Methods of Functional Analysis and Topology
Vol. 31 (2025), no. 3, pp. 204–221
doi.org/10.31392/MFAT-npu26_3.2025.05

SPECTRAL PROPERTIES OF ESSENTIAL PSEUDOSPECTRA UNDER POLYNOMIALLY NON-STRICT SINGULAR PERTURBATIONS

BILEL ELGABEUR

Abstract. This paper investigates the essential pseudospectra of closed linear operators in Banach spaces, focusing on perturbations induced by polynomially non-strictly singular operators, a class that extends the concept of polynomially strictly singular operators. New results are presented regarding the behavior of the essential pseudospectra under these perturbations. In particular, we explore the impact on the left (resp. right) Weyl and Fredholm essential pseudospectra. Additionally, we examine the essential pseudospectra of the sum of two bounded linear operators and apply the results to characterize the pseudo-Fredholm spectra of 2×2 block operator matrices.

1. Introduction

Let X and Y be two Banach spaces. By an operator A from X into Y we mean a linear operator with domain $\mathcal{D}(A) \subseteq X$ and range contained in Y. We denote by $\mathcal{C}(X,Y)$ (resp., $\mathcal{L}(X,Y)$) the set of all closed, densely defined (resp., bounded) linear operators from X to Y. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. If $A \in \mathcal{C}(X,Y)$, we write $N(A) \subset X$ and $R(A) \subset Y$ for the null space and the range of A. We set $\alpha(A) := \dim N(A)$ and $\beta(A) := \operatorname{codim} R(A)$. Let $A \in \mathcal{C}(X,Y)$ with closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X,Y)$) if $\alpha(A) < \infty$, and then A is a Φ_--operator ($A \in \Phi_-(X,Y)$) if $\beta(A) < \infty$.$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ is the class of Fredholm operators while $\Phi_\pm(X,Y)$ denotes the set $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$. For $A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{K}(X,Y), \mathcal{C}(X,Y), \Phi_+(X,Y), \Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced, respectively, by $\mathcal{L}(X), \mathcal{K}(X), \mathcal{C}(X), \Phi_+(X), \Phi_\pm(X)$ and $\Phi(X)$. Let $A \in \mathcal{C}(X)$, the spectrum of A will be denoted by $\sigma(A)$. The resolvent set of $A, \rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in $\Phi_{+A}, \Phi_{-A}, \Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X), \Phi_-(X), \Phi_\pm(X)$ or $\Phi(X)$, respectively. Let $F \in \mathcal{L}(X,Y)$.F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$. F is called an upper (resp., lower) Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp., $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp., $U \in \Phi_-(X,Y)$). The set of Weyl operators is defined as $\mathcal{W}(X,Y) = \{A \in \Phi(X,Y) : i(A) = 0\}$. Sets of left and right Fredholm operators, respectively, are defined as:

$$\Phi_l(X) := \{A \in \mathcal{L}(X) : R(A) \text{ is a closed and complemented subspace of } X,\ \alpha(A) < \infty\}.$$

$$\Phi_r(X) := \{A \in \mathcal{L}(X) : N(A) \text{ is a closed and complemented subspace of } X,\ \beta(A) < \infty\}.$$

An operator $A \in \mathcal{L}(X)$ is left (right) Weyl if A is left (right) Fredholm operator and $i(A) \leq 0 (i(A) \geq 0)$. We use $\mathcal{W}_l(X)(\mathcal{W}_r(X))$ to denote the set of all left(right) Weyl operators. It is Known that the sets $\Phi_l(X)$ and $\Phi_r(X)$ are open satisfying the following inclusions:

2020 *Mathematics Subject Classification.* 47B06, 47D03, 47A10, 47A53, 34K08.
Keywords. Pseudo spectrum, Essential pseudospectra, strict singular operators, polynomially non-strict singular operators.

$$\Phi(X) \subset \mathcal{W}_l(X) \subset \Phi_l(X) \text{ and } \Phi(X) \subset \mathcal{W}_r(X) \subset \Phi_r(X).$$

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y), \mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively. In general, we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}_+(X,Y) \subseteq \mathcal{F}(X,Y)$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{F}_-(X,Y) \subseteq \mathcal{F}(X,Y).$$

