinement of P_m and P_n. Let us now consider a strictly increasing real-valued function g on the interval I. Then, for the partition P of I, we define
$g(P) = \{g(a) = g(t_0), g(t_1), ..., g(t_{n-1}), g(t_n) = g(b) \subset g(I)\}$ and $\triangle g_j = g(t_j) - g(t_{j-1})$.
We note that $\triangle g_j$ is positive and $\sum_{j=1}^{n} \triangle g_j = g(b) - g(a)$.
Moreover, $g(P)$ is a partition of $[g(a), g(b)]_{\mathbb{R}}$. In what follows, for the particular case $g(t) = t$, we obtain the Riemann sums for delta integrals studied by Mozyrska, Pawluszewicz, and Torres [8].

Definition 1.1. Let $f : [a, b] \longrightarrow \mathbb{R}$ and let $P = \{(t_i, [t_{j-1}, t_j]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$. The Riemann sum $S(P, f)$ of f on P is defined by

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$$S(P,f)=\sum_{i=1}^{n} f(t_i)(t_j-t_{j-1}). \tag{1.1}$$

A positive $\delta:[a,b]\longrightarrow\mathbb{R}$ i.e. $\delta>0$ for all t in $[a,b]$) is known as a gauge on $[a,b]$.

Definition 1.2. A function $f:[a,b]\longrightarrow\mathbb{R}$ is Riemann integrable on $[a,b]$ if there exists a number L such that for each $\varepsilon>0$ there exists a constant $\delta>0$ such that $|S(P,f)-L|<\varepsilon$ for all tagged partitions P of $[a,b]$ with norm of P less than δ.

Definition 1.3. [8] Let f be a real-valued and bounded function on the interval I and $P=t_0,t_1,...,t_n$ of I. Denote $I_{\triangle_j}=[t_j-t_{j-1}],\quad j=1,2,...,n$ and

$$m_{\triangle_j}=\inf_{t\in I_{\triangle_j}} f(t) \tag{1.2}$$

and

$$M_{\triangle_j}=\sup_{t\in I_{\triangle_j}} f(t) \tag{1.3}$$

The upper Riemann-Stieltjes-$\triangle$-sum of f with respect to a monotone increasing function g with the partition P, denoted by $U_{\triangle}(P,f,g)$ is defined by

$$U_{\triangle}(P,f,g)=\sum_{j=1}^{n} M_{\triangle_j}\Delta g_j, \tag{1.4}$$

while the lower Riemann-Stieltjes-$\triangle$-sum of f with respect to a monotone increasing function g with the partition P, denoted by $L_{\triangle}(P,f,g)$, is defined by

$$L_{\triangle}(P,f,g)=\sum_{j=1}^{n} m_{\triangle_j}\Delta g_j \tag{1.5}$$

Definition 1.4. [8] Let $I=[a,b]$, where $a,b\in I$. The upper Riemann-Stieltjes -$\triangle$-integral from a to b with respect to function g is defined by

$$\overline{\int_a^b} f(t)\triangle g(t)=\inf_{t\in P(I)} U_{\triangle}(P,f,g); \tag{1.6}$$

while the upper Riemann-Stieltjes-$\triangle$-integral from a to b with respect to function g is defined by

$$\underline{\int_a^b} f(t)\triangle g(t)=\sup_{t\in P(I)} L_{\triangle}(P,f,g). \tag{1.7}$$

If the upper Riemann-Stieltjes-$\triangle$-integral coincides with the lower Darboux-Stieltjes$-\triangle$-integral, we say that f is Riemann-Stieltjes-$\triangle$-integrable with respect to g on I, and the common value of the integrals, denoted by

$$\int_a^b f(t)\triangle g(t) \tag{1.8}$$

is called the Riemann-Stieltjes (or simply Stieltjes)-$\triangle$-integral of f with respect to g on I.

2. The Main Results

In this section, we shall give some inequalities of Riemann-Stieltjes integral with applications in Hilbert spaces on time scales.

A time scale is simply a non-empty, closed subset $\mathbb{T}$ of the real numbers. Let denote a time scales by $\mathbb{T}$ and let $a,b\in\mathbb{T}, a<b$. We distinguish $[a,b]$ as a real interval and define $I=[a,b]_{\mathbb{T}}=[a,b]\cap\mathbb{T}$. In this sense, $[a,b]=[a,b]_{\mathbb{R}}$. Let I be a nonempty, closed, and bounded interval consisting points from a time scales $\mathbb{T}$. Moreover, if $I=[a,b]_{\mathbb{T}}$, then

define $I_{\triangle=[a,\rho(b)]_{\mathbb{T}}}$ and $I_{\bigtriangledown=[\sigma(a),b]_{\mathbb{T}}}$.
The forward jump operator is the function $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(a_i) = b_i$ for all i and $\sigma(t) = t$ for all $t \in \mathbb{T}$ that are not a right-hand endpoint of a contiguous interval. (Note that if $t_0 = sup\mathbb{T}$ is finite, then definition requires that $\sigma(t_0) = t_0$ which is the usual convention. A function $f : \mathbb{T} \to \mathbb{R}$ is continuous (continuous at a point) if it is continuous in the usual relative sense (i.e., using the topology that $\mathbb{T}$ inherits as subset of $\mathbb{R}$). The set of points $\{a_i\}$ from $\mathbb{T}$ is called the right-scattered points. The set of points $\{b_i\}$ from $\mathbb{T}$ is called the left-scattered points.

