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EXISTENCE RESULT FOR SOME COUPLED NONLINEAR PARABOLIC SYSTEMS IN ORLICZ-SOBOLEV SPACES

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Abstract. Consider the nonlinear parabolic system

$$\begin{cases} \frac{\partial b_i(x,u_i)}{\partial t} - \operatorname{div}\Big(\mathcal{A}(x,t,u_i,\nabla u_i) + \Phi_i(x,t,u_i)\Big) + f_i(x,u_1,u_2) = 0 & \text{in } Q_T \\ u_i = 0 & \text{on } \Gamma \\ b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) & \text{in } \Omega, \end{cases}$$

where $i = 1, 2$. In this paper we deal with the renormalized solution for the above system in Orlicz-Sobolev spaces where f_i is a Carathéodory function satisfying some growth assumptions. The main term which contains the space derivatives and a non-coercive lower order term are considered in divergence form satisfying only the original Orlicz growths.

1. Introduction

The analysis of partial differential equations (PDEs) is one of the main fields of mathematics, PDEs with nonlinearities involving modular functions have attracted an increasing amount of attention in recent years. Systems of nonlinear PDEs present some new and interesting phenomena, which are not present in the study of a single equation. Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $Q_T = \Omega \times (0,T)$ where T is a positive real number and M is an Orlicz function. Let $\mathrm{A}(u) := -\operatorname{div} \mathcal{A}(x,t,u,\nabla u)$ be a so-called Leray-Lions type operator whose prototype is the p-Laplacian operator and $b_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $b_i(x,\cdot)$ is a strictly increasing C^1-function for any fixed $x \in \Omega$ with $b_i(x,0) = 0$.

Consider for $i = 1, 2$, the following parabolic system

$$\begin{cases} \dfrac{\partial b_i(x,u_i)}{\partial t} + \mathrm{A}(u_i) - \operatorname{div} \Phi_i(x,t,u_i) + f_i(x,u_1,u_2) = 0 & \text{in } Q_T \\ u_i = 0 & \text{on } \Gamma \\ b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) & \text{in } \Omega, \end{cases} \tag{1.1}$$

A model of applications of these operators is the Boussinesq's system:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u - 2\operatorname{div}(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) & \text{in } Q_T \\ \dfrac{\partial b(\theta)}{\partial t} + u \cdot \nabla b(\theta) - \triangle\theta = 2\mu(\theta)|\varepsilon(u)|^2 & \text{in } Q_T \\ u = 0, \quad \theta = 0 & \text{on } \Gamma \\ u(t=0) = u_0 \quad b(\theta)(t=0) = b(\theta_0) & \text{in } \Omega, \end{cases}$$

where the first equation is the motion conservation equation, the unknowns are the fields of displacement $u : Q_T \to \mathbb{R}^N$ and temperature $\theta : Q_T \to \mathbb{R}$, the field $\varepsilon(\nabla u) = \dfrac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain rate tensor. Also, for applications to fluid mechanics models one can see [17].

The problem (1.1), with a single equation, has been investigated in different particular directions. As example, in the classical Sobolev spaces, i.e $M = t^p$, for $\Phi \equiv 0$, b is a maximal monotone graph on $\mathbb{R}$ and $\mathcal{A}(x,t,s,\xi)$ is independent of s, existence and uniqueness of a renormalized solution have been studied by Blanchard and Murat in [5]

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and by Blanchard and Porretta in the case where $\mathcal{A}(x,t,s,\xi)$ is independent of t in [6]. In [1], Bennouna et al. have studied problem (1.1) for a measure $\mu = f - \operatorname{div}(F)$, with $f \in L^1(Q_T)$, $F \in (L^{p'}(Q_T))^N$ and Φ satisfies the condition

$$|\Phi(x,t,s)| \le c(x,t)|s|^\gamma,$$

with $c(x,t) \in L^\tau(Q_T)$ for some $\tau = \dfrac{N+p}{p-1}$ and $\gamma = \dfrac{N+2}{N+p}(p-1)$.

In what concerns contributions in Orlicz spaces with a single equation, Azroul et al. have proved in [3] the existence of renormalized solution, where Φ depends only on u and $b(x,u) = b(u)$, the same result has been shown by Redwane in [23] where $b(x,u)$ depends on x and u. In [21], the authors have proved existence of renormalized solution under the assumptions, $f \in L^1(Q_T)$ and Φ satisfies a growth condition described by an N-function P that increases essentially less rapidly than the appropriate Orlicz function M,

$$|\Phi(x,t,s)| \le \overline{P}^{-1}(P(|s|)) \text{ with } P \prec\prec M. \tag{1.2}$$

The previous result has been enhanced in [7] under the likely growth condition in the elliptic case,

$$|\Phi(x,s)| \le \gamma(x) + \overline{M}^{-1}(M(|s|)), \text{ with } \gamma \in E_{\overline{M}}(\Omega). \tag{1.3}$$

Turn now to the doubly equation, in the classical Sobolev spaces, the system (1.1) has been solved by Azroul et al. in [4] in the case where b_i and Φ_i are independent of x. For the study of (1.1) in some particular cases one can consult [8, 9, 10, 19, 22].

The approach of this paper is how to deal with the existence of renormalized solutions for system (1.1) in Orlicz spaces where Φ_i satisfies the original Orlicz growth condition

$$|\Phi_i(x,t,s)| \le \gamma(x,t) + \overline{M}^{-1}(M(|s|)), \text{ where } \gamma \in E_{\overline{M}}(Q_T), \tag{1.4}$$

without assuming any restriction on the modular function M neither on its complementary $\overline{M}$, the described problem lives in non reflexive Orlicz spaces. The existence result in this context generalizes all works mentioned above.

In dealing with this problem, we have encountered some difficulties, essentially, under the growth condition (1.4), it's difficult to prove existence of solution for the regularized problem and proving its convergence, which are the basic results in the proof of such solutions. The novelty in the main proofs follows thanks to an algebraic trick combined with the convexity of M and Young's inequality on a well-chosen positive quantities.

This article is organized as follows, in section 2, we recall some well-known preliminaries, results and properties of Orlicz-Sobolev spaces and inhomogeneous Orlicz-Sobolev spaces. Section 3 is devoted to basic assumptions, problem setting and the proof of the main result.

2. Preliminaries

2.1. **Orlicz-Sobolev spaces.** Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and convex function with:

$$M(t) > 0 \text{ for } t > 0, \lim_{t\to 0} \frac{M(t)}{t} = 0 \text{ and } \lim_{t\to+\infty} \frac{M(t)}{t} = +\infty.$$

The function M is said an N-function or an Orlicz function, the N-function complementary to M is defined as

$$\overline{M}(t) = \sup\Big\{st - M(s), s \ge 0\Big\}.$$

We recall that (see [2])

$$M(t) \le t\overline{M}^{-1}(M(t)) \le 2M(t) \quad \text{for all } t \ge 0 \tag{2.5}$$

and the Young's inequality: for all $s,t \ge 0$,

$$st \le \overline{M}(s) + M(t).$$

We said that M satisfies the Δ_2-condition if for some $k > 0$,

$$M(2t) \leq kM(t) \quad \text{for all } t \geq 0, \tag{2.6}$$

and if (2.6) holds only for $t \geq t_0$, then M is said to satisfy the Δ_2-condition near infinity.

Let M_1 and M_2 be two N-functions. The notation $M_1 \prec\prec M_2$ means that M_1 grows essentially less rapidly than M_2, i.e.

$$\forall \epsilon > 0, \quad \lim_{t\to\infty} \frac{M_1(t)}{M_2(\epsilon t)} = 0,$$

that is the case if and only if

$$\lim_{t\to\infty} \frac{(M_2)^{-1}(t)}{(M_1)^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < \infty \quad (\text{resp.} \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0).$$

Endowed with the Luxemburg norm

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)dx \leq 1\Big\},$$

and the so-called Orlicz norm, that is

$$|||u|||_{M,\Omega} = \sup_{\|v\|_{\overline{M}}\leq 1} \int_\Omega |u(x)\, v(x)|\, dx,$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The Orlicz-Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$).

This is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\Big)\, dx \to 0 \quad \text{for all } |\alpha| \leq 1;$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with norm convergence. Recall that the norm $\|Du\|_M$ defined on $W_0^1L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [13]).

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open Ω has the segment property then the space $\mathfrak{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [13]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined. For more details one can see for example [2] or [16].

2.2. **Inhomogeneous Orlicz-Sobolev spaces.** As in Section 2.1 of Preliminaries, let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q_T = \Omega \times (0, T)$. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \Omega$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \Big\{ u \in L_M(Q_T) : D_x^\alpha u \in L_M(Q_T) \quad \text{for all} \quad |\alpha| \le 1 \Big\},$$

and

$$W^{1,x}E_M(Q_T) = \Big\{ u \in E_M(Q_T) : D_x^\alpha u \in E_M(Q_T) \quad \text{for all} \quad |\alpha| \le 1 \Big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm,

$$\|u\| = \sum_{|\alpha| \le 1} \|D_x^\alpha u\|_{M,Q_T}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which have as many copies as there is α-order derivatives, $|\alpha| \le 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}}))$. If $u \in W^{1,x}L_M(Q_T)$ then the function : $t \mapsto u(t) = u(t, \cdot)$ is defined on $(0, T)$ with values in $W^1 L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then the concerned function is a $W^1 E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_M(Q_T) \subset L^1(0, T; W^1 E_M(\Omega))$. The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $u \in W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0, T)$. However, the scalar function $t \mapsto \| u(t) \|_{M,\Omega}$ is in $L^1(0, T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathfrak{D}(Q_T)$. It is proved that when Ω has the segment property, then each element u of the closure of $\mathfrak{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q_T)$, of some subsequence $(u_n) \subset \mathfrak{D}(Q_T)$ for the modular convergence; i.e., if, for some $\lambda > 0$, such that for all $|\alpha| \le 1$;

$$\int_{Q_T} M\Big(\frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda} \Big) dx\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This implies that the sequence (u_n) converges to u in $W^{1,x}L_M(Q_T)$ in the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$. Consequently,

$$\overline{\mathfrak{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathfrak{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore,

$$W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M.$$

We have then the following complementary system

$$\Big(W_0^{1,x}L_M(Q_T), F, W_0^{1,x}E_M(Q_T), F_0 \Big)$$

F being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is shown that,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \Big\{ f = \sum_{|\alpha| \le 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q_T) \Big\},$$

this space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M},Q_T},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{M}}(Q_T).$$

The space F_0 is then given by,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \Big\{ f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q_T) \Big\},$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

2.3. Technical lemmas.

Lemma 2.1. [14] *Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function: continuous and convex function with,*

$$M(t) > 0 \text{ for } t > 0, \lim_{t\to 0} \frac{M(t)}{t} = 0 \text{ and } \lim_{t\to +\infty} \frac{M(t)}{t} = +\infty.$$

Let $u_k, u \in L_M(\Omega)$. If $u_k \to u$ for the modular convergence, then $u_k \to u$ for $\sigma(L_M, L_{\overline{M}})$.