If $X = Y$ we write $\mathcal{F}(X), \mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X,X), \mathcal{F}_+(X,X)$ and $\mathcal{F}_-(X,X)$, respectively. Let $\Phi^b(X,Y), \Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$, respectively. If in Definition 1.1 we replace $\Phi(X,Y), \Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by $\Phi^b(X,Y), \Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y), \mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$. These classes of operators were introduced and investigated in [6]. In particular, it is shown that $\mathcal{F}^b(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$ and $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_+(X,Y) \subseteq \mathcal{F}^b(X,Y)$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_-(X,Y) \subseteq \mathcal{F}^b(X,Y)$$

Let $A \in \mathcal{C}(X)$. It follows from the closeness of A that $\mathcal{D}(A)$ endowed with the graph norm $\|\cdot\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \leqslant \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \quad \beta(\hat{A}) = \beta(A), \quad R(\hat{A}) = R(A) \\ \alpha(\hat{A}+\hat{B}) = \alpha(A+B), \\ \beta(\hat{A}+\hat{B}) = \beta(A+B) \text{ and } R(\hat{A}+\hat{B}) = R(A+B) \end{cases} \tag{1.1}$$

In this paper we are concerned with the following essential spectra of $A \in C(X)$:

$\sigma_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi(X)\}$: the Fredholm spectrum of A.

$\sigma^l_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_l(X)\}$: the left Fredholm spectrum of A.

$\sigma^r_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_r(X)\}$: the right Fredholm spectrum of A.

$\sigma_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}(X)\}$: the Weyl spectrum of A.

$\sigma^l_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_l(X)\}$: the left Weyl spectrum of A.

$\sigma^r_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_r(X)\}$: the right Weyl spectrum of A.

$\sigma_{\text{eap}}(A) := \mathrm{C} \backslash \rho_{\text{eap}}(A)$: the essential approximate point spectrum of A.

$\sigma_{e\delta}(A) := \mathrm{C} \backslash \rho_{e\delta}(T)$: the essential defect spectrum of A.

where

$$\rho_{\text{eap}}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_+(X) \text{ and } i(\lambda - A) \leq 0\},$$

and

$$\rho_{e\delta}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_-(X) \text{ and } i(\lambda - A) \geq 0\}$$

The definition of pseudo spectrum of a closed densely linear operator A for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} : \left\|(\lambda - A)^{-1}\right\| > \frac{1}{\varepsilon}\right\}. \tag{1.2}$$

By convention, we write $\left\|(\lambda - A)^{-1}\right\| = \infty$ if $(\lambda - A)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(A)$. In [8], Davies defined another equivalent of the pseudo spectrum, one that is in terms of perturbations of the spectrum. In fact for $A \in C(X)$, we have

$$\sigma_\varepsilon(A) := \bigcup_{\|D\|<\varepsilon} \sigma(A+D). \tag{1.3}$$

Inspired by the notion of pseudospectra, Ammar and Jeribi in their works [5, 6], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator as follows: for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo-upper (resp. lower) semi-Fredholm operator if $A + D$ is an upper (resp. lower) semi-Fredholm operator and it is called a pseudo semi-Fredholm operator if $A + D$ is a semi-Fredholm operator. A is called a pseudo-Fredholm operator if $A + D$ is a Fredholm operator. They are noted by $\Phi^\varepsilon(X)$ the set of pseudo-Fredholm operators, by $\Phi^\varepsilon_\pm(X)$ the set of pseudo-semi-Fredholm operator and by $\Phi^\varepsilon_+(X)$ (resp. $\Phi^\varepsilon_-(X)$) the set of pseudo-upper semi-Fredholm (resp. lower semi-Fredholm) operator. A complex number λ is in $\Phi^\varepsilon_{\pm A}$, $\Phi^\varepsilon_{+A}, \Phi^\varepsilon_{-A}$ or Φ^ε_A if $\lambda - A$ is in $\Phi^\varepsilon_\pm(X), \Phi^\varepsilon_+(X), \Phi^\varepsilon_-(X)$ or $\Phi^\varepsilon(X)$.
F. Abdmouleh and B. Elgabeur in [3] defining the concept of pseudo left (resp. right)-Fredholm, for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo left (resp. right) Fredholm operator if $A + D$ is an left (resp. right) Fredholm operator they are denoted by $\Phi^\varepsilon_l(X)$ (resp. $\Phi^\varepsilon_r(X)$). In this paper we are concerned with the following essential pseudospectra of $A \in C(X)$:

$$\begin{aligned}
\sigma_{e1,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_+(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{+A},\\
\sigma_{e2,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_-(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{-A},\\
\sigma_{e3,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_\pm(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{\pm A},\\
\sigma_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon(X)\} = \mathbb{C}\backslash\Phi^\varepsilon_A,\\
\sigma_{eap,\varepsilon}(A) &:= \sigma_{e1,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall\|D\| < \varepsilon\},\\
\sigma_{e\delta,\varepsilon}(A) &:= \sigma_{e2,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall\|D\| < \varepsilon\},\\
\sigma^l_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_l(X)\},\\
\sigma^r_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_r(X)\},\\
\sigma^l_{w,\varepsilon}(A) &:= \sigma^l_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall\|D\| < \varepsilon\},\\
\sigma^r_{w,\varepsilon}(A) &:= \sigma^r_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall\|D\| < \varepsilon\},\\
\sigma_{w,\varepsilon}(A) &:= \sigma_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) = 0, \forall\|D\| < \varepsilon\}.
\end{aligned}$$

Note that if ε tends to 0, we recover the usual definition of the essential spectra of a closed operator A. The subsets σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra [15], σ_{e3} is the Kato essential spectrum,[18] σ_e is the Wolf essential spectrum [15], σ_{e5} is the Schechter essential spectrum[25], σ_{eap} is the essential approximate point spectrum [23], $\sigma_{e\delta}$ is the essential defect spectrum [24], $\sigma^l_e(A)$(resp.$\sigma^r_e(A)$) is the left (resp. right) Fredholm essential spectra and $\sigma^l_w(A)$(resp.$\sigma^r_w(A)$) is the left (resp. right) Weyl essential spectra [14, 28, 29].
As a concept, pseudospectra and essential pseudospectra are interesting because they offer more information than spectra, especially about transients rather than just asymptotic behavior. Moreover, they perform more efficiently than spectra in terms of convergence and approximation. These include the existence of approximate eigenvalues far from the spectrum and the instability of the spectrum even under small perturbations. Various applications of pseudospectra and essential pseudospectra have been developed as a result of the analysis of pseudospectra and essential pseudospectra.

In this paper, we extend our study of the essential pseudospectra in Banach spaces, focusing on a broader class of operators known as polynomially non-strictly singular operators, which generalize several well-established classes of perturbations, including Fredholm perturbations, polynomially Fredholm perturbations, and strictly singular operators. These classes have been foundational in the work of K. Latrach et al. [20, 21], particularly in the context of essential spectra. The study of these operators has attracted significant attention in spectral theory, with numerous contributions from various authors. The first objective of this paper is to extend the stability results of essential pseudospectra obtained in earlier works [1, 2, 3, 5, 6, 10, 11] by considering perturbations from polynomially non-strictly singular operators for densely defined closed linear operators. The second aim is to describe the essential pseudospectrum of the sum of two bounded linear operators within the framework of these new perturbations.

Let us now outline the structure of the paper:

Section 2 provides a review of essential definitions and notation related to Fredholm operators and their essential spectra. Additionally, we investigate the properties of polynomially non-strictly singular operators, presenting novel results.

Section 3 focuses on stability results and introduces a new characterization of the left (resp. right) Weyl and left (resp. right) Fredholm essential pseudospectra within the class of polynomially non-strictly singular operators.

Section 4 presents a key result concerning the essential pseudospectra of the sum of two bounded linear operators, based on the concept of polynomially non-strictly singular perturbations.

Finally, in Section 5, we extend the results to define pseudo-left (right)-Fredholm spectra for $2 \rtimes 2$ block operator matrices, using polynomially non-strictly singular operators as the basis for the analysis.

We now list some of the known facts about left and right Fredholm operators in Banach space which will be used in the sequel.