Definition 2.1. ([3]). A function $f : \mathbb{T} \to \mathbb{R}$ is said to have delta derivative $f^{\Delta}(t)$ at a point $t \in \mathbb{T}$ provided that for every $\varepsilon > 0$ there is a neighbourhood $(t-\delta, t+\delta) \cap \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \tag{2.1}$$

for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$.

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

Definition 2.2. A mapping $F : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if:
(i) F is continuous at each right-dense point of $\mathbb{T}$
(ii) at each left-dense point $t \in \mathbb{T}$, $\lim_{s\to t^-} g(s) = g(t^-)$ exists.
Define the time scale interval in $\mathbb{T}$ by

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} \text{ such that } a \leq t \leq b\}. \tag{2.2}$$

Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}, a < b$, and we define the closed interval $I = [a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. The open and half-open intervals are defined in a similar way. A partition of I is any finite ordered subset $P = t_0, t_1, ..., t_n \subset [a, b]_{\mathbb{T}}$,
where $a = t_0 < t_1 < ... < t_n = b$. Each partition $P = t_0, t_1, ..., t_n$ of I decomposes I into subintervals $I_{\Delta j} = [t_{j-1}, t_j]_{\Delta}, j = 1, 2, ..., n$, such that $I_{\Delta_j} \cap I_{\Delta_k} = \varnothing$ for any $k \neq j$. By $\triangle t_j = t_j - t_{j-1}$, we denote the length of the j^{th} subinterval in the partition P; by $P(I)$ the set of all partitions of I.

Let us now consider a strictly increasing real-valued function g on the interval I. Then, for the partition P of I, we define

$$g(P) = \{g(a) = g(t_0), g(t_1), ..., g(t_{n-1}), g(t_n) = g(b) \subset g(I)\} \text{ and } \triangle g_j = g(t_j) - g(t_{j-1}). \tag{2.3}$$

We note that $\triangle g_j$ is positive and $\sum_{j=1}^{n} \triangle g_j = g(b) - g(a)$.

Let $\mathcal{C}([a, b])$ be the space of continuous functions on $[a, b]$. Then,

$$\mathcal{C}_0 = \{f \in \mathcal{C}([a, b]) : f(a) = 0\}. \tag{2.4}$$

Then $\mathcal{C}_0$ is a Banach space under the Alexiewicz norm

$$\|f\| = \|f\|_\infty = \sup_{t\in[a,b]} |f(t)| = \max_{t\in[a,b]} |f(t)|. \tag{2.5}$$

An inner product space or pre-Hilbert space over a scalar $\mathbb{K}$ is a pair $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ consisting of a linear space $\mathcal{H}$ over K and a functional $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$, called the inner product of $\mathcal{H}$, with the following properties:

$$(i) \quad \langle f, f\rangle \geq 0, \ \forall \ f \in \mathcal{H}, \text{ and } \langle f, f\rangle = 0 \ iff \ f = 0;$$
$$(ii) \quad \langle f, g\rangle = \overline{\langle g, f\rangle}, \ \forall \ f, g \in \mathcal{H};$$
$$(iii) \quad \langle kf, g\rangle = k\langle f, g\rangle, \ \forall \ f, g \in \mathcal{H}, k \in \mathbb{K};$$
$$(iv) \quad \langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle, \ \forall \ f, g, h \in \mathcal{H}.$$

For $f, g \in \mathcal{H}$, then $\langle f, g\rangle$ is called the inner product of f and g. For $f \in \mathcal{H}$, define $\|f\|$ by $\|f\| = \sqrt{\langle f, f\rangle}$. Then, $\|\cdot\|$ is a norm on $\mathcal{H}$, where $(\mathcal{H}, \|\cdot\|)$ is a normed space. $\|\cdot\|$ is called the norm induced by the inner product $\langle\cdot,\cdot\rangle$. A complete inner product space is called Hilbert space.