Lemma 2.2. *Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function: continuous and convex function with,*

$$M(t) > 0 \text{ for } t > 0, \lim_{t\to 0} \frac{M(t)}{t} = 0 \text{ and } \lim_{t\to +\infty} \frac{M(t)}{t} = +\infty.$$

If $u_n \to u$ for the modular convergence with every $\lambda > 0$ in $L_M(\Omega)$, then $u_n \to u$ strongly in $L_M(\Omega)$.

Proof. We will use the Orlicz norm, for all $\lambda > 0$ we have $\int_\Omega M\left(\frac{|u_k(x) - u(x)|}{\lambda}\right) dx \to 0$ as $k \to \infty$. Thus $M\left(\frac{|u_k(x)-u(x)|}{\lambda}\right)$ tends to 0 strongly in $L^1(\Omega)$ and so for a subsequence, still indexed by k, we can assume that $u_k \longrightarrow u$ a.e. in Ω. For an arbitrary $v \in L_{\overline{M}}(\Omega)$, there exists $\lambda_v > 0$ such that $\overline{M}\left(\frac{v}{\lambda_v}\right) \in L^1(\Omega)$. By Young's inequality and the convexity of $\overline{M}$ we can write

$$|(u_k(x) - u(x))v(x)| \leq M\left(2\lambda_v|u_k(x) - u(x)|\right) + \frac{1}{2}\overline{M}\left(\frac{v(x)}{\lambda_v}\right).$$

Applying Vitali's theorem we obtain $\int_\Omega |(u_k(x) - u(x))v(x)|\, dx \to 0$ for all $v \in L_{\overline{M}}(\Omega)$ and so

$$|||u_k - u|||_{M,\Omega} = \sup_{\|v\|_{\overline{M}}\leq 1} \int_\Omega |(u_k(x) - u(x))v(x)|\, dx \to 0 \text{ as } k \to \infty,$$

which yields the result. ☐

Lemma 2.3. [13] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be a Orlicz function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then, $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & a.e. \text{ in } \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \text{ in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.4. [13] *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly lipschitzian, with $F(0)=0$. Let M be a Orlicz function. we assume that the set of discontinuity points D of F' is finite, then the mapping $F: W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Lemma 2.5. [12] *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, satisfying the segment property, then*

$$\Big\{u\in W_0^{1,x}L_M(Q_T): \frac{\partial u}{\partial t}\in W^{-1,x}L_{\overline{M}}(Q_T)+L^1(Q_T)\Big\}\subset \mathcal{C}\Big([0,T],L^1(\Omega)\Big).$$

Lemma 2.6. [13] *(Integral Poincaré's type inequality in inhomogeneous Orlicz spaces) Let Ω be a bounded open subset of $\mathbb{R}^N$ and M is an Orlicz function, then there exists two positive constants $\delta,\lambda>0$ such that*

$$\int_{Q_T} M(\delta|u(x,t)|)\,dx\,dt \leq \int_{Q_T}\lambda M(|\nabla u(x,t)|)\,dx\,dt \quad \forall u\in W_0^1L_M(Q_T).$$

Lemma 2.7. [20, Lemma 2.4] *If $f_n\subset L^1(\Omega)$ with $f_n\to f\in L^1(\Omega)$ a.e. in Ω, $f_n,f\geq 0$ a. e. in Ω and $\displaystyle\int_\Omega f_n(x)\,dx\to\int_\Omega f(x)\,dx$, then $f_n\to f$ in $L^1(\Omega)$.*

Lemma 2.8. [14] *Suppose that Ω satisfies the segment property and let $u\in W_0^1L_M(\Omega)$. Then, there exists a sequence $(u_n)\subset\mathfrak{D}(\Omega)$ such that $u_n\to u$ for the modular convergence in $W_0^1L_M(\Omega)$. Furthermore, if $u\in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$ then*

$$\|u_n\|_\infty\leq\ (N+1)\|u\|_\infty.$$

Lemma 2.9. *(cf. [11]) Let M be an N-function. Let (u_n) be a sequence of $W^{1,x}L_M(Q_T)$ such that, $u_n\rightharpoonup u$ weakly in $W^{1,x}L_M(Q_T)$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ and $\dfrac{\partial u_n}{\partial t}=h_n+k_n$ in $\mathfrak{D}'(Q_T)$ with h_n is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$ and k_n is bounded in $L^1(Q_T)$. Then, $u_n\to u$ strongly in $L^1_{Loc}(Q_T)$. If further, $u_n\in W_0^{1,x}L_M(Q_T)$ then $u_n\to u$ strongly in $L^1(Q_T)$.*

3. Basic assumptions and main result

Through this paper Ω is a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, satisfying the segment property. Let $Q_T=(0,T)\times\Omega$ be the cylinder of $\mathbb{R}^N$, $\Gamma:=(0,T)\times\partial\Omega$ and M is an Orlicz function. Consider a Carathéodory function $b_i:\Omega\times\mathbb{R}\to\mathbb{R}$ such that for every $x\in\Omega$, $b_i(x,s)$ is a strictly increasing C^1-function with $b_i(x,0)=0$ and for any $k>0$, there exists a constant $\lambda_k^i>0$, a function $A_k^i\in L^\infty(\Omega)$ and a function $\widetilde{A}_k^i\in L_\varphi(\Omega)$ such that,

$$\lambda_k^i\leq\frac{\partial b_i(x,s)}{\partial s}\leq A_k^i(x) \quad\text{and}\quad \Big|\nabla_x\Big(\frac{\partial b_i(x,s)}{\partial s}\Big)\Big|\leq\widetilde{A}_k^i(x). \tag{3.7}$$

Let $A: D(A)\subset W_0^{1,x}L_M(Q_T)\to W^{-1,x}L_{\overline{M}}(Q_T)$ an operator of Leray-Lions type of the form:

$$\mathrm{A}(u):=-\operatorname{div}\ \mathcal{A}(x,t,u,\nabla u),$$

This work aims to prove the existence of renormalized solutions in the setting of Orlicz spaces to the nonlinear parabolic system

$$\begin{cases} \frac{\partial b_i(x,u_i)}{\partial t}-\operatorname{div}\Big(\mathcal{A}(x,t,u_i,\nabla u_i)+\Phi_i(x,t,u_i)\Big)+f_i(x,u_1,u_2)=0 & \text{in } Q_T\\ u_i=0 & \text{on } \Gamma\\ b_i(x,u_i)(t=0)=b_i(x,u_{i,0}) & \text{in } \Omega, \end{cases} \tag{3.8}$$

where $i=1,2$. The vector $\mathcal{A}:Q_T\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $(x,t)\in Q_T$ and for all $s\in\mathbb{R},\xi,\eta\in\mathbb{R}^N(\xi\neq\eta)$ the following conditions

(H_1): There exists a function $c(x,t) \in E_{\overline{M}}(Q_T)$ and some positive constants k_1, k_2, k_3 and an Orlicz function $P \prec\prec M$ such that

$$|\mathcal{A}(x,t,s,\xi)| \leq \beta\Big[c(x,t) + k_1 \overline{M}^{-1}(P(k_2|s|)) + \overline{M}^{-1}(M(k_3|\xi|))\Big].$$

(H_2): $\mathcal{A}$ is strictly monotone,

$$\Big(\mathcal{A}(x,t,s,\xi) - \mathcal{A}(x,t,s,\eta)\Big) \cdot \Big(\xi - \eta\Big) > 0.$$

(H_3): $\mathcal{A}$ is coercive,

$$\mathcal{A}(x,t,s,\xi) \cdot \xi \geq \alpha M(|\xi|).$$

For the lower order term, we assume $\Phi_i : Q_T \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function satisfying:

(H_4): For all $s \in \mathbb{R}$ and for almost every $x \in \Omega$,

$$|\Phi_i(x,t,s)| \leq \gamma(x,t) + \overline{M}^{-1}(M(|s|)) \text{ where } \gamma \in E_{\overline{M}}(Q_T).$$

Moreover, we suppose that for $i = 1, 2$, $b_i : \Omega \times \mathbb{R} \to \mathbb{R}$ is a strictly increasing C^1-function with $b_i(x,0) = 0$ and $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with

$$f_1(x,0,s) = f_2(x,s,0) = 0 \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}. \tag{3.9}$$

and for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,

$$sign(s_i) f_i(x,s_1,s_2) \geq 0. \tag{3.10}$$

The growth assumptions on f_i are as follows: For each $k > 0$, there exists $\sigma_k > 0$ and a function F_k in $L^1(\Omega)$ such that

$$|f_1(x,s_1,s_2)| \leq F_k(x) + \sigma_k\ |b_2(x,s_2)| \tag{3.11}$$

a.e. in Ω, for all s_1 such that $|s_1| \leq k$, for all $s_2 \in \mathbb{R}$. For each $k > 0$, there exists $\lambda_k > 0$ and a function G_k in $L^1(\Omega)$ such that

$$|f_2(x,s_1,s_2)| \leq G_k(x) + \lambda_k\ |b_1(x,s_1)| \tag{3.12}$$

for almost every $x \in \Omega$, for every s_2 such that $|s_2| \leq k$, and for every $s_1 \in \mathbb{R}$. Finally, we assume the following condition on the initial data $u_{i,0}$:

$$u_{i,0} \text{ is a measurable function such that } b_i(x,u_{i,0}) \in L^1(\Omega), \text{ for } i = 1, 2. \tag{3.13}$$

