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ON THE CLASSES OF NULL ALMOST L-WEAKLY COMPACT AND NULL ALMOST M-WEAKLY COMPACT OPERATORS

SAFAE EL FILALI AND KHALID BOURAS

Abstract. In this paper, we establish conditions under which each positive Null almost L-weakly compact operator is Null almost M-weakly compact and conversely. Moreover, we provide the necessary and sufficient conditions under which any positive Null almost L-weakly compact operator $T : E \to F$ admits a Null almost M-weakly compact adjoint $T' : F' \to E'$. Finally, we give some connections between the class of Null almost L-weakly compact (resp. Null almost M-weakly compact) operators and the class of L-weakly compact (resp. M-weakly compact).

1. Introduction

The classes of L- and M-weakly compact operators are two special classes of weakly compact operators and were introduced and studied by P. Meyer-Nieberg [5].

– An operator T from a Banach lattice E into a Banach space X is *M-weakly compact* if for each disjoint bounded sequence (x_n) of E, we have $\|T(x_n)\| \to 0$.
– An operator T from a Banach space X into a Banach lattice E is called *L-weakly compact* if for each disjoint bounded sequence (y_n) in the solid hull of $T(B_E)$, we have $\|y_n\| \to 0$.

In [4] Bouras and El aloui introduced two classes of operators:

- An operator T from a Banach space X into a Banach lattice F is called *Null almost L-weakly compact* if for every weakly null sequence (x_n) of X and every disjoint sequence (f_n) of $B_{F'}$, we have $f_n(T(x_n)) \to 0$.
- An operator T from a Banach lattice E into a Banach space Y is called *Null almost M-weakly compact* if $f_n(T(x_n)) \to 0$ for every disjoint sequence (x_n) of B_E and every weakly null sequence (f_n) of Y'.

It should be noted that the class of Null almost L-weakly compact (resp. Null almost M-weakly compact) operators contains that of L-weakly compact (resp. M-weakly compact) operators, but the converse is not true in general. For instance, the identity $Id_{\ell^1} : \ell^1 \to \ell^1$ (resp. $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$) is Null almost L-weakly compact (resp. Null almost M-weakly compact) operator since ℓ^1 has the positive Schur property. But it is not L-weakly compact (resp. M-weakly compact) as (e_n) is a norm bounded disjoint sequence of ℓ^1 (resp. ℓ^∞) satisfying $\|e_n\|_1 = \|e_n\|_\infty = 1 \not\to 0$.

A Null almost L-weakly compact operator between Banach lattices is not necessarily Null almost M-weakly compact and conversely. For instance, according to Proposition 2.2 and Corollary 2.1 of [4], let the identity operator $Id_{\ell^1} : \ell^1 \to \ell^1$ (resp. $Id_{c_0} : c_0 \to c_0$) is Null almost L-weakly compact (resp. Null almost M-weakly compact) as $\ell^1 = (c_0)'$ has the positive Schur property. But it is not Null almost M-weakly compact (resp. Null almost L-weakly compact) because $(\ell^1)'$ (resp. (c_0)) does not have the positive Schur property.

Contrary to L- and M-weakly compact operators, which are in duality with each other, the situation is different for Null almost L-weakly compact and Null almost M-weakly

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compact operators. Recently, the authors in [4] proved that an operator $T : E \to Y$, its adjoint T' is Null almost L-weakly compact, if and only if T is Null almost M-weakly compact, and for an operator $T : X \to F$, if its adjoint T' is Null almost M-weakly compact, then T is Null almost L-weakly compact. However, in general, the adjoint of Null almost L-weakly compact operator need not be Null almost M-weakly compact.