Definition 2.3. Let $\mathcal{H}$ be Hilbert space and $f : [a, b]_{\mathbb{T}} \to \mathcal{H}$. Let g be a non-decreasing function defined on $[a, b]_{\mathbb{T}}$ and let $P = \{t_0, t_1, ..., t_n\}$ be a tagged partition of $[a, b]_{\mathbb{T}}$. The Riemann-Stieltjes-Δ-sum $\Delta_S(P_\delta, f, q)$ of f with respect to g on partition P, is defined by

$$\Delta_S(P_\delta, f, g) = \sum_{j=1}^{n} f(\xi_j)[g(t_j) - g(t_{j-1})]. \tag{2.6}$$

Since $\Delta_{g_j} = g(t_j) - g(t_{j-1})$, therefore, the Riemann-StieltjesΔ-sum can be written as

$$\Delta_S(P_\delta, f, g) = \sum_{j=1}^{n} f(\xi_j)\Delta_{g_j}. \tag{2.7}$$

Definition 2.4. Let $\mathcal{H}$ be Hilbert space and $f : [a, b]_{\mathbb{T}} \to \mathcal{H}$. A function f is said to be Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a, b]_{\mathbb{T}}$ if f is continuous $f \in \mathcal{C}_0$ such that the Riemann-Stieltjes-Δ- derivative of f is α. We denote the Riemann-Stieltjes-Δ-integral of f with respect to a monotone increasing function g on $[a, b]_{\mathbb{T}}$

$$\int_a^b f(t)\Delta g(t) = \alpha. \tag{2.8}$$

Let $f : [a, b]_{\mathbb{T}} \to \mathcal{C}$ be continuous on $[a, b]_{\mathbb{T}}$ and $g : [a, b]_{\mathbb{T}} \to \mathcal{C}$ be of bounded variation. Denote $\vee_a^b(g)$ the total variation of g on $[a, b]_{\mathbb{T}}$, The following sharp inequality holds

$$\left\| \int_a^b f(t)\Delta g(t) \right\| \leq \max_{t\in[a,b]_{\mathbb{T}}} |f(t)| \vee_a^b (g) \tag{2.9}$$

Let $L > 0$ be Lipschitzian constant and g be nondecreasing function such that

$$\|g(t) - g(s)\| \leq L|t - s|$$

for any $t, s \in [a, b]_{\mathbb{T}}$, then we have

$$\|g(t) - g(s)\| \leq L \int_a^b |f(t)|\Delta g(t) \tag{2.10}$$

for any Riemann integrable function $f : [a, b]_{\mathbb{T}} \to \mathcal{C}$.

Moreover, we have the modular inequality

$$\|g(t) - g(s)\| \leq \int_a^b |f(t)|\Delta g(t) \tag{2.11}$$

if $g : [a, b]_{\mathbb{T}} \to \mathcal{C}$ is monotone increasing on $[a, b]_{\mathbb{T}}$.

Theorem 2.5. *Let $h : [a, b]_{\mathbb{T}} \to \mathcal{C}$ be Lipschitzian with the constant $L > 0$, $p : [a, b]_{\mathbb{T}} \to \mathcal{C}$ be Lipschitzian with the constant $K > 0$, $f : [a, b]_{\mathbb{T}} \to \mathcal{C}$ is continuous on $[a, b]_{\mathbb{T}}$ and $g : [a, b]_{\mathbb{T}} \to \mathcal{C}$ is monotone nondecreasing function on $[a, b]_{\mathbb{T}}$. Then we have the inequality*

$$\left\| \int_a^b f(t)\Delta g(h(t)p(t)) \right\| \leq K \int_a^b |f(t)h(t)|\Delta g(t) + L \int_a^b |f(t)p(t)|\Delta g(t) \tag{2.12}$$

$$\leq \max(K, L) \int_a^b |f(t)|(|h(t)| + |p(t)|)\Delta g(t). \tag{2.13}$$

The inequalities (2.12) and (2.13) are sharp.

Proof: Let $q : [a,b]_{\mathbb{T}} \to \mathcal{C}$ is continuous and $v : [a,b]_{\mathbb{T}} \to \mathcal{C}$ is Lipschitzian with the constant $L > 0$, then the Riemann-Stieltjes-Δ-integral $\int_a^b q(s)\Delta v(s)$ exists and we have the inequality

$$\left\|\int_a^b q(s)\Delta v(s)\right\| \leq L\int_a^b |q(s)|\Delta v(s). \tag{2.14}$$

Let $h, p : [a,b]_{\mathbb{T}} \to \mathcal{C}$ be two functions of bounded variation and such that the Riemann-Stieltjes-Δ-integral $\int_a^b h(t)\Delta g(t)$ exists, if $f : [a,b]_{\mathbb{T}} \to \mathcal{C}$ is continuous and $g : [a,b]_{\mathbb{T}} \to \mathcal{C}$ is monotone nondecreasing function on $[a,b]_{\mathbb{T}}$, then the Riemann-Stieltjes-Δ-integral $\int_a^b f(t)\Delta g(h(t)p(t))$ exists and

$$\int_a^b f(t)\Delta g(h(t)p(t)) = \int_a^b f(t)h(t)\Delta g(p(t)) + \int_a^b f(t)p(t)\Delta g(h(t)). \tag{2.15}$$