Definition 3.1. A couple of functions (u_1, u_2) defined on Q_T is called a renormalized solution of system (3.8) if for $i = 1, 2$ the function u_i satisfies

$$T_k(u_i) \in W_0^{1,x} L_M(Q_T) \quad \text{and} \quad b_i(x,u_i) \in L^\infty(0,T; L^1(\Omega)), \tag{3.14}$$

$$\lim_{m\to\infty} \int_{\{m \leq |u_i(x,t)| \leq m+1\}} \mathcal{A}(x,t,u_i,\nabla u_i)\nabla u_i \, dx\, dt = 0, \tag{3.15}$$

and if, for every function r in $W^{1,\infty}(\mathbb{R})$ such that r' has a compact support, we have

$$\begin{aligned} &\tfrac{\partial B_{i,r}(x,u_i)}{\partial t} - \text{div}\,(r'(u_i)\mathcal{A}(x,t,u_i,\nabla u_i)) + r''(u_i)\mathcal{A}(x,t,u_i,\nabla u_i)\nabla u_i \\ &-\text{div}\,(r'(u_i)\Phi_i(x,t,u_i)) + r''(u_i)\Phi_i(x,t,u_i)\nabla u_i + f_i(x,u_1,u_2)r'(u_i) = 0 \end{aligned} \tag{3.16}$$

in $\mathfrak{D}'(Q_T)$, and

$$B_{i,r}(x,u_i)(t=0) = B_{i,r}(x,u_{i,0}) \quad \text{in } \Omega, \tag{3.17}$$

where $B_{i,r}(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b_i(x,s)}{\partial s} r'(s)\, ds$ in Ω.

Remark 3.2. [21, 23] For every nondecreasing function $r \in W^{2,\infty}(\mathbb{R})$ such that supp(r') $\subset [-k,k]$ and (3.7), we have

$$\lambda_k^i |r(s_1) - r(s_2)| \leq |B_{i,r}(x,s_1) - B_{i,r}(x,s_2)| \leq \|A_k^i\|_{L^\infty(\Omega)} |r(s_1) - r(s_2)|,$$

for almost every $x \in \Omega$ and for every $s_1, s_2 \in \mathbb{R}$.

Lemma 3.3. [18] *Under assumptions (H_1)-(H_3), let (Z_n) be a sequence in $W_0^{1,x}L_M(Q_T)$ such that*

$$Z_n \rightharpoonup Z \quad \text{in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M(Q_T), \Pi E_{\overline{M}}(Q_T)), \tag{3.18}$$

$$\Big(\mathcal{A}(x,t,Z_n,\nabla Z_n)\Big)_n \quad \text{is bounded in } \Big(L_{\overline{M}}(Q_T)\Big)^N, \tag{3.19}$$

$$\lim_{n,s\to\infty} \int_{Q_T} \Big(\mathcal{A}(x,t,Z_n,\nabla Z_n) - \mathcal{A}(x,t,Z_n,\nabla Z\chi_s)\Big) \cdot \Big(\nabla Z_n - \nabla Z\chi_s\Big) dxdt = 0, \tag{3.20}$$

where χ_s is the characteristic function of the set $\Omega_s = \Big\{x \in \Omega : |\nabla Z| \le s\Big\}$. Then,

$$\nabla Z_n \to \nabla Z \quad \text{a.e. in } Q_T, \tag{3.21}$$

$$\lim_{n\to\infty} \int_{Q_T} \mathcal{A}(x,t,Z_n,\nabla Z_n)\nabla Z_n \, dx = \int_{Q_T} \mathcal{A}(x,t,Z,\nabla Z)\nabla Z \, dxdt, \tag{3.22}$$

$$M(|\nabla Z_n|) \longrightarrow M(|\nabla Z|) \quad \text{in } L^1(Q_T). \tag{3.23}$$

In what follows, we will use the following real function of a real variable, called the truncation at height $k > 0$,

$$T_k(s) = \max\Big(-k, \min(k,s)\Big) = \begin{cases} s & \text{if } |s| \le k \\ k\dfrac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and its primitive is defined by

$$\widetilde{T}_k(s) = \int_0^s T_k(t)\, dt.$$

Note that $\widetilde{T}_k$ has the properties: $\widetilde{T}_k(s) \ge 0$ and $\widetilde{T}_k(s) \le k|s|$.

The following theorem is our main result.

Theorem 3.4. *Assume that the assumptions $(H_1) - (H_4)$ and (3.9)-(3.13) hold true, then there exists at least one solution (u_1, u_2) for the parabolic system (3.8) in sense of Definition 3.1.*

The proof of the above theorem is divided into four steps.

Step 1: Approximate problems. For each $n \in \mathbb{N}^*$, put

$$b_{i,n}(x,s) = b_i(x, T_n(s)) + \frac{1}{n}s, \tag{3.24}$$

$$\mathcal{A}_n(x,t,s,\xi) = \mathcal{A}(x,t,T_n(s),\xi) \text{ a.e } (x,t) \in Q_T, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N,$$

and

$$\Phi_{i,n}(x,t,s) = \Phi_i(x,t,T_n(s)) \text{ a.e } (x,t) \in Q_T, \forall s \in \mathbb{R},$$

$$f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),s_2) \text{ a.e in } \Omega, \forall s_1, s_2 \in \mathbb{R}, \tag{3.25}$$

$$f_{2,n}(x,s_1,s_2) = f_2(x,s_1,T_n(s_2)) \text{ a.e in } \Omega, \forall s_1, s_2 \in \mathbb{R}. \tag{3.26}$$

And let $u_{i,0n} \in C_0^\infty(\Omega)$ such that

$$\| b_{i,n}(x,u_{i,0n}) \|_{L^1} \le \| b_i(x,u_{i,0}) \|_{L^1} \text{ and } b_{i,n}(x,u_{i,0n}) \longrightarrow b_i(x,u_{i,0}) \text{ in } L^1(\Omega).$$

Consider the following regularized problem

$$\begin{cases} \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t} - \operatorname{div}\Big(\mathcal{A}_n(x,t,u_{i,n},\nabla u_{i,n}) + \Phi_{i,n}(x,t,u_{i,n})\Big) & \\ +f_{i,n}(x,u_{1,n},u_{2,n}) = 0 & \text{in } Q_T \\ u_{i,n} = 0 & \text{on } \Gamma \\ b_{i,n}(x,u_{i,n})(t=0) = b_{i,n}(x,u_{i,0n}) & \text{in } \Omega. \end{cases} \tag{3.27}$$

From (3.24), for $i = 1, 2$, we have

$$\frac{\partial b_{i,n}(x,s)}{\partial s} \geq \frac{1}{n}, \quad |b_{i,n}(x,s)| \leq \max_{|s|\leq n} |b_i(x,s)| + 1 \quad \forall s \in \mathbb{R},$$

thanks to (3.11) and (3.12) , $f_{1,n}$ and $f_{2,n}$ satisfy: there exists $F_n, G_n \in L^1(\Omega)$ and $\sigma_n > 0$, $\lambda_n > 0$ such that

$$|f_{1,n}(x, s_1, s_2)| \leq F_n(x) + \sigma_n \max_{|s|\leq n} |b_i(x,s)| \text{ a.e. in } \Omega, \forall s_1, s_2 \in \mathbb{R},$$

$$|f_{2,n}(x, s_1, s_2)| \leq G_n(x) + \lambda_n \max_{|s|\leq n} |b_i(x,s)| \text{ a.e. in } \Omega, \forall s_1, s_2 \in \mathbb{R},$$

Let $z_n(x, t, u_{i,n}, \nabla u_{i,n}) = \mathcal{A}_n(x, t, u_{i,n}, \nabla u_{i,n}) + \Phi_{i,n}(x, t, u_{i,n})$, which satisfies the quoted assumptions (A_1), (A_2), (A_3) and (A_4) of [15]. Indeed, it remains to prove (A_4) (the others assumptions follow immediately from the hypothesis of our problem), to this end, we use Young's inequality as follows

$$\begin{aligned}
|\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}| &\leq |\gamma(x,t)||\nabla u_{i,n}| + \overline{M}^{-1}(M(|T_n(u_{i,n})|))|\nabla u_{i,n}| \\
&= \frac{\alpha^2}{\alpha+2}\frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla u_{i,n}| \\
&\quad + \frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_n(u_{i,n})|))\frac{\alpha}{\alpha+1}|\nabla u_{i,n}| \\
&\leq \frac{\alpha^2}{\alpha+2}\Big(\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + M\Big(|\nabla u_{i,n}|\Big)\Big) \\
&\quad + \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_n(u_{i,n})|))\Big) + M\Big(\frac{\alpha}{\alpha+1}|\nabla u_{i,n}|\Big).
\end{aligned}$$

While $\frac{\alpha}{\alpha+1} < 1$, using the convexity of M and since $\overline{M}$ and $\overline{M}^{-1} \circ M$ are increasing functions, one has

$$\begin{aligned}
|\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}| &\leq \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + \frac{\alpha^2}{\alpha+2}M\Big(|\nabla u_{i,n}|\Big) \\
&\quad + \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) + \frac{\alpha}{\alpha+1}M\Big(|\nabla u_{i,n}|\Big).
\end{aligned}$$

Then we get

$$\begin{aligned}
\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} &\geq -\Big(\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1}\Big)M\Big(|\nabla u_{i,n}|\Big) - \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) \\
&\quad - \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big).
\end{aligned}$$

Using this last inequality and (H_3) we obtain

$$\begin{aligned}
z_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} &\geq \Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1}\Big)M\Big(|\nabla u_{i,n}|\Big) \\
&\quad - \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) - \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) \\
&\geq \frac{\alpha^2}{(\alpha+1)(\alpha+2)}M\Big(|\nabla u_{i,n}|\Big) - \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(n))\Big) \\
&\quad - \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big).
\end{aligned}$$

Since $\gamma \in E_{\overline{M}}(Q_T)$, $\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) \in L^1(Q_T)$. Thus, from [12], the approximate problem (3.27) has at least one weak solution $u_{i,n} \in W_0^{1,x}L_M(Q_T)$.