In this regard, we give some characterizations of Null almost L-weakly compact and Null almost M-weakly compact operators (Propositions 2.1,2.2 and 2.3). Also, we provide necessary conditions under which each positive Null almost L- weakly compact operator is Null almost M-weakly compact. More precisely, we show that every positive Null almost L-weakly compact operator $T : E \to F$ is Null almost M-weakly compact, then the norm of E' or F' is order continuous (Theorem 2.5). Furthermore, we establish some conditions on a pair of Banach lattices E and F that tell us when every Null almost M-weakly compact operator $T : E \to F$ is Null almost L-weakly compact. Mainly, we prove that if every positive Null almost M-weakly compact operator $T : E \to F$ is Null almost L-weakly compact, then the lattice operations in E are weakly sequentially continuous or the norm of F is order continuous (Theorem 2.6).

In the second goal of this paper, we investigate sufficient and necessary conditions under which the adjoint operator of every positive Null almost L-weakly compact operator is Null almost M-weakly compact (see Theorem 2.10 and Proposition 2.8). Finally, we present some conditions for which the class Null almost L-weakly compact (resp. Null almost M-weakly compact) operators coincides with that of L-weakly compact (resp. M-weakly compact) operators. Specifically, we show that if every positive Null almost L-weakly compact from E into F is L-weakly compact, then the norm of E' is order continuous or dim $F < \infty$ (see Theorem 2.12). Moreover, we prove that for a Banach lattice F such that F is Dedekind σ-complet, if every positive Null almost M-weakly compact from E into F is M-weakly compact, then E is finite dimensional or the norm of F is order continuous (see Theorem 2.15).

In this paper X, Y will denote real Banach spaces and E, F will denote real Banach lattices. The unit ball of E will be denoted by B_E. We will use the term operator for any $T : X \to Y$ between two Banach spaces to mean a bounded linear mapping. We refer the reader to [1, 5, 8] for notation and terminology concerning Banach lattices and operators.

2. Main Results

We start with the following characterizations of Null almost L-weakly compact operator (resp. Null almost M-weakly compact).

Proposition 2.1. *Let X be a Banach space and F be a Banach lattice. If one of the following statements holds:*

(1) *F or X has the Schur property.*
(2) *F'' has an order continuous norm and X has the Dunford-Pettis property.*
(3) *F has an order continuous norm and X has the Dunford-Pettis* property.*

Then each operator $T : X \to F$ is Null almost L-weakly compact.

Proof. (1) Let (x_n) be a weakly null sequence of X and (f_n) be a disjoint sequence of $B_{F'}$. The sequence $(T(x_n))$ converges weakly to zero, as F has the Schur property (resp. X has the Schur property), then $\|T(x_n)\| \to 0$ (resp. $\|x_n\| \to 0$) and hence $f_n(T(x_n)) \to 0$.
(2) Let (x_n) be a weakly null sequence of X and (f_n) be a disjoint sequence of $B_{F'}$, as F'' has an order continuous, then it follows from Theorem 2.4.14 of [5] that $f_n \overset{\sigma(F',F'')}{\to} 0$, and so $T'(f_n) \overset{w}{\to} 0$ in X'. Since X has the Dunford-Pettis property, then $f_n(T(x_n)) = T'(f_n)(x_n) \to 0$.
(3) The proof is similar to that of (2).

□

Proposition 2.2. *Let E be a Banach lattice and Y be a Banach space. If one of the following statements holds:*

(1) *E' or Y' has the Schur property.*
(2) *Y has the Dunford-Pettis property and E' has an order continuous norm.*
(3) *E has the weak Dunford-Pettis property and E' has an order continuous norm.*

Then every operator $T : E \to Y$ is Null almost M-weakly compact.

Proof. (1) Let (x_n) be a disjoint sequence of E and (f_n) be a weakly null sequence of Y', the sequence $(T'(f_n))$ converges weakly to 0. As E' has the Schur property (resp. Y' has the Schur property), then $\|T'(f_n)\| \to 0$ (resp. $\|f_n\| \to 0$) and hence $f_n(T(x_n)) \to 0$.