Taking the norm of (2.15), we have

$$\begin{aligned}\left\|\int_a^b f(t)\Delta g(h(t)p(t))\right\| &\leq \left\|\int_a^b f(t)h(t)\Delta g(p(t))\right\| + \left\|\int_a^b f(t)p(t)\Delta g(h(t))\right\| \\ &\leq K\int_a^b |f(t)h(t)|\Delta g(t) + L\int_a^b |f(t)p(t)|\Delta g(t) \\ &\leq \max\{K,L\}\int_a^b |f(t)|(|h(t)|+|p(t)|)\Delta g(t).\end{aligned}$$

The inequality (2.12) is proved.
Let the functions $f, p : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and $f(t) = p(t) = |t - \frac{a+b}{2}|$. The functions f and p are Lipschizian with the constant $L = 1$. For any $t, s \in [a,b]_{\mathbb{T}}$, we have

$$\begin{aligned}\|f(t) - f(t)\| &= \left\|t - \frac{a+b}{2}\right\| - \left\|s - \frac{a+b}{2}\right\| \\ &\leq |t-s|\end{aligned}$$

which shows that the function f is Lipschitzian with the constant $L = 1$.
Now

$$\begin{aligned}\left\|\int_a^b f(t)\Delta(h(t)p(t)\right\| &= \int_a^b f(t)\Delta g\left(\left(t - \frac{a+b}{2}\right)^2\right) \\ &= 2\left\|\int_a^b f(t)\left(t - \frac{a+b}{2}\right)\Delta g(t)\right\|\end{aligned}$$

and

$$K\int_a^b |f(t)h(t)|\Delta g(t) + L\int_a^b |f(t)p(t)|\Delta g(t) = 2\int_a^b |f(t)|\left|t - \frac{a+b}{2}\right|\Delta g(t) \tag{2.16}$$

and the inequality (2.12) becomes

$$\left\|\int_a^b f(t)\left(t - \frac{a+b}{2}\right)\Delta g(t)\right\| \leq \int_a^b |f(t)|\left|t - \frac{a+b}{2}\right|\Delta g(t). \tag{2.17}$$

Equality holds if $f : [a.b]_{\mathbb{T}} \to \mathbb{R}, h(t) = t - \dfrac{a+b}{2}$.

Theorem 2.6. ***Cauchy-Schwarz Inequality***
Let $\mathcal{H}$ be Hilbert space. For all vectors x, y in an inner product space $\mathcal{H}$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|. \tag{2.18}$$

Equality holds if and only if x and y are collinear.

Proof. Let x and y be nonzero vectors, by applying positive definite property to $x-ty$ for $t \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leq \langle x-ty, x-ty\rangle = \langle x, x-ty\rangle - t\langle y, x-ty\rangle \\ &= \langle x, x\rangle - t\langle x, y\rangle - \langle x, y\rangle + t^2\langle y, y\rangle \\ &= \|x\|^2 - 2t\langle x, y\rangle + t^2\|y\|^2. \end{aligned}$$

Put $t = \langle x, y\rangle/\|y\|^2$ to obtain

$$0 \leq \|x\|^2 - \frac{\langle x, y\rangle^2}{\|y\|^2}. \tag{2.19}$$

Therefore,

$$\langle x, y\rangle^2 \leq \|x\|^2\|y\|^2 \tag{2.20}$$

Hence the inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

holds.

Corollary 2.7. *Let $\mathcal{H}$ be Hilbert space and $f, h : [a, b]_{\mathbb{T}} \to \mathcal{H}$. f and h are Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a, b]_{\mathbb{T}}$ if f and h are continuous such that*

$$\left\|\int_a^b f(t)h(t)\Delta g(t)\right\| \leq \left(\int_a^b f(t)^2\Delta g(t)\right)^{\frac{1}{2}} \left(\int_a^b h(t)^2\Delta g(t)\right)^{\frac{1}{2}}. \tag{2.21}$$

The L^2 norms on $\mathcal{C}[a, b]_{\mathbb{T}}$ are indeed norms.