Step 2: A Priori Estimates.

Proposition 3.5. *Suppose that the assumptions* $(H_1)-(H_4)$*, (3.9)-(3.13) hold true and let* $u_{i,n}$ *be a solution of the approximate problem (3.27). Then, for all* $k>0$*, there exists a constant* $C_{i,k}$*,* $\widehat{C}_k$ *(not depending on* n*), such that:*

$$\| T_k(u_{i,n}) \|_{W_0^{1,x}L_M(Q_T)} \leq C_{i,k}, \tag{3.28}$$

$$\int_\Omega B^n_{i,k}(x,u_{i,n})(\sigma)\,dx \leq \widehat{C}_k + k\|b_i(x,u_{i,0}\|_{L^1(\Omega)}, \tag{3.29}$$

for almost any $\sigma\in(0,T)$ *where* $B^n_{i,k}(x,\tau)=\displaystyle\int_0^\tau T_k(s)\frac{\partial b_{i,n}(x,s)}{\partial s}\,ds$*, and*

$$\lim_{k\to\infty} meas\Big\{(x,t)\in Q_T : |u_{i,n}|>k\Big\}=0. \tag{3.30}$$

Proof. Let us take the test function $T_k(u_{i,n})\chi_{(0,\sigma)}$ in the approximate problem (3.27), one has for every $\sigma\in(0,T)$

$$\begin{aligned}
&\int_\Omega B^n_{i,k}(x,u_{i,n})(\sigma)\,dx + \int_{Q_\sigma}\mathcal{A}_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt\\
&\quad+\int_{Q_\sigma}\Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt+\int_{Q_\sigma} f_{i,n}(x,u_{1,n},u_{2,n})T_k(u_{i,n})\,dx\,dt\\
&\qquad=\int_\Omega B^n_{i,k}(x,u_{i,0n})\,dx.
\end{aligned} \tag{3.31}$$

First, let us remark that $\Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})$ is different from zero only on the set $\{|u_{i,n}|\leq k\}$ where $T_k(u_{i,n})=u_{i,n}$. Thanks to (H_4) and Young's inequality with an algebraic trick for the constant of coercivity $\alpha>0$, we have

$$\begin{aligned}
&\int_{Q_\sigma}\Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt\\
&\leq\int_{Q_\sigma}|\gamma(x,t)||\nabla T_k(u_{i,n})|\,dx\,dt\\
&\quad+\int_{Q_\sigma}\overline{M}^{-1}(M(|T_k(u_{i,n})|))|\nabla T_k(u_{i,n})|\,dx\,dt\\
&=\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma}\frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla T_k(u_{i,n})|\,dx\,dt\\
&\quad+\int_{Q_\sigma}\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_k(u_{i,n})|))\frac{\alpha}{\alpha+1}|\nabla T_k(u_{i,n})|\,dx\,dt\\
&\leq\frac{\alpha^2}{\alpha+2}\Big(\int_{Q_\sigma}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt+\int_{Q_\sigma}M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\Big)\\
&\quad+\int_{Q_\sigma}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|T_k(u_{i,n})|))\Big)\,dx\,dt\\
&\qquad+\int_{Q_\sigma}M\Big(\frac{\alpha}{\alpha+1}|\nabla T_k(u_{i,n})|\Big)\,dx\,dt.
\end{aligned}$$

Since $\gamma\in E_{\overline{M}}(Q_\sigma)$, then $\displaystyle\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt=\gamma_0<+\infty$ and while $\displaystyle\frac{\alpha}{\alpha+1}<1$, using the convexity of M and the fact that $\overline{M}$ and $\overline{M}^{-1}\circ M$ are increasing functions, we get

$$\begin{aligned}&\int_{Q_\sigma} \Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt\\&\le \gamma_0+\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\\&\quad+\int_{Q_\sigma}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(k))\Big)\,dx\,dt\\&\quad+\frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt.\end{aligned}\tag{3.32}$$

Since Ω is bounded, there exists some constant $C_{k,\alpha}$ such that

$$\int_{Q_\sigma}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(k))\Big)\,dx\,dt=C_{k,\alpha}.$$

Which gives the estimate

$$\begin{aligned}&\int_{Q_\sigma} \Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt\\&\le \gamma_0+\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\\&\quad+C_{k,\alpha}+\frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt.\end{aligned}\tag{3.33}$$

On the other hand, due to (3.10), we have

$$\int_{Q_\sigma} f_{i,n}(x,u_{1,n},u_{2,n})T_k(u_{i,n})\,dx\,dt\ge 0.\tag{3.34}$$

Concerning the first integral in (3.31), we have by construction of $B^n_{i,k}(x,u_{i,n})$,

$$\int_\Omega B^n_{i,k}(x,u_{i,n})(\sigma)\,dx\ge 0\tag{3.35}$$

and

$$0\le\int_\Omega B^n_{i,k}(x,u_{i,0n})\,dx\le k\int_\Omega |b_{i,n}(x,u_{i,0n})|\,dx\le k\|b_i(x,u_{i,0})\|_{L^1(\Omega)}.\tag{3.36}$$

Combining (3.31), (3.33), (3.34), (3.35) and (3.36) we get

$$\begin{aligned}&\int_{Q_\sigma}\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\,dx\,dt\\&\le\gamma_0+k\overline{C}+C_{k,\alpha}+\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\\&\quad+\frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt,\end{aligned}\tag{3.37}$$

where $\overline{C}=\|b_i(x,u_{i,0})\|_{L^1(\Omega)}$. Thanks to (H_3), we deduce

$$\int_{Q_\sigma}\Big(\alpha-\frac{\alpha^2}{\alpha+2}-\frac{\alpha}{\alpha+1}\Big)M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\le\gamma_0+k\overline{C}+C_{k,\alpha}.\tag{3.38}$$

Since $\Big(\alpha-\frac{\alpha^2}{\alpha+2}-\frac{\alpha}{\alpha+1}\Big)=\frac{\alpha^2}{(\alpha+1)(\alpha+2)}>0$, finally we have

$$\int_{Q_T} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\le(\gamma_0+k\overline{C}+C_{k,\alpha})\frac{(\alpha+1)(\alpha+2)}{\alpha^2}=C_{i,k}.\tag{3.39}$$

To prove (3.29), we combine (3.31), (3.33), (3.34), (3.36), (3.37) and (3.39) with $\widehat{C}_k = \gamma_0 + C_{k,\alpha} + (\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1})C_{i,k}$. Finally, we prove (3.30), to this end, since $T_k(u_{i,n})$ is bounded in $W_0^{1,x}L_M(Q_T)$ there exists $\lambda > 0$ and a constant $C_{i,0}(k)$ such that

$$\int_{Q_T} M\Big(\frac{|T_k(u_{i,n})|}{\lambda}\Big)\,dx\,dt \le C_{i,0}(k)$$

Case 1: if $C_{i,0}(k) \le 1$. By using Young's inequality, we obtain

$$\begin{aligned}
\text{meas}\Big\{|u_{i,n}| > k\Big\} &= \frac{1}{k}\int_{\{|u_{i,n}|>k\}} k\,dx\,dt \le \frac{1}{k}\int_{Q_T} |T_k(u_{i,n})|\,dx\,dt \\
&\le \frac{\lambda}{k}\Big(\int_{Q_T} M\Big(\frac{|T_k(u_{i,n})|}{\lambda}\Big)\,dx\,dt + \int_{Q_T} \overline{M}(1)\,dx\,dt\Big) \\
&\le \frac{\lambda}{k}\Big(1 + \overline{M}(1)|Q_T|\Big) \quad \forall n, \quad \forall k > 0, \\
&\longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned} \tag{3.40}$$

Case 2: if $C_{i,0}(k) > 1$, we think to use the convexity of M with $\dfrac{1}{C_{i,0}(k)} < 1$ and Young's inequality for $P \prec\prec M$ appearing in assumption (H_1) which implies that $\forall \epsilon > 0,$ there exist a constant $d_\epsilon : P(t) \le M(\epsilon t) + d_\epsilon$, we obtain for $\epsilon < \dfrac{1}{C_{i,0}(k)} \le 1$

$$\begin{aligned}
\text{meas}\Big\{|u_{i,n}| > k\Big\} &= \frac{1}{k}\int_{\{|u_{i,n}|>k\}} k\,dx\,dt \le \frac{1}{k}\int_{Q_T} |T_k(u_{i,n})|\,dx\,dt \\
&\le \frac{\lambda}{k}\Big(\int_{Q_T} P\Big(\frac{|T_k(u_{i,n})|}{\lambda}\Big)\,dx\,dt + \int_{Q_T} \overline{P}(1)\,dx\,dt\Big) \\
&\le \frac{\lambda}{k}\Big(\int_{Q_T} M\Big(\epsilon\frac{|T_k(u_{i,n})|}{\lambda}\Big)\,dx\,dt + \int_{Q_T} (\overline{P}(1) + d_\epsilon)\,dx\,dt\Big) \\
&\le \frac{\lambda}{k}\Big(1 + (\overline{P}(1) + d_\epsilon)|Q_T|\Big) \quad \forall n, \quad \forall k > 0, \\
&\longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{aligned} \tag{3.41}$$

□

Lemma 3.6. *Let $u_{i,n}$ be a solution of the approximate problem (3.27), then:*
(i) $u_{i,n} \longrightarrow u_i$ *a.e. in* Q_T,
(ii) $b_{i,n}(x, u_{i,n}) \longrightarrow b_i(x, u_i)$ *a.e. in* Q_T,
(iii) $b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega))$.