(2) and (3) Let (x_n) be a disjoint sequence of E and (f_n) be a weakly null sequence of Y'. Since E' has an order continuous norm, it follows from Theorem 116.1 of [8] that $x_n \overset{w}{\to} 0$ in E, then $T(x_n) \overset{w}{\to} 0$ in Y. On the other hand $T'(f_n) \overset{w}{\to} 0$ in E'. As E admits the weak Dunford-Pettis property (resp. Y admits the Dunford-Pettis property), hence $f_n(T(x_n)) = T'(f_n)(x_n) \to 0$.

□

As per [5] a linear operator $T : E \to F$ between two vector lattices is said to be disjointness preserving if T sends disjoint elements in E to disjoint elemnet in F (i.e, $x \perp y$ in E implies $Tx \perp Ty$ in F).

Proposition 2.3. *Let E and F be two Banach lattices and Y be a Banach space. Consider the scheme of operators $E \overset{S}{\to} Y \overset{R}{\to} F$.*

(1) *If S is Null almost M-weakly compact operator, then $R \circ S$ is likewise Null almost M-weakly compact.*
(2) *If R is Null almost M-weakly compact operator and S is disjointness preserving then $R \circ S$ is likewise Null almost M-weakly compact.*

Proof. (1) Let (x_n) be a disjoint sequence of B_E and (f_n) be a weakly null sequence of F', then $R'(f_n) \overset{\sigma(Y',Y'')}{\to} 0$. Since S is Null almost M-weakly compact. So, $R'(f_n)(S(x_n)) = f_n(R(S(x_n))) = f_n(R \circ S(x_n)) \to 0$ and hence $R \circ S$ is Null almost M-weakly compact operator.
(2) Let (x_n) be a disjoint sequence of B_E and (f_n) be a weakly null sequence of F'. Since the operator S is disjointness preserving, then the sequence $S(x_n)$ is disjoint of B_Y. On the other hand R is Null almost M-weakly compact, then $f_n(R \circ S(x_n)) = f_n(R(S(x_n))) \to 0$.
□

The following lemma establishes the characterization of the order continuity of the norm of the topological dual of a Banach lattice by a Null almost M-weakly compact operator.

Lemma 2.4. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(1) *Every positive operator $T : E \to \ell^1$ is Null almost M-weakly compact.*
(2) *The norm of E' is order continuous.*

Proof. (2) ⇒ (1) Follows from Proposition 2.2.
(1) ⇒ (2) Suppose that the norm of E' is not order continuous, according to Theorem 2.4.14 of [5], E contains a vector sublattice isomorphic to ℓ^1, then there is a positive projection P on E whose range is a lattice isomorphic copy of ℓ^1. Clearly, P is not Null almost M-weakly compact. □

Now we are in a position to give necessary conditions under which each positive Null almost L-weakly compact operator is Null almost M-weakly compact.

Theorem 2.5. *Let E and F be two Banach lattices. If each positive Null almost L-weakly compact operator from E into F is Null almost M-weakly compact, then one of the following assertions holds:*

(1) *The norm of E' is order continuous.*
(2) *The norm of F' is order continuous.*

Proof. To complete the proof, it suffices to show that if the norm of F' is not order continuous, then the norm of E' must be order continuous. By Lemma 2.4, there exists a positive operator $P : F \to \ell^1$ that is not Null almost M-weakly compact. Consider now the composed operator $P \circ T : E \to F \to \ell^1$.

Assuming that the norm on F' is not order continuous, it follows from the proof of Lemma 2.4 that there exists a sublattice H of F that is isomorphic to ℓ^1. Let $i : \ell^1 \to F$ be the canonical injection of ℓ^1 into F and consider an arbitrary positive operator $T : E \to \ell^1$. Since ℓ^1 has the Schur property, then it follows from Proposition 2.1 that T is a Null almost L-weakly compact operator and hence it is Null almost M-weakly compact by our assumption. Finally, Lemma 2.4 finishes the proof. □

There exist operators which are Null almost M-weakly compact but not Null almost L-weakly compact. For example, let the identity operator $Id_{c_0} : c_0 \to c_0$ is Null almost M-weakly compact as $(c_0)'$ has the positive Schur property. But it is not Null almost L-weakly compact because (c_0) does not have the positive Schur property.