Theorem 2.8. *Let f, h : $[a, b]_{\mathbb{T}\to\mathbb{C}}$ be continuous in $[a, b]_{\mathbb{T}}$. If g : $[a, b]^{\mathbb{T}} \to \mathbb{C}$ is a dominated function with monotonic increasing functions λ and μ. for any continuous nonnegative function $\Psi : [a, b]_{\mathbb{T}} \to [0, \infty)$ we have*

$$\left|\int_a^b \Psi f h\Delta g(t)\right|^2 \leq \int_a^b \Psi|f|^2\Delta\lambda(t) \int_a^b \Psi|h|^2\Delta\mu(t). \tag{2.22}$$

Proof. Since the Riemann-Stieltjes-Δ-integral $\int_a^b \Psi f h\Delta g(t)$ exists, a partition of I is any finite ordered subset $P = t_0, t_1, ..., t_n \subset [a, b]_{\mathbb{T}}$, where $I_n^{(n)} : a = t_0 < t_1 < ... < t_n = b$. Each partition $P = t_0, t_1, ..., t_n$ of I decomposes I into subintervals $I_{\Delta j} = [t_{j-1}, t_j]_{\Delta}, j = 1, 2, ..., n$ with the norm

$$\mu\left(I_n^{(n)}\right) := \max_{j\in\{0,\cdots,n-1\}} \left(t_{j+1}^n - t_j^{(n)}\right) \to 0$$

as $n \to \infty$, and for any intermediate points $\xi_j^{(n)} \in [t_j^{(n)}, t_{j+1}^{(n)}], \quad j \in \{0, \cdots, n-1\}$, we have

$$\begin{aligned}
\int_a^b \Psi f h \Delta g(t) &= \left| \lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) f(\xi_j^{(n)}) h(\xi_j^{(n)}) [g\left(t_{j+1}^n\right) - g\left(t_j^{(n)}\right)] \right| && (2.23)\\
&\leq \lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) |f(\xi_j^{(n)})| |h(\xi_j^{(n)})| \left| g\left(t_{j+1}^n\right) - g\left(t_j^{(n)}\right) \right| && (2.24)\\
&\leq \lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) |f(\xi_j^{(n)})| |h(\xi_j^{(n)})| && (2.25)\\
&\times \left| \mu\left(t_{j+1}^n\right) - \mu\left(t_j^{(n)}\right) \right|^{\frac{1}{2}} \left| \mu\left(t_{j+1}^n\right) - \mu\left(t_j^{(n)}\right) \right|^{\frac{1}{2}} := I. && (2.26)
\end{aligned}$$

Applying the weighted Cauchy-Bunyakovsky-Schwarz discrete inequality

$$\sum_{k=1}^{n} \Psi_k a_k b_k \leq \left(\sum_{k=1}^{n} \Psi_k a_k^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \Psi_k b_k^2 \right)^{\frac{1}{2}} \tag{2.27}$$

where $\Psi_k, a_k, b_k \geq 0$ for $k \in \{1, ..., n\}$, we have

$$\begin{aligned}
I &\leq \left(\lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) |f(\xi_j^{(n)})|^2 \left[\left| \mu\left(t_{j+1}^n\right) - \mu\left(t_j^{(n)}\right) \right|^{\frac{1}{2}} \right]^2 \right)^{\frac{1}{2}} && (2.28)\\
&\times \left(\lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) |h(\xi_j^{(n)})|^2 \left[\left| \nu\left(t_{j+1}^n\right) - \nu\left(t_j^{(n)}\right) \right|^{\frac{1}{2}} \right]^2 \right)^{\frac{1}{2}} && (2.29)\\
&= \left(\lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) |f(\xi_j^{(n)})|^2 \left[\mu\left(t_{j+1}^n\right) - \mu\left(t_j^{(n)}\right) \right] \right)^{\frac{1}{2}} && (2.30)\\
&\times \left(\lim_{\mu(I_n^{(n)}) \to 0} \sum_{j=0}^{n-1} \Psi(\xi_j^{(n)}) |h(\xi_j^{(n)})|^2 \left[\nu\left(t_{j+1}^n\right) - \nu\left(t_j^{(n)}\right) \right] \right)^{\frac{1}{2}} && (2.31)\\
&= \left(\int_a^b \Psi |f|^2 \Delta\mu(t) \right)^{\frac{1}{2}} \left(\int_a^b \Psi |h|^2 \Delta\nu(t) \right)^{\frac{1}{2}} && (2.32)
\end{aligned}$$

Using the inequalities (2.23)-(2.26) and (2.28)-(2.32), we have the required result (2.21).

3. Applications in Self-Adjoint Operators

Let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}; \langle \cdot, \cdot \rangle)$. Let $X \in \mathcal{B}(\mathcal{H})$ be selfadjoint and let Ψ_τ be defined for all $\tau \in \mathbb{R}$ as follows

$$\Psi_\tau(s) = \begin{cases} 1, & for \quad -\infty < s \leq \tau, \\ 0, & for \quad \tau < s < +\infty. \end{cases}$$

Then for every $\tau \in \mathbb{R}$ the operator

$$P_\tau := \Psi_\tau(X) \tag{3.1}$$

is a projection which reduces X. The spectral representation of bounded selfadjoint operators in Hilbert spaces ([6]) show case the properties of the projection (3.1).