Proof. To prove (**i**) and (**ii**), we adapt the same way as in [21, Lemma 5.3], we take a $C^2(\mathbb{R})$ nondecreasing function Γ_k such that $\Gamma_k(s) = \begin{cases} s & \text{for } |s| \le \frac{k}{2} \\ k & \text{for } |s| \ge k \end{cases}$ and multiply the approximate problem (3.27) by $\Gamma'_k(u_{i,n})$ we obtain

$$\begin{aligned}
\frac{\partial B^{i,n}_{\Gamma_k}(x, u_{i,n})}{\partial t} &= \text{div}\,\Big(\mathcal{A}_n(x, t, u_{i,n}, \nabla u_{i,n})\Gamma'_k(u_{i,n})\Big) \\
-\mathcal{A}_n(x, t, u_{i,n}, \nabla u_{i,n})\Gamma''_k(u_{i,n})\nabla u_{i,n} \quad &+ \text{div}\,\Big(\Gamma'_k(u_{i,n})\Phi_{i,n}(x, t, u_{i,n})\Big) \\
&-\Gamma''_k(u_{i,n})\Phi_{i,n}(x, t, u_{i,n})\nabla u_{i,n} - f_{i,n}\Gamma'_k(u_{i,n}),
\end{aligned} \tag{3.42}$$

where $B^{i,n}_{\Gamma_k}(x, \tau) = \displaystyle\int_0^\tau \frac{\partial b_{i,n}(x, s)}{\partial s}\Gamma'_k(s)\,ds.$
Remark that $\overline{M}^{-1} \circ M$ is an increasing function, $\gamma \in E_{\overline{M}}(Q_T)$, $\text{supp}(\Gamma'_{\text{k}})$, $\text{supp}(\Gamma''_{\text{k}}) \subset$

[−k, k] and using Young's inequality we get

$$\begin{aligned}
&\Big|\int_{Q_T} \Gamma_k'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\,dx\,dt\Big| \\
&\le \|\Gamma_k'\|_{L^\infty}\Big(\int_{Q_T} |\gamma(x,t)|\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(|T_k(u_{i,n})|))\,dx\,dt\Big) \\
&\le \|\Gamma_k'\|_{L^\infty}\Big(\int_{Q_T} \Big(\overline{M}(|\gamma(x,t)|) + M(1)\Big)\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(k))\,dx\,dt\Big) \\
&< C_{1,k},
\end{aligned} \tag{3.43}$$

Also, form Young's inequality and estimate (3.39) we have

$$\begin{aligned}
&\Big|\int_{Q_T} \Gamma_k''(u_{i,n})\Phi_n(x,t,u_{i,n})\nabla u_{i,n}\,dx\,dt\Big| \\
&\le \|\Gamma_k''\|_{L^\infty}\Big(\int_{Q_T} |\gamma(x,t)|\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(|T_k(u_{i,n})|))|\nabla T_k(u_{i,n})|\,dx\,dt\Big) \\
&\le \|\Gamma_k''\|_{L^\infty}\Big[\int_{Q_T} \Big(\overline{M}(|\gamma(x,t)|) + M(1)\Big)\,dx\,dt + \int_{Q_T} M(k)\,dx\,dt \\
&\qquad\qquad + \int_{Q_T} M(|\nabla T_k(u_{i,n})|)\,dx\,dt\Big] \\
&< C_{2,k},
\end{aligned} \tag{3.44}$$

where $C_{1,k}$ and $C_{2,k}$ are two positive constants independent of n. Then each term in the right-hand side of (3.42) is bounded either in $L^1(Q_T)$ or in $W^{-1,x}L_{\overline{M}}(Q_T)$, which implies that

$$\frac{\partial B^{i,n}_{\Gamma_k}(x,u_{i,n})}{\partial t} \text{ is bounded in } L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T). \tag{3.45}$$

Moreover, due to the properties of Γ_k' and (3.7), we have

$$|\nabla B^{i,n}_{\Gamma_k}(x,u_{i,n})| \le \|A^i_k\|_{L^\infty(\Omega)}|\nabla T_k(u_{i,n})|\|\Gamma_k'\|_{L^\infty(\Omega)} + k\|\Gamma_k'\|_{L^\infty(\Omega)}\widetilde{A}^i_k(x),$$

which implies, thanks to (3.28), that

$$B^{i,n}_{\Gamma_k}(x,u_{i,n}) \text{ is bounded in } W^{1,x}_0 L_M(Q_T).$$

Arguing as in [21, 23], we get ***(i)*** and ***(ii)*** of Lemma 3.6.

For ***(iii)***, use ***(ii)*** and we pass to the limit-inf in (3.29) as $n \longrightarrow +\infty$, we get

$$\frac{1}{k}\int_\Omega B_{i,k}(x,u_i)(\sigma)\,dx \le \frac{\widehat{C}_k}{k} + \Big(\|b_i(x,u_{i,0}\|_{L^1(\Omega)}\Big),$$

for almost any $\sigma \in (0,T)$. Thanks to the definition of $B_{i,k}(x,s)$ and the convergence of $\frac{1}{k}\int_\Omega B_{i,k}(x,u_i)$ to $b_i(x,u_i)$ as k goes to $+\infty$, this gives that $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$. □

The next lemma will be used later, proving it now.

Lemma 3.7. *Let $u_{i,n}$ be a solution of the approximate problem (3.27), then:*
(i) $\{\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\}_n$ *is bounded in* $(L_{\overline{M}}(Q_T))^N$,
(ii) $\displaystyle\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_{i,n}|\le m+1\}} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx = 0.$

Proof. ***(i)*** Let $\phi \in (E_M(Q_T))^N$ be an arbitrary function. From (H_2) we can write

$$\Big(\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - \mathcal{A}(x,t,T_k(u_{i,n}),\phi)\Big)\cdot\Big(\nabla T_k(u_{i,n}) - \phi\Big) \ge 0.$$

Which gives:

$$\begin{aligned}&\int_{Q_T}\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\phi\,dx\,dt\\&\le\int_{Q_T}\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\,dx\,dt\\&\quad+\int_{Q_T}\mathcal{A}(x,t,T_k(u_{i,n}),\phi)(\phi-\nabla T_k(u_{i,n}))\,dx\,dt.\end{aligned}\tag{3.46}$$

Let us denote by J_1 and J_2 the first and the second integral respectively in the right-hand side of (3.42), so that

$$J_1=\int_{Q_T}\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\,dx\,dt.$$

Going back to (3.37), we obtain

$$\begin{aligned}J_1\le\gamma_0+k\overline{C}+C_{k,\alpha}+\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma}M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\\+\frac{\alpha}{\alpha+1}\int_{Q_\sigma}M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt,\end{aligned}\tag{3.47}$$

And thanks to (3.28), there exists a positive constant C_{J_1} independent of n such that

$$J_1\le C_{J_1}.\tag{3.48}$$

Now we estimate the integral J_2, to this end, remark that

$$\begin{aligned}J_2&=\int_{Q_T}\mathcal{A}(x,t,T_k(u_{i,n}),\phi)(\phi-\nabla T_k(u_{i,n}))\,dx\,dt\\&\le\int_{Q_T}|\mathcal{A}(x,t,T_k(u_{i,n}),\phi)||\phi|\,dx\,dt+\int_{Q_T}|\mathcal{A}(x,t,T_k(u_{i,n}),\phi)||\nabla T_k(u_{i,n})|\,dx\,dt.\end{aligned}$$

On the other hand, let η be large enough, from (H_1) and the convexity of $\overline{M}$, we get:

$$\begin{aligned}&\int_{Q_T}\overline{M}\Big(\frac{|\mathcal{A}(x,t,T_k(u_{i,n}),\phi)|}{\eta}\Big)\,dx\,dt\\&\le\int_{Q_T}\overline{M}\Big(\frac{c(x,t)+\overline{M}^{-1}(P(k_1|T_k(u_{i,n})|)+\overline{M}^{-1}(M(k_2|\phi|))}{\eta}\Big)\,dx\,dt\\&\le\frac{1}{\eta}\int_{Q_T}\overline{M}(c(x,t))\,dx\,dt+\frac{1}{\eta}\int_{Q_T}\overline{M}\Big(\overline{M}^{-1}(P(k_1|T_k(u_{i,n})|))\Big)\,dx\,dt\\&\quad+\frac{1}{\eta}\int_{Q_T}\overline{M}\Big(\overline{M}^{-1}(M(k_2|\phi|))\Big)\,dx\,dt\\&\le\frac{1}{\eta}\int_{Q_T}\overline{M}(c(x,t))\,dx\,dt+\frac{1}{\eta}\int_{Q_T}P(k_1k)\,dx\,dt\\&\quad+\frac{1}{\eta}\int_{Q_T}M(k_2|\phi|)\,dx\,dt.\end{aligned}\tag{3.49}$$

Since $\phi\in(E_M(Q_T))^N$, $c(x,t)\in E_{\overline{M}}(Q_T)$, we deduce that $\{\mathcal{A}(x,t,T_k(u_{i,n}),\phi)\}$ is bounded in $(L_{\overline{M}}(Q_T))^N$ and we have $\{\nabla T_k(u_{i,n})\}$ is bounded in $(L_M(Q_T))^N$, consequently, $J_2\le C_{J_2}$, where C_{J_2} is a positive constant not depending on n. And then we obtain

$$\int_{Q_T}\mathcal{A}(x,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\phi\,dx\,dt\le C_{J_1}+C_{J_2}.\quad\text{for all }\phi\in(E_M(Q_T))^N.\tag{3.50}$$

By Banach-Steinhaus theorem, $\{\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\}_n$ is bounded in $(L_{\overline{M}}(Q_T))^N$.