The following result gives the conditions under which each positive Null almost M-weakly compact operator T from E to F is Null almost L-weakly compact.

Theorem 2.6. *Let E and F be two Banach lattices. If every positive Null almost M-weakly compact from E into F is Null almost L-weakly compact, then one of the following conditions is valid:*

(1) *The lattice operations in E are weakly sequentially continuous.*
(2) *The norm of F is order continuous.*

Proof. Suppose by way of contradiction that the lattice operations in E are not weakly sequentially continuous and the norm of F is not order continuous. To finish the proof, we have to construct a Null almost M-weakly compact operator from E into F that is not Null almost L-weakly compact.

As the lattice operations of E are not weakly sequentially continuous, there exists a weakly null sequence (x_n) of E, $f \in (E')^+$, $h, h_n \in [-f, f]$ which satisfy $h_n \to h$ for the topology $\sigma(E', E)$ and $h_n(x_n) \geq \epsilon$ for all n and some $\epsilon > 0$ (see Theorem 2 of [6]).

On the other hand, since the norm of F is not order continuous Theorem 2.4.2 of [5] implies that there exists a positive order bounded disjoint sequence $(y_n) \subset F$ which does not converge to zero in norm. We may assume that there exist $y \in F^+$ such that $0 \leq y_n \leq y$ and $\|y_n\| = 1$ for all n. Thus, by Lemma 3.4 of [2], there exists a positive disjoint sequence (f_n) of F' with $\|f_n\| \leq 1$ such that $f_n(y_m) = 1$ for all $n = m$ and $f_n(y_m) = 0$ for $n \neq m$.

Now, consider the operator $S : E \to c_0$ defined by

$$S(x) = (h_n(x) - h(x))_n$$

holds for each $x \in E$. And define the operator $R : c_0 \to F$ by

$$R(\lambda_n) = \sum_{n=1}^{\infty} \lambda_n y_n.$$

As $(c_0)'$ has the Schur property, then by Proposition 2.2 and 2.3, S is Null almost M-weaky compact and hence $T = R \circ S$ defined by

$$T(x) = \sum_{n=1}^{\infty}(h_n(x) - h(x))y_n \quad \text{for all} \quad x \in E$$

is Null almost M-weakly compact operator. But T is not Null almost L-weakly compact. Otherwise, $f_n(T(x_n)) \to 0$, i.e, by the following equality

$$f_n(T(x_n)) \;\; = \;\; f_n(\sum_{k=1}^{\infty}(h_k(x_n) - h(x_n))y_k) = (h_n(x_n) - h(x_n))$$

$h_n(x_n) \to 0$. This leads to a contradiction. And this finish the proof.

□

Remark 2.7. The conditions stated in Theorem 2.6 are insufficient to ensure that each positive Null almost M-weakly compact operator $T : E \to F$ is Null almost L-weakly compact. In fact, if we consider $E = F = c_0$, it is clear that the identity operator $Id_{c_0} : c_0 \to c_0$ is Null almost M-weakly compact, but it is not Null almost L-weakly compact. However, c_0 has weakly sequentially continuous lattice operations and the norm of c_0 is order continuous.

The authors in [4] gave an example of a Null almost L-weakly compact operator such that its adjoint is not Null almost M-weakly compact. Noting that the Lorentz space $\wedge(\omega, 1)$ has the positive Schur property but its bidual does not, they concluded that the identity operator of the Lorentz space $\wedge(\omega, 1)$ is Null almost L-weakly compact, but its adjoint is not Null almost M-weakly compact.

In the following result, we present sufficient conditions under which the adjoint of every Null almost L-weakly compact operator is Null almost M-weakly compact.

Proposition 2.8. *Let E and F be two Banach lattices. If one of the following assertions is valid:*

1. *F'' has the Schur property.*
2. *F'' has an order continuous norm and E' has the Dunford-Pettis property.*
3. *$dim\ F < \infty$.*

Then the adjoint of every Null almost L-weakly compact $T : E \to F$ is Null almost M-weakly compact.