Proposition 3.1. *Let X be a selfadjoint operator on Hilbert space $\mathcal{H}$ and let $m = \min\{\tau \mid \tau \in Sp(X)\}$ and $M = \max\{\tau \mid \tau \in Sp(X)\}$. Let $\Psi : [a,b]_{\mathbb{T}} \to \mathcal{H}$, and Ψ is Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a,b]_{\mathbb{T}}$. Then there exists a family of projections $\{P_\tau\}_\tau \in \mathbb{R}$, called the spectral family of X, with the following properties:*
(i) $P_\tau \leq P_\tau, for\ \tau \leq \tau^{'}$;
(ii) $P_{m-0} = 0, P_M = I$ and $P_{\tau+0} = P_\tau$ for all $\tau \in \mathbb{R}$;
we the representation

$$X = \int_{m-0}^{M} \tau \Delta g(P_\tau). \tag{3.2}$$

Moreover, for every continuous complex-valued function Ψ defined on $\mathbb{R}$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| \Psi(X) - \sum_{k=1}^{n} \Psi(\tau_k^{'})[P_{\tau_k} - P_{\tau_{k-1}}] \right\| \leq \varepsilon \tag{3.3}$$

whenever

$$\begin{cases} \tau_0 < m = \tau_1 < \cdots < \tau_{n-1} < \tau_n = M, \\ \tau_k - \tau_{k-1} \leq \delta \quad for\ 1 \leq k \leq n, \\ \tau_k^{'} \in [\tau_{k-1}, \tau_k] \quad for\ 1 \leq k \leq n \end{cases}$$

which means that

$$\Psi(X) = \int_{m-0}^{M} \Psi(\tau) \Delta g(P_\tau), \tag{3.4}$$

which is the integral of Riemann-Stieltjes type.

Proof. Let $X \in \mathcal{B}(\mathcal{H})$ be selfadjoint operator and $P_\tau = \Psi_\tau(X)$ be projection which reduces X. With the assumptions in Proposition 3.1 for X, P_τ and Ψ, we have the following representations:

$$\Psi(X)s = \int_{m-0}^{M} \Psi(\tau) \Delta g(P_\tau)s \quad \forall s \in \mathcal{H} \tag{3.5}$$

and

$$\langle \Psi(X)s, t \rangle = \int_{m-0}^{M} \Psi(\tau) \Delta g \langle P_\tau s, t \rangle \quad \forall\ s, t \in \mathcal{H}. \tag{3.6}$$

Thus,

$$\langle \Psi(X)s, s \rangle = \int_{m-0}^{M} \Psi(\tau) \Delta g \langle P_\tau s, s \rangle \quad \forall\ s \in \mathcal{H}. \tag{3.7}$$

Hence, we obtain the inequality

$$\|\Psi(X)s\|^2 = \int_{m-0}^{M} |\Psi(\tau)|^2 \Delta g \|P_\tau\|^2 s \quad \forall\ s \in \mathcal{H} \tag{3.8}$$

Proposition 3.2. *Let X be a selfadjoint operator on Hilbert space $\mathcal{H}$ and let $m = \min\{\tau \mid \tau \in Sp(X)\}$ and $M = \max\{\tau \mid \tau \in Sp(X)\}$. Let $f, h : [a,b]_{\mathbb{T}} \to \mathcal{H}$, f and h are Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a,b]_{\mathbb{T}}$ if f and h are continuous functions on $[m, M]$, then we have the inequality*

$$|\langle f(X)h(X)x, y \rangle|^2 \leq \langle |f(X)|^2 x, x \rangle \langle |h(X)|^2 y, y \rangle \tag{3.9}$$

for any $x, y \in \mathcal{H}$.

Proof. Let $\mathcal{H}$ be Hilbert space and let $x, y \in \mathcal{H}$. For $\varepsilon > 0$, the functions $u, v, w : [m-\varepsilon, M] \to \mathcal{C}$ are defined by $v(t) = \langle P_t x, y\rangle$, $v(t) = \langle P_t x, x\rangle$ and $w(t) = \langle P_t y, y\rangle$ where $\{P_\tau\}_{\tau\in\mathbb{R}}$ is the spectral family of the bounded selfadjoint operator X. Using Theorem 2.8 for nonnegative operator f and for $t, s \in [m-\varepsilon, M]$ with $t > s$ such that

$$|\langle fx, y\rangle|^2 \leq \langle fx, x\rangle\langle fy, y\rangle, \tag{3.10}$$

we have

$$\begin{aligned} |u(t) - u(s)|^2 &= |\langle (P_t - P_s)x, y\rangle|^2 \leq \langle (P_t - P_s)x, x\rangle\langle (P_t - P_s)y, y\rangle \\ &= (v(t) - v(s))(w(t) - w(s)). \end{aligned}$$

It shows that u is dominated by the nondecreasing functions (v, w) on $[m-\varepsilon, M]$.
By using Theorem 2.8 for f, h, u, v on $[m-\varepsilon, M$, we have

$$\left|\int_{m-\varepsilon}^{M} f(t)\Delta g(\langle P_t x, y\rangle)\right|^2 \leq \int_{m-\varepsilon}^{M} |f(t)|\Delta g(\langle P_t x, x\rangle)\int_{m-\varepsilon}^{M} |f(t)|\Delta g(\langle P_t y, y\rangle) \tag{3.11}$$

for any $x, y \in \mathcal{H}$.
By using the representation of continuous functions of selfadjoint operators and letting $\varepsilon \to 0^+$, we have the required result (3.9).