(ii) Testing (3.27) by $\theta_m(u_{i,n}) = T_{m+1}(u_{i,n}) - T_m(u_{i,n})$, we have

$$\begin{aligned}&\int_\Omega B_m(x, u_{i,n})(T)\,dx + \int_{Q_T} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla\theta_m(u_{i,n})\,dx\,dt\\&\quad+\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})\,dx\,dt = \int_\Omega B_m(x,u_{i,0n})\,dx\\&\quad+\int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt,\end{aligned}\tag{3.51}$$

where $B_m(x,\tau) = \int_0^\tau \frac{\partial b_i(x,s)}{\partial s}\theta_m(s)\,ds$. Since $B_m(x,u_{i,n})(T) \geq 0$, hence from (H_3) and (H_4), it follows

$$\begin{aligned}&\alpha\int_{Q_T} M(|\nabla\theta_m(u_{i,n})|)\,dx\,dt\\&\leq \int_{Q_T}\overline{M}^{-1}(M(|u_{i,n}|))|\nabla\theta_m(u_{i,n})|\,dx\,dt + \int_{Q_T}|\gamma(x,t)||\nabla\theta_m(u_{i,n})|\,dx\,dt\\&\quad+\int_\Omega B_m(x,u_{i,0n})\,dx + \int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt.\end{aligned}\tag{3.52}$$

That means, knowing that $\nabla\theta_m(u_{i,n}) = \nabla u_{i,n}\chi_{E_m}$ a.e. in Q_T where

$$E_m := \Big\{(x,t)\in Q_T : m \leq |u_{i,n}| \leq m+1\Big\},$$

and following the same argument as in the proof of (3.28) of proposition 3.5, we get

$$\begin{aligned}&\alpha\int_{Q_T} M(|\nabla\theta_m(u_{i,n})|)\,dx\,dt\\&\leq \int_{Q_T}\overline{M}^{-1}(M(|u_{i,n}|))|\nabla u_{i,n}|\chi_{E_m}\,dx\,dt + \int_{E_m}|\gamma(x,t)||\nabla\theta_m(u_{i,n})|\,dx\,dt\\&\quad+\int_\Omega B_m(x,u_{i,0n})\,dx + \int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt\\&\leq \int_{Q_T}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|u_{i,n}|)\Big)\chi_{E_m}\,dx\,dt + \int_{Q_T} M\Big(\frac{\alpha}{\alpha+1}|\nabla\theta_m(u_{i,n})|\Big)\,dx\,dt\\&\quad+\frac{\alpha^2}{\alpha+2}\Big(\int_{E_m}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt + \int_{Q_T} M\Big(|\nabla\theta_m(u_{i,n})|\Big)\,dx\,dt\Big)\\&\quad+\int_\Omega B_m(x,u_{i,0n})\,dx + \int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt.\end{aligned}\tag{3.53}$$

let $C^\alpha_{max} := \max\Big((\alpha+1), \frac{(\alpha+1)(\alpha+2)}{\alpha^2}\Big)$, it follows

$$\begin{aligned}&\int_{Q_T} M(|\nabla\theta_m(u_{i,n})|)\,dx\,dt\\&\leq C^\alpha_{max}\Big[\int_{E_m}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt + \int_\Omega B_m(x,u_{i,0n})\,dx\\&\quad+\int_{E_m}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|u_{i,n}|)\Big)\,dx\,dt + \int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt\Big].\end{aligned}\tag{3.54}$$

Now, let us concentrate on the convergence as $n\to\infty$ of each integral in (3.54), which can be treated by the same way (Lebesgue's dominated convergence theorem), take for example the first one:

$$\int_{\{m\leq|u_{i,n}|\leq m+1\}}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt = \int_\Omega\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\leq|u_{i,n}|\leq m+1\}}\,dx\,dt$$

Put $g_n = \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\le|u_{i,n}|\le m+1\}}$, since χ is continuous, then

$$g_n \longrightarrow g = \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\le|u_i|\le m+1\}} \quad \text{a.e. in } Q_T.$$

And we have $|g_n| \le \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)$ which is integrable on Q_T, since $\gamma \in E_{\overline{M}}(Q_T)$. From Lebesgue's dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_{Q_T} g_n\,dx\,dt = \int_{Q_T}\lim_{n\to\infty} g_n\,dx\,dt = \int_{Q_T}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\chi_{\{m\le|u_i|\le m+1\}}\,dx\,dt.$$

Passing to the limit as $n\to\infty$ in (3.54), we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T} M(|\nabla\theta_m(u_{i,n})|)\,dx\,dt\\
&\le C_{max}^{\alpha}\Big[\int_{\{m\le|u_i|\le m+1\}}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x)|\Big)\,dx\,dt + \int_\Omega B_m(x,u_{i0})\,dx\\
&\qquad + \int_{\{m\le|u_i|\le m+1\}}\overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(|u_i|)\Big)\,dx\,dt\\
&\qquad + \int_{Q_T} f_i\theta_m(u_i)\,dx\,dt\Big].
\end{aligned} \tag{3.55}$$

Now, we will pass to the limit as $m\to\infty$, by Lebesgue's theorem each integral in (3.55) goes to zero as m goes to ∞, which gives

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{Q_T} M(|\nabla\theta_m(u_{i,n})|)\,dx\,dt = 0. \tag{3.56}$$

Our aim here is to prove that $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})\,dx\,dt = 0$, to this end, Young's inequality allows us to get

$$\begin{aligned}
\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})\,dx\,dt \quad &\le \int_{Q_T} M(|\nabla\theta_m(u_{i,n})|)\,dx\,dt\\
&\quad + \int_{E_m}\overline{M}(\Phi_{i,n}(x,t,u_{i,n}))\,dx\,dt.
\end{aligned} \tag{3.57}$$

We have already proved that the first integral in the right-hand side of (3.57) goes to zero as m and n go to ∞, it remains to show that the second one goes to zero again. Indeed, note that, for $n \ge m+1 \ge |u_{i,n}|$ we have $T_n(u_{i,n}) = T_{m+1}(u_{i,n}) = u_{i,n}$, then, from (H_4) and the convexity of $\overline{M}$ we obtain

$$\begin{aligned}
&\int_{\{m\le|u_{i,n}|\le m+1\}}\overline{M}(\Phi_{i,n}(x,t,u_{i,n}))\,dx\,dt\\
&= \int_{\{m\le|u_{i,n}|\le m+1\}}\overline{M}(|\Phi_{i,n}(x,t,T_{m+1}(u_{i,n}))|)\,dx\,dt\\
&\le \int_{\{m\le|u_{i,n}|\le m+1\}}\overline{M}(\overline{M}^{-1}(M(|T_{m+1}(u_{i,n})|))\,dx\,dt\\
&\le \int_{\{m\le|u_{i,n}|\le m+1\}} M(|T_{m+1}(u_{i,n})|)\,dx\,dt\\
&\le \int_{Q_T} M(m+1)\,dx\,dt.
\end{aligned} \tag{3.58}$$

We deduce that

$$\begin{aligned}
&\int_{\{m\le|u_{i,n}|\le m+1\}}\overline{M}(|\Phi_{i,n}(x,t,T_{m+1}(u_{i,n}))|)\,dx\,dt\\
&= \int_{Q_T}\overline{M}(|\Phi_{i,n}(x,t,T_{m+1}(u_{i,n})|)\,\chi_{\{m\le|u_{i,n}|\le m+1\}}dx\,dt \le C_{0,m}.
\end{aligned} \tag{3.59}$$

Let us denote $G_n^m = \overline{M}(|\Phi_{i,n}(x,t,T_{m+1}(u_{i,n})|)\,\chi_{\{m\le |u_{i,n}|\le m+1\}} \longrightarrow G^m$ a.e. in Ω where

$$G^m = \overline{M}(|\Phi_i(x,t,T_{m+1}(u_i)|)\,\chi_{\{m\le |u_i|\le m+1\}},$$

since $\overline{M}$ is continuous and Φ_i is a Carathéodory function. From (3.59), G_n^m is bounded independently of n, using Lebesgue's theorem, it follows, as $n \longrightarrow \infty$

$$\int_{\{m\le |u_{i,n}|\le m+1\}} \overline{M}(|\Phi_{i,n}(x,t,u_{i,n})|)\,dx\,dt \longrightarrow \int_{\{m\le |u_i|\le m+1\}} \overline{M}(|\Phi_i(x,t,u_i)|)\,dx\,dt. \tag{3.60}$$

And then

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le |u_{i,n}|\le m+1\}} \overline{M}(|\Phi_{i,n}(x,t,u_{i,n})|)\,dx\,dt = 0 \tag{3.61}$$

Combining (3.56), (3.57) and (3.61) we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla\theta_m(u_{i,n})\,dx\,dt = 0 \tag{3.62}$$

Finaly, let $m,n \longrightarrow \infty$ in (3.51), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{m\le |u_{i,n}|\le m+1\}} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt = 0. \tag{3.63}$$

□

Step 3: Almost everywhere convergence of the gradients.

Proposition 3.8. *Let $u_{i,n}$ be a solution of the approximate problem (3.27). Then, for all $k\ge 0$ we have (for a subsequence still denoted by $u_{i,n}$): as $n\to+\infty$,*
(i) $\nabla u_{i,n}\to\nabla u_i$ *a.e. in* Q_T,
(ii) $\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) \rightharpoonup \mathcal{A}(x,t,T_k(u_i),\nabla T_k(u_i))$ *weakly in* $(L_{\overline{M}}(Q_T))^N$,
(iii) $M(|\nabla T_k(u_{i,n})|)\to M(|\nabla T_k(u_i)|)$ *strongly in* $L^1(Q_T)$.

Proof. Let $\theta_{i,j}\in\mathfrak{D}(Q_T)$ be a sequence such that $\theta_{i,j}\longrightarrow u_i$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and let $\psi_{i,j}\in\mathfrak{D}(\Omega)$ be a sequence which converges strongly to $u_{i,0}$ in $L^1(\Omega)$.