Proof. (1) Let T be a Null almost L-weakly compact operator , as F'' has the Schur property then T'' is Null almost L-weakly compact and Theorem 2.1 (1) of [4] follows that T' is Null almost M-weakly compact operator.

(2) To show that T' is Null almost M-weakly compact. Let (f_n) be a disjoint sequence of $B_{F'}$ and (ψ_n) be a weakly null sequence in E'', we have to prove that $\psi_n(T'(f_n)) \to 0$. Since F'' has an order continuous norm, then it follows from Theorem 2.4.14 of [5] that $f_n \to 0$ for $\sigma(F', F'')$ and hence $T'(f_n) \to 0$ for $\sigma(E', E'')$, as E' has Dunford-Pettis property then $\psi_n(T'(f_n)) \to 0$, as desired.

(3) In this case, every operator is L-weakly compact, then its adjoint T' is M-weakly compact, and hence T' is Null almost M-weakly compact. □

The assertions (1) and (2) in the previous proposition are not necessary as shown in the following example.

Example 2.9. An operator $T : \ell^1 \to \ell^\infty$ is Null almost L-weakly compact if and only if it adjoint T' is Null almost M-weakly compact. In fact, let $T : \ell^1 \to \ell^\infty$ defined by $T((\lambda_n)_n) = (\sum_{n=1}^{\infty} \lambda_n)e$, where $e = (1, 1, 1, ...)$ is Null almost L-weakly compact as compact (it rank one). Hence, it is clear that its adjoint $T' : \ell^1 \to \ell^\infty$ is Null almost M-weakly compact.

The next result tells us when the adjoint of every positive Null almost L-weakly compact operator, between two Banach lattices, is Null almost M-weakly compact.

Theorem 2.10. *Let E and F be two Banach lattices such that F'' has an order continuous norm. If each positive Null almost L-weakly compact operator $T : E \to F$ admits a Null almost M-weakly compact adjoint $T' : F' \to E'$, then one of the following properties is valid:*

(1) *E' has the weak Dunford-Pettis property.*
(2) *F is KB-space.*

Proof. Assume by way of contradiction that E' does not have weak Dunford-Pettis property and F is not a KB-space. We need to construct a positive Null almost L-weakly compact operator from E into F such that its adjoint is not Null almost M-weakly compact.
Since the Banach lattice F is not a KB-space, then by Theorem 4.61 of [1] c_0 is lattice embeddable in F. By virtue of Proposition 0.5.1 [7] we have $F = \overline{\text{span}}\{y_n : n \in \mathbb{N}\}$ for some sequence $(y_n) \subset F$ of positive disjoint elements. We may assume that $\|y_n\| = 1$ and $c = \sup_k \left\|\sum_{n=1}^k y_k\right\| < \infty$. Hence, the operator $S : c_0 \to F$ defined by

$$S\left((\lambda_n)_n\right) = \sum_{n=1}^{\infty} \lambda_n y_n$$

is a linear homomorphism from c_0 onto F. On the other hand, according to Theorem 116.3 of [8] there exists a disjoint sequence (f_n) of positive elements in the unit ball of F' such that $f_n(y_n) = 1$ for all n and $f_n(y_m) = 0$ for $m \neq n$.

As E' does not have weak Dunford-Pettis property, there exist a weakly null sequence (ψ_n) in E'' and a disjoint weakly null sequence (ϕ_n) in $(E')^+$ such that $(\psi_n(\phi_n))$ does not converge to zero. Then, by passing to a subsequence, we can assume that for some $\epsilon > 0$ we have

$$|\psi_n(\phi_n)| > \epsilon$$

for all n. Consider the operator $R : E \to c_0$ defined by $R(x) = (\phi_n(x))_{n=1}^{\infty}$.