Proposition 3.3. *Let X be a selfadjoint operator on Hilbert space $\mathcal{H}$ and let $m = \min\{\tau \mid \tau \in Sp(X)\}$ and $M = \max\{\tau \mid \tau \in Sp(X)\}$. Let $f, h : [a, b]_{\mathbb{T}} \to \mathcal{H}$, f and h are Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a, b]_{\mathbb{T}}$ if f and h are continuous functions on $[m, M]$. Then for any $x, y \in \mathcal{H}$ with $x, y \neq 0$, we have the inequality*

$$|C(f, h; X, x, y)|^2 \leq C(f; X, x)C(h; X, y) \tag{3.12}$$

Proof. The proof of proposition 3.3 follows from a similar argument of proposition 3.2. Therefore, it is omitted.

The continuous functions $f, h : [a, b]_{\mathbb{T}} \to \mathcal{C}$ and the selfadjoint operator X have the following functionals:

$$F(f, h; X, x, y) = \langle x, y\rangle\langle f(X)h(X)x, y\rangle - \langle f(X)x, y\rangle\langle h(X)x, y\rangle, \tag{3.13}$$

$$G(f; x) = \|x\|^2\langle |f(X)|^2 x, x\rangle - |\langle f(X)x, x\rangle|^2 (\geq 0) \tag{3.14}$$

and

$$G(f; x, y) = \|y\|^2\langle |f(X)|^2 x, x\rangle + \|x\|^2\langle |f(X)|^2 y, y\rangle - 2Re((f(X)x, x)\langle \overline{f}(X)y, y\rangle)(\geq 0), \tag{3.15}$$

for any $x, y \in \mathcal{H}$.

Corollary 3.4. *Let X be a selfadjoint operator on Hilbert space $\mathcal{H}$ and let $m = \min\{\tau \mid \tau \in Sp(X)\}$ and $M = \max\{\tau \mid \tau \in Sp(X)\}$. Let $f, h : [a, b]_{\mathbb{T}} \to \mathcal{H}$, f and h are Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a, b]_{\mathbb{T}}$ if f and h are continuous functions on $[m, M]$. Then for any $x, y \in \mathcal{H}$ with $x, y \neq 0$, we have the inequality*

$$|F(f, h; X, x, y)|^2 \leq \frac{1}{2}[G(f, x)G(f; x, y)G(h; x, y)G(h; y)]. \tag{3.16}$$

4. Applications for Unitary Operators

Let $(\mathcal{H}, \langle , . \rangle)$ be a complex Hilbert space. The bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ on the Hilbert space $\mathcal{H}$ is unitary if and only if $T^* = T^{-1}$. If T is a unitary operator, then there exists a family of projections $\{P_\tau\}_{\tau_{\in[0,2\pi]}}$, called the spectral family of T with the following properties:
(i) $P_\tau \leq P_\nu for \ \ 0 \leq \tau \leq \nu \leq 2\pi$
(ii) $P_0 = 0$ and $P_{2\pi} = I$, the identity operator on $\mathcal{H}$.
(iii) $P_{\tau+0} = P_\tau$ for $0 \leq \tau \leq 2\pi$;
(iv) $T = \int_0^{2\pi} e^{i\tau} \Delta g(P_\tau)$ where the integral is of Riemann-Stieltjes-Δ-type.

Let $\{\mathcal{F}_\tau\}_{\tau_{\in[0,2\pi]}}$ be family of projections satisfying the above properties (i)-(iv), for the operator T such that $\mathcal{F}_\tau = P_\tau$ for all $\tau \in [0, 2\pi]$.
Thus, for every continuous complex-valued function $F : \mathcal{C}(0, 1) \to \mathcal{C}$ on the complex unit circle $\mathcal{C}(0, 1)$, we have

$$F(T) = \int_0^{2\pi} F(e^{i\tau}) \Delta g(P_\tau) \tag{4.1}$$

where the integral is of the Riemann-Stieltjes-Δ-type.
The following inequalities follow from (4.1):

$$F(T)x = \int_0^{2\pi} F(e^{i\tau}) \Delta g(P_\tau x), \tag{4.2}$$

$$\langle F(T)x, y\rangle = \int_0^{2\pi} F(e^{i\tau}) \Delta g \langle P_\tau x, y\rangle \tag{4.3}$$

and

$$|F(T)x|^2 = \int_0^{2\pi} |F(e^{i\tau})|^2 \Delta g \|P_\tau x\|^2 \tag{4.4}$$

for any $x, y \in \mathcal{H}$.