Put $Z_{i,j}^\mu = T_k(\theta_{i,j})_\mu + \mathrm{e}^{-\mu t}T_k(\psi_{i,j})$ where $T_k(\theta_{i,j})_\mu$ is the mollification with respect to the time of $T_k(\theta_{i,j})$, notice that $Z_{i,j}^\mu$ is a smooth function having the following properties:

$$\frac{\partial Z_{i,j}^\mu}{\partial t} = \mu(T_k(\theta_{i,j}) - Z_{i,j}^\mu),\quad Z_{i,j}^\mu(0) = T_k(\psi_{i,j})\quad\text{and } |Z_{i,j}^\mu|\le k,$$

$$Z_{i,j}^\mu \longrightarrow T_k(u_i)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_{i,j}),\quad \text{in } W_0^{1,x}L_M(Q_T)\quad \text{modularly as } j\longrightarrow\infty,$$

$$T_k(u_i)_\mu + \mathrm{e}^{-\mu t}T_k(\psi_{i,j}) \longrightarrow T_k(u_i),\quad \text{in } W_0^{1,x}L_M(Q_T)\quad \text{modularly as } \mu\longrightarrow\infty.$$

Let now the function h_m defined on $\mathbb{R}$ for any $m\ge k$ by:

$$h_m(r) = \begin{cases} 1 & \text{if } |r|\le m\\ -|r|+m+1 & \text{if } m\le |r|\le m+1\\ 0 & \text{if } |r|\ge m+1.\end{cases}$$

Put $E_m = \Big\{(x,t) \in Q_T : m \leq |u_{i,n}| \leq m+1\Big\}$ and let us test the approximate problem (3.27) by the test function $\varphi^{\mu,i}_{n,j,m} = (T_k(u_{i,n}) - Z^{\mu}_{i,j})h_m(u_{i,n})$, we get

$$\begin{aligned}\Big\langle \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, \varphi^{\mu,i}_{n,j,m}\Big\rangle &+ \int_{Q_T} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})h_m(u_{i,n})\,dx\,dt\\ &+ \int_{Q_T} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\nabla u_{i,n} h'_m(u_{i,n})\,dx\,dt\\ &+ \int_{E_m} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\,dx\,dt\\ &+ \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})h_m(u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})\,dx\,dt\\ &= -\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})\varphi^{\mu,i}_{n,j,m}\,dx\,dt.\end{aligned} \tag{3.64}$$

For simplicity, denote by $\epsilon(n,j,\mu,i)$ and $\epsilon(n,j,\mu)$ any quantities such that

$$\lim_{i\to+\infty}\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,\mu,i) = 0,$$

$$\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,\mu) = 0.$$

We have the following lemma which can be found in [21, 23].

Lemma 3.9. *(cf. [21, 23]) Let $\varphi^{\mu,i}_{n,j,m} = (T_k(u_{i,n}) - Z^{\mu}_{i,j})h_m(u_{i,n})$, then for any $k \geq 0$ we have:*

$$\Big\langle \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, \varphi^{\mu,i}_{n,j,m}\Big\rangle \geq \epsilon(n,j,\mu,i), \tag{3.65}$$

where $\Big\langle \cdot,\cdot\Big\rangle$ denotes the duality pairing between $L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T)$ and $L^\infty(Q_T) \cap W^{1,x}_0 L_M(Q_T)$.

To complete the proof of Proposition 3.8, we establish the results below, for any fixed $k \geq 0$, we have:

(r_1) $\displaystyle\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})\varphi^{\mu,i}_{n,j,m}\,dx\,dt = \epsilon(n,j,\mu).$

(r_2) $\displaystyle\int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})h_m(u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})\,dx\,dt = \epsilon(n,j,\mu).$

(r_3) $\displaystyle\int_{E_m} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\,dx\,dt = \epsilon(n,j,\mu).$

(r_4) $\displaystyle\int_{Q_T} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\nabla u_{i,n} h'_m(u_{i,n})\,dx\,dt \leq \epsilon(n,j,\mu,m).$

(r_5) $$\begin{aligned}\int_{Q_T} &[\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - \mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_i)\chi_s)]\\ &\times[\nabla T_k(u_{i,n}) - \nabla T_k(u_i)\chi_s]\,dx\,dt \leq \epsilon(n,j,\mu,m,s).\end{aligned}$$

The proofs of (r_1), (r_4) and (r_5) are the same as in [21, 23].

To prove (r_2), to this end, for $n \geq m+1$, we have

$$\Phi_{i,n}(x,t,u_{i,n})h_m(u_{i,n}) = \Phi_i(x,t,T_{m+1}(u_{i,n}))h_m(T_{m+1}(u_{i,n})) \text{ a.e in } Q_T.$$

put $P_{i,n} = \overline{M}\Big(\dfrac{|\Phi_i(x,t,T_{m+1}(u_{i,n})) - \Phi_i(x,t,T_{m+1}(u_i))|}{\eta}\Big)$. Since Φ_i is continuous with respect to its third argument and $u_{i,n} \longrightarrow u_i$ a.e in Q_T, then $\Phi_i(x,t,T_{m+1}(u_{i,n})) \to \Phi_i(x,t,T_{m+1}(u_i))$ a.e in Ω as n goes to infinity, besides $\overline{M}(0) = 0$, it follows

$$P_{i,n} \longrightarrow 0, \quad \text{a.e in } \Omega \text{ as } n \to \infty. \tag{3.66}$$

Using now the convexity of $\overline{M}$ and (H_4), we have for every $\eta > 0$ and $n \geq m+1$:

$$\begin{aligned} P_{i,n} &= \overline{M}\Big(\frac{|\Phi_i(x,t,T_{m+1}(u_{i,n})) - \Phi_i(x,t,T_{m+1}(u_i))|}{\eta}\Big) \\ &\leq \overline{M}\Big(\frac{|\Phi_i(x,t,T_{m+1}(u_{i,n}))| + |\Phi_i(x,t,T_{m+1}(u_i))|}{\eta}\Big) \\ &\leq \overline{M}\Big(\frac{2}{\eta}|\gamma(x,t)| + \frac{2}{\eta}\overline{M}^{-1}(M(m+1))\Big) \\ &= \overline{M}\Big(\frac{1}{2}\frac{4}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{4}{\eta}\overline{M}^{-1}(M(m+1))\Big) \\ &\leq \frac{1}{2}\overline{M}(\frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{M}(\frac{4}{\eta}\overline{M}^{-1}(M(m+1))). \end{aligned} \tag{3.67}$$

We put $C_m^\eta(x,t) = \frac{1}{2}\overline{M}(\frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{M}(\frac{4}{\eta}\overline{M}^{-1}(M(m+1)))$. Since $\gamma \in E_{\overline{M}}(Q_T)$, we have $C_m^\eta \in L^1(Q_T)$, Then by Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty} \int_{Q_T} P_{i,n}\, dx\, dt = \int_{Q_T} \lim_{n\to\infty} P_{i,n}\, dx\, dt = 0. \tag{3.68}$$

This implies that $\{\Phi_i(x,t,T_{m+1}(u_{i,n}))\}$ converges modularly to $\Phi_i(x,t,T_{m+1}(u_i))$ as $n \to \infty$ in $(L_{\overline{M}}(Q_T))^N$. Moreover, both $\Phi_i(x,t,T_{m+1}(u_{i,n}))$ and it's limit $\Phi_i(x,t,T_{m+1}(u_i))$ belong to $(E_{\overline{M}}(Q_T))^N$, indeed, from (H_4) we have for every $\eta > 0$

$$\begin{aligned} &\int_{Q_T} \overline{M}\Big(\frac{|\Phi_i(x,t,T_{m+1}(u_{i,n}))|}{\eta}\Big)\, dx\, dt \\ &\leq \int_{Q_T} \overline{M}\Big(\frac{1}{\eta}|\gamma(x,t)| + \frac{1}{\eta}\overline{M}^{-1}(M(|T_{m+1}(u_{i,n})|))\Big)\, dx\, dt \\ &\leq \int_{Q_T} \overline{M}\Big(\frac{1}{2}\frac{2}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{2}{\eta}\overline{M}^{-1}(M(m+1))\Big)\, dx\, dt \\ &\leq \int_{Q_T} \frac{1}{2}\overline{M}(\frac{2}{\eta}|\gamma(x,t)|)\, dx\, dt + \int_{Q_T} \frac{1}{2}\overline{M}\Big(\frac{2}{\eta}\overline{M}^{-1}(M(m+1))\Big)\, dx\, dt \\ &< \infty \text{ since } \gamma \in E_{\overline{M}}(Q_T) \text{ and } \Omega \text{ is bounded,} \end{aligned}$$

we use the same arguments for $\Phi_i(x,t,T_{m+1}(u_i))$. Thanks to Lemma 2.2, we deduce that

$$\Phi_i(x,t,T_{m+1}(u_{i,n})) \longrightarrow \Phi_i(x,t,T_{m+1}(u_i)) \text{ strongly in } (E_{\overline{M}}(Q_T))^N.$$

On the other hand, $\nabla T_k(u_{i,n}) \rightharpoonup \nabla T_k(u_i)$ weakly in $(L_M(Q_T))^N$ as n goes to infinity, it follows that

$$\begin{aligned} &\lim_{n\to\infty} \int_{Q_T} \Phi_i(x,t,u_{i,n}) h_m(u_{i,n})[\nabla T_k(u_{i,n}) - \nabla Z_{i,j}^\mu]\, dx\, dt \\ &= \int_{Q_T} \Phi_i(x,t,u_i) h_m(u_i)[\nabla T_k(u_i) - \nabla Z_{i,j}^\mu]\, dx\, dt. \end{aligned} \tag{3.69}$$