Next, we consider the operator $T = S \circ R : E \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} \phi_n(x).y_n$$

holds for all $x \in E$. It is clear that its adjoint $T' : F' \to E'$ defined by

$$T'(h) = \sum_{n=1}^{\infty} h_n(y_n).\phi_n$$

for all $h \in F'$. As F'' has an order continuous norm and c_0 has Dunford-Pettis property, then by Proposition 2.1, S is Null almost L-weakly compact and by Theorem 2.6(i) of [4], the composed operator T is Null almost L-weakly compact. But the operator T' is not Null almost M-weakly compact. Indeed, note that (ψ_n) is a weakly null sequence in E''

and (f_n) a disjoint sequence of F'. From,

$$\begin{aligned}\psi_n(T'(f_n)) &= \psi_n(\sum_{k=1}^{\infty} f_n(y_k).\phi_k) \\ &= \sum_{n=1}^{\infty} f_n(y_k).\psi_n(\phi_k) \\ &= f_n(y_n).\psi_n(\phi_n)\end{aligned}$$

and so, $|\psi_n T'(f_n)| = |\psi_n(\phi_n)| \geq \epsilon$ for all n. We conclude that $\psi_n(T'(f_n) \not\to 0$, then T' is not Null almost M-weakly compact. This is a contradiction and the proof is finished.

□

Remark 2.11. The second assertion in the necessary condition of the previous theorem is not sufficient to guarantee that the adjoint of every positive Null almost L-weakly compact operator $T : E \to F$ is Null almost M-weakly compact. In fact, the identity operator of the Lorentz space $\wedge(\omega, 1)$ is Null almost L-weakly compact but its adjoint is not Null almost M-weakly compact. However, $\wedge(\omega, 1)$ is a KB-space.

We mentioned before that each Null almost L-weakly compact operator is not L-weakly compact. However, we have the following theorem.

Theorem 2.12. *Let E and F be two Banach lattices. If every positive Null almost L-weakly compact from E into F is L-weakly compact, then one of the following conditions is valid:*

(1) *The norm of E' is order continuous.*
(2) *F is finite-dimensional.*

Proof. Suppose that neither (1) not (2) holds. To finish the proof, we have to construct a positive Null almost L-weakly compact operator $T : E \to F$ that is not L-weakly compact.

As the norm of E' is not order continuous, it follows from Theorem 116.1 of [8] that there is a norm bounded disjoint sequence (u_n) of positive elements of E which does not converge weakly to zero. Hence, we may assume that $||u_n|| \leqslant 1$ for all n and also that for some $0 \leqslant \phi \in E'$ satisfying $\phi(u_n) = 1$ for all n. Then, it follows from Theorem 116.3 of [8] that the components ϕ_n of ϕ in the carriers C_{u_n} form an order bounded disjoint sequence in $(E')^+$ such that $\phi_n(u_n) = \phi(u_n)$ for all n and $\phi_n(u_m) = 0$ if $n \neq m$.

Define the positive operator $R : E \to \ell_1$ by

$$R(x) = \left(\frac{\phi_n(x)}{\phi(u_n)}\right)_{n=1}^{\infty} \quad \text{for all} \quad x \in E.$$

Since $\sum_{n=1}^{\infty} |\frac{\phi_n(x)}{\phi(u_n)}| \leqslant \frac{1}{\epsilon} \sum_{n=1}^{\infty} \phi_n(|x|) \leqslant \frac{1}{\epsilon}\phi(|x|)$ holds for each $x \in E$ the operator R is well defined. On the other hand, F is infinite-dimensional then there exists a disjoint sequence (y_n) of positive elements in F such that $\|y_n\| = 1$ for all n. Now, consider the positive operator $S : \ell_1 \to F$ defined by

$$S((\lambda_n)_n) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for all} \quad (\lambda_n) \in \ell_1.$$

Clearly, the operator S is well defined. Next, we consider the composed operator $T = S \circ R : E \to \ell_1 \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} \frac{\phi_n(x)}{\phi(u_n)} y_n \quad \text{for all} \quad x \in E.$$

Since ℓ_1 has the Schur property, then T is a Null almost L-weakly compact operator, but it is not L-weakly compact. To see this, from $T(u_n) = y_n$ for all n. Note that (y_n) is a disjoint sequence in the solid hull of $T(B_E)$. But the L-weak compactness of T implies that $\|y_n\| \to 0$, which is a contradiction. □