Theorem 4.1. *Let T be a unitary operator on the Hilbert space $\mathcal{H}$. Let $F, H : [a, b]_{\mathbb{T}} \to \mathcal{H}$, F and H are Riemann-Stieltjes-Δ-integrable with respect to a monotone increasing function g on $[a, b]_{\mathbb{T}}$. Then for every continuous complex-valued function $F, H : \mathcal{C}(0, 1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0, 1)$, we have*

$$|\langle F(T)H(T)x, y\rangle|^2 \leq \langle |F(T)|^2 x, x\rangle \langle |H(T)|^2 y, y\rangle \tag{4.5}$$

for any $x, y \in \mathcal{H}$.

Proof. Let $\{P_\tau\}_{\tau_{\in[0,2\pi]}}$ be the spectral family of the unitary operator T. Define the function $J, G, H : [0, 2\pi] \to \mathbb{C}$ given by

$$J(t) = \langle J_t x, y\rangle, \quad G(t) = \langle P_t x, x\rangle \quad and \quad H(t0 = \langle P_t y, y\rangle.$$

for any $x, y \in \mathcal{H}$ and for $t, s \in [0, 2\pi]$.
Applying the Schwarz inequality for nonnegative operator T

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle \langle Ty, y\rangle, \tag{4.6}$$

we have

$$\begin{aligned} |J(t) - J(s)|^2 &= |\langle (P_t - P_s)x, y\rangle|^2 \leq \langle (P_t - P_s)x, x\rangle \langle (P_t - P_s)y, y\rangle \\ &= (G(t) - G(s))(H(t) - H(s)). \end{aligned}$$

This shows that J is dominated by the monotonic increasing functions (G, H) on $[0, 2\pi]$. By utilizing Theorem 2.8 for $F(e^{it}), J, G$ and H on $[0, 2\pi]$, we have

$$\left|\int_0^{2\pi} F(e^{it})H(e^{it})\Delta g(\langle P_t x, y\rangle)\right| \leq \int_0^{2\pi} |F(e^{it})|^2 \Delta g(\langle P_t x, x\rangle)\int_0^{2\pi} |H(e^{it})|^2 \Delta g(\langle P_t y, y\rangle), \tag{4.7}$$

for any $x, y \in \mathcal{H}$.
By the representation of continuous function of unitary operators, we have the required result

$$|\langle F(T)H(T)x, y\rangle|^2 \leq \langle|F(T)|^2 x, x\rangle\langle|H(T)|^2 y, y\rangle. \tag{4.8}$$

5. Conclusion

We used the definition and properties of Riemann-Stieltjes-Δ-integral to give suitable applications to self-adjoint and unitary operators in Hilbert spaces. Thees results are obtained on time scales which unify both discrete and continuous problems.

Conflict of Interests

The authors declare that they have no conflict of interests among them.

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References

[1] D. A. Afariogun, H. O. Olaoluwa. *Refinement of Henstock-Kurzweil-Stieltjes-◇-Integral for topological Vector Space-valued functions on time scales*. J. Sci. Research and Develop. (2020), Vol. 19 (2): 1-12.
[2] R. Bartle, *Return to the Riemann integral*, Amer. Math. Monthly 103 (1996), 625-632.
[3] M. Bohner and G. Guseinov, *Riemann and Lebesgue integration, in advances in dynamic equations on time scales*, 117-163, Birkhauser, Boston, (2003).
[4] S. S. Dragomir, *Inequalities of Gruss type for the Stieltjes integral and applications*, Kragujevac J. Math. pp 89-122.
[5] S. S. Dragomir, *Inequalities for the Riemann-Stieltjes integral of S-dominated Integrators with Applications (I)*, Preprint RGMIA Res. Rep. Call. 16(2013).
[6] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc., New York, (1969), pp. 256-276.
[7] E. McShane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, vol. 88. Amer. Math. Soc. (1969).
[8] D. Mozyrska, E. Pawluszewicz, and D. F. M. Torres, *The Riemann-Stieltjes integral on time scales*, Austr. J. Math. Anal. Appl. (2009).
[9] C. W. Swartz and D. S. Kurtz, *Theories of Integration: The integrals of Riemann, Lebesgue, Henstock-Kurzweil, and McShane*, World Scientific, (2004).
[10] B. Thompson, *Henstock-Kurzweil integrals on time scales*, Panamer. Math. J. No. 1, 18(2008), 1-19.
[11] A. K. Zareen, J. Fahd, K. Aziz, K. Hasib. *Derivation of dynamical integral inequalities based on two-dimensional time scales theory*. J. Ineq. Appl. (2020): 209, 1-17.

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