Using the modular convergence of $Z_{i,j}^\mu$ as $j \longrightarrow \infty$ and then $\mu \longrightarrow \infty$, we get (r_2). Now we prove (r_3), remark that for $n \geq m+1$, we have

$$\nabla u_{i,n} h_m'(u_{i,n}) = \nabla T_{m+1}(u_{i,n}) \text{ a.e in } Q_T.$$

By the almost everywhere convergence of $u_{i,n}$, we have $T_k(u_{i,n}) - Z_{i,j}^\mu$ converges to $T_k(u_i) - Z_{i,j}^\mu$ in $L^\infty(Q_T)$ weak-$*$ and we have already proved that $\Phi_i(x,t,T_{m+1}(u_{i,n})) \longrightarrow \Phi_i(x,t,T_{m+1}(u_i))$ strongly in $(E_{\overline{\varphi}}(Q_T))^N$ then,

$$\Phi_i(x,t,T_{m+1}(u_{i,n}))\left(T_k(u_{i,n}) - Z_{i,j}^\mu\right) \longrightarrow \Phi_i(x,t,T_{m+1}(u_i))\left(T_k(u_i) - Z_{i,j}^\mu\right),$$

strongly in $E_{\overline{\varphi}}(Q_T)$ as $n \longrightarrow \infty$. Using again the fact that, $\nabla T_{m+1}(u_{i,n}) \rightharpoonup \nabla T_{m+1}(u_i)$ weakly in $(L_\varphi(Q_T))^N$ as n tends to $+\infty$ we obtain

$$\begin{aligned}&\int_{E_{m,n}} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\,dx\,dt\\ &\longrightarrow \int_{E_m} \Phi_i(x,t,u_i)\nabla u_i(T_k(u_i) - Z^{\mu}_{i,j})\,dx\,dt \text{ as } n \longrightarrow \infty.\end{aligned}$$

Using the modular convergence of $Z^{\mu}_{i,j}$ as $j \longrightarrow +\infty$ and letting μ tends to infinity, we get (r_3). As a consequence of Lemma 3.3, the results of Proposition 3.8 follow. □

Step 4: Passing to the limit.
The limit u_i of the approximate solution $u_{i,n}$ of (3.27) satisfies the renormalization identity,

$$\lim_{m\to\infty} \int_{\{m\le |u_i|\le m+1\}} \mathcal{A}(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt = 0. \tag{3.70}$$

Proof. Fix $m > 0$ and we can write

$$\begin{aligned}&\int_{\{m\le |u_{i,n}|\le m+1\}} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\ &= \Big(\int_{Q_T} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})(\nabla T_{m+1}(u_{i,n}) - \nabla T_m(u_{i,n}))\,dx\,dt\Big)\\ &= \Big(\int_{Q_T} \mathcal{A}(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n}))\nabla T_{m+1}(u_{i,n})\,dx\,dt\\ &\qquad - \int_{Q_T} \mathcal{A}(x,t,T_m(u_{i,n}),\nabla T_m(u_{i,n}))\nabla T_m(u_{i,n}))\,dx\,dt\Big).\end{aligned}$$

From ***(ii)***, ***(iii)*** of Proposition 3.8 and passing to the limit as n goes to infinity for fixed m, we get

$$\begin{aligned}&\lim_{n\to\infty} \int_{\{m\le |u_{i,n}|\le m+1\}} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\\ &= \int_{\{m\le |u_i|\le m+1\}} \mathcal{A}(x,t,u_i,\nabla u_i)\nabla u_i\,dx.\end{aligned}$$

Finally, we pass to the limit as m goes to infinity and then we use (3.63), it follows

$$\begin{aligned}&\lim_{m\to\infty}\lim_{n\to\infty} \int_{\{m\le |u_{i,n}|\le m+1\}} \mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\ &= \lim_{m\to\infty} \int_{\{m\le |u_i|\le m+1\}} \mathcal{A}(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt = 0.\end{aligned}$$

Which give the desired result. □

Now, we will pass to the limit. Let us take in the approximate problem (3.27) the test function $r'(u_{i,n})$ with $r \in W^{1,\infty}(\mathbb{R})$ such that r' have a compact support such that for $k > 0$, $\mathrm{supp}(\mathrm{r}') \subset [-\mathrm{k},\mathrm{k}]$ we get

$$\begin{aligned}&\frac{\partial B^n_{i,r}(x,u_{i,n})}{\partial t} - \mathrm{div}\,(r'(u_{i,n})\mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})) + r''(u_{i,n})\mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\\ &\quad -\mathrm{div}\,(r'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})) + r''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}\\ &\qquad = -f_{i,n}(x,u_{1,n},u_{2,n})r'(u_{i,n}) \text{ in } \mathfrak{D}'(Q_T),\end{aligned} \tag{3.71}$$

for $i = 1, 2$, where $B^n_{i,r}(x,\tau) = \displaystyle\int_0^\tau \frac{\partial b_{i,n}(x,s)}{\partial s} r'(s)\,ds$.

Our aim here is to pass to the limit in each term in the previous equality, let us start by the terms of the left-hand side:

Limit of the first term $\dfrac{\partial B^n_{i,r}(x,u_{i,n})}{\partial t}$, since r is bounded and $B^n_{i,r}(x,u_{i,n}) \longrightarrow B_{i,r}(x,u_i)$ a.e in Q_T and in $L^\infty(Q_T)$ weak*, then

$$\frac{\partial B^n_{i,r}(x,u_{i,n})}{\partial t} \longrightarrow \frac{\partial B_{i,r}(x,u_i)}{\partial t} \quad \text{in} \quad \mathfrak{D}'(Q_T) \quad \text{as} \quad n \to \infty.$$

Remark that, since r' and r'' have a compact support in $\mathbb{R}$, there exists $k > 0$ such that $\mathrm{supp(r')}, \mathrm{supp(r'')} \subset [-\mathrm{k},\mathrm{k}]$. For n large enough, we have:

$$r'(u_{i,n})\mathcal{A}(x,t,u_{i,n},\nabla u_{i,n}) = r'(u_{i,n})\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) \quad \text{a.e. in } Q_T,$$

$$\begin{aligned} &r''(u_{i,n})\mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \\ &= r''(u_{i,n})\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n}) \quad \text{a.e. in } Q_T, \end{aligned}$$

$$r'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) = r'(T_k(u_{i,n}))\Phi_{i,n}(x,t,T_k(u_{i,n})),$$

$$r''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = r''(T_k(u_{i,n}))\Phi_{i,n}(x,t,T_k(u_{i,n}))\nabla T_k(u_{i,n}).$$

For the second term of (3.71), Since $r'(u_{i,n}) \to r'(u_i)$ a.e in Q_T as $n \to \infty$, r' is bounded and $(ii),(iii)$ of proposition 3.8 we have

$$r'(u_{i,n})\mathcal{A}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) \rightharpoonup r'(u_i)\mathcal{A}(x,t,T_k(u_i),\nabla T_k(u_i))$$

weakly in $(L_{\overline{M}}(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$, then

$$r'(u_{i,n})\mathcal{A}(x,t,u_{i,n},\nabla u_{i,n}) \rightharpoonup r'(u_i)\mathcal{A}(x,t,u_i,\nabla u_i) \text{ weakly in } (L_{\overline{M}}(Q_T))^N.$$

Concerning the third term of (3.71), Since $r'(u_{i,n}) \to r''(u_i)$ a.e in Q_T as $n \to \infty$, r'' is bounded and $(ii),(iii)$ of proposition 3.8 we obtain, as $n \to \infty$

$$r''(u_{i,n})\mathcal{A}(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} \rightharpoonup r''(u_i)\mathcal{A}(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i)$$

weakly in $L^1(Q_T)$. And then

$$r''(u_i)\mathcal{A}(x,t,T_k(u_i),\nabla T_k(u_i))\nabla T_k(u_i) = r''(u_i)\mathcal{A}(x,t,u_i,\nabla u_i)\nabla u_i \quad \text{a.e. in } Q_T.$$

Arguing similarly, we get the limit of the fourth term of (3.71),

$$r'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \to r'(u_i)\Phi_i(x,t,u_i) \text{ strongly in } (E_M(Q_T))^N.$$

For the remaining term of the left-hand side, we have $r''(u_{i,n})$ converges to $r''(u_i)$ and $\nabla T_k(u_{i,n}) \rightharpoonup \nabla T_k(u_i)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$, while $\Phi_{i,n}(x,T_k(u_{i,n}))$ is uniformly bounded with respect to n and converges a.e. in Q_T to $\Phi_i(x,T_k(u_i))$ as n tends to $+\infty$. Therefore

$$r''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \rightharpoonup r''(u_i)\Phi_i(x,t,u_i)\nabla u_i \text{ weakly in } L_M(Q_T).$$

Concerning the right-hand side of (3.71), due to (3.11),(3.12), (3.25) and (3.26), we have

$$f_{i,n}(x,u_{1,n},u_{2,n})r'(u_{i,n}) \longrightarrow f_i(x,u_1,u_2)r'(u_i) \quad \text{strongly in } L^1(Q_T) \text{ as } n \to \infty.$$

Now, we are ready to pass to the limit as $n \to \infty$ in each term of (3.71) to conclude that u_i satisfies (3.16). It remains to show that $B_{i,r}(x,u_i)$ satisfies the initial condition of (3.27). To do this, recall that, r'' has a compact support, we have $B^n_{i,r}(x,u_{i,n})$ is bounded in $L^\infty(Q_T)$. Moreover, (3.71) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial B^n_{i,r}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T)$. As a consequence, an Aubin's type Lemma (cf [24, Corollary 4]) and Lemma 2.5 imply that $B^n_{i,r}(x,u_{i,n})$ is in a compact set of $C^0([0,T];L^1(\Omega))$. It follows that, $B^n_{i,r}(x,u_{i,n})(t=0)$ converges to $B_{i,r}(x,u_i)(t=0)$ strongly in $L^1(\Omega)$. Due to Remark 3.2 and the fact that $b_{i,n}(x,u_{i,0n}) \longrightarrow b_i(x,u_{i,0})$ in $L^1(\Omega)$, we conclude that $B^n_{i,r}(x,u_{i,n})(t=0) = B^n_{i,r}(x,u_{i,0n})$ converges to $B_{i,r}(x,u_i)(t=0)$ strongly in $L^1(\Omega)$. Then we conclude that $B_{i,r}(x,u_i)(t=0) = B_{i,r}(x,u_{i,0})$ in Ω, which conclude the full proof of the main result.

Declarations

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