Corollary 2.13. *Let E and F be two Banach lattices such that the norm of F is order continuous then the following statements are equivalent:*

(1) *Each positive Null almost L-weakly compact operator $T : E \to F$ is L-weakly compact.*
(2) *One of the following conditions is valid:*
  (a) *The norm of E' is order continuous.*
  (b) *F is finite-dimensional.*

Proof. (1) ⇒ (2) Follows from Theorem 2.12.
$(2a) \Rightarrow (1)$ Follows from Theorem 3.6.17 of [5].
$(2b) \Rightarrow (1)$ In this case, every operator T is L-weakly compact.

□

Corollary 2.14. *Let F be a Banach lattice. Then the following assertions are equivalent:*

(1) *Every Null almost L-weakly compact from ℓ_1 into F is L-weakly compact.*
(2) *dim $F < \infty$.*

Note that each M-weakly compact operator is Null almost M-weakly compact but the converse is not true.

Theorem 2.15. *Let E and F be two Banach lattices such that F is Dedekind σ-complet. If every positive Null almost M-weakly compact from E into F is M-weakly compact, then one of the following conditions is valid:*

(1) *E is finite dimensional.*
(2) *The norm of F is order continuous.*

Proof. By way of contradiction, let us assume that E is infinite-dimensional and the norm of F is not order continuous. We need to construct an operator from E into F that is Null almost M-weakly compact but not M-weakly compact.

Since the Banach lattice E is infinite-dimensional, by Lemma 2.3 and Lemma 2.5 of [3], there exists a positive disjoint sequence (x_n) of E^+ with $\|x_n\| = 1$ for all n and there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| = 1$, such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

Consider the positive operator $R : E \to \ell^\infty$ defined by:

$$R(x) = (g_n(x))_n$$

for each $x \in E$. On the other hand, as the norm of F is not order continuous, by Theorem 4.14 of [1] there exists some $y \in F^+$ and there exists a disjoint sequence $(y_n) \subset [0, y]$ which does not converge to zero in norm, we can assume that $\|y_n\| = 1$ for all n. As F is Dedekind σ-complet, it can be inferred from the proof of Theorem 117.3 of [8] that the operator $S : \ell^\infty \to F$ defined by:

$$S((\lambda_n)_n) = \sum_{n=1}^{\infty} \lambda_n y_n.$$

Next, we consider the composed operator $T = S \circ R : E \to \ell^\infty \to F$ defined by

$$T(x) = \sum_{n=1}^{\infty} g_n(x) y_n$$

for each $x \in E$, the operator T is well defined. As $(\ell^\infty)'$ has the Schur property, then it follows from Propositions 2.2, that T is Null almost M-weakly compact. But T is not

M-weakly compact. To see this, since (x_n) is a disjoint sequence of B_E^+ we have

$$\|T(x_n)\| = \|y_n\| = 1$$

for all n, we conclude that E is a finite-dimensional Banach lattice, and the proof is complete. □

Remark 2.16. The assumption "F is Dedekind σ-complet" is essential. Indeed, if we take $E = \ell^\infty$ and $F = c$. It follows from proposition 2.2 that $T : \ell^\infty \to c$ is Null almost M-weakly compact. As ℓ^∞ is an AM-space, then by Theorem 5.62 of [1], T is M-weakly compact. Neither of the two possible conditions is holds.

As a consequence, we obtain :

Corollary 2.17. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(1) *Every positive Null almost M-weakly compact operator $T : E \to \ell^\infty$ is M-weakly compact.*
(2) *E is finite dimensional.*

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Safae El Filali: safaeelfilali8@gmail.com
MMA, FPL, Abdelmalek Essaadi University, Tetouan, Morocco.

Khalid Bouras: k.bouras@uae.ac.ma
MMA, FPL, Abdelmalek Essaadi University, Tetouan, Morocco.

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