Proposition 1.1. [17, propositon 2.3] *Let X, Y and Z be three Banach spaces.*
(i) If $A \in \Phi^b(Y,Z)$ and $T \in \Phi^b_l(X,Y)$ (resp. $T \in \Phi^b_r(X,Y)$), then $AT \in \Phi^b_l(X,Z)$ (resp. $AT \in \Phi^b_r(X,Z)$).

(ii) If $A \in \Phi^b(Y,Z)$ and $T \in \Phi^b_l(X,Y)$ (resp. $T \in \Phi^b_r(X,Y)$), then $TA \in \Phi^b_l(X,Z)$ (resp. $TA \in \Phi^b_r(X,Z)$). ♢

Theorem 1.2. [22, 25] *Let X, Y and Z be three Banach spaces, $A \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$. (i) If $A \in \Phi^b(Y,Z)$ and $T \in \Phi^b(X,Y)$, then $AT \in \Phi^b(X,Z)$ and $i(AT) = i(A) + i(T)$.*

(ii) If $X = Y = Z$, $AT \in \Phi^b(X)$ and $TA \in \Phi^b(X)$, then $A \in \Phi^b(X)$ and $T \in \Phi^b(X)$. ♢

Lemma 1.3. [14, Theorem 2.3] *Let $A \in \mathcal{L}(X)$, then*

(i) $A \in \Phi^b_l(X)$ if and only if, there exist $A_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $A_l A = I - K$.

(ii) $A \in \Phi^b_r(X)$ if and only if, there exist $A_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $AA_r = I - K$. ♢

Lemma 1.4. [14, Theorem 2.7] *Let $A \in \mathcal{L}(X)$.*

If $A \in \Phi^b_l(X)$(resp.$\Phi^b_r(X)$) and $K \in \mathcal{K}(X)$, then $A + K \in \Phi^b_l(X)$(resp.$\Phi^b_r(X)$) and $i(A+K) = i(A)$. ♢

Lemma 1.5. [14, Theorem 2.5] *Let $A, B \in \mathcal{L}(X)$.*

If $A \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) and* $B \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) then* $AB \in \Phi_l^b(X)$*(resp.*$\Phi_r^b(X)$*) and*

$$i(A+B) = i(A) + i(B). \qquad \diamondsuit$$

We close with the following Lemma.

Lemma 1.6. [7, Lemma 3.4] *Let* $A \in \mathcal{L}(X)$*.*

(i) *If* $AB \in \Phi_l^b(X)$ *then* $B \in \Phi_l^b(X)$*.*

(ii) *If* $AB \in \Phi_r^b(X)$ *then* $A \in \Phi_r^b(X)$*.*

Definition 1.7. Let X be a Banach space.

(i) An operator $A \in \mathcal{L}(X)$ is said to have a left Fredholm inverse if there exists $A_l \in \mathcal{L}(X)$ such that $I - A_l A \in \mathcal{K}(X)$.

(ii) An operator $A \in \mathcal{L}(X)$ is said to have a right Fredholm inverse if there exists $A_r \in \mathcal{L}(X)$ such that $I - AA_r \in \mathcal{K}(X)$. $\diamondsuit$

We know by the classical theory of Fredholm operators, see for example [18], that A belong to $\Phi(X)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.

We define these sets $\mathcal{I}nvF_A^l(X)$ and $\mathcal{I}nvF_A^r(X)$ by:

$$\mathcal{I}nv_{A,l}^F(X) := \{A_l \in \mathcal{L}(X) : A_l \text{ is a left Fredholm inverse of } A\},$$

$$\mathcal{I}nv_{A,r}^F(X) := \{A_r \in \mathcal{L}(X) : A_r \text{ is a right Fredholm inverse of } A\}.$$

2. Polynomially Non-Strictly Singular Perturbations

In this section, we introduce the class of *polynomially non-strictly singular perturbations* associated with a closed linear operator T on a Banach space X. This class generalizes the notion of polynomially strictly singular perturbations by relaxing the strict singularity condition through two complementary sets of perturbations characterized by their resolvent behavior and polynomial bounds. These perturbations play a crucial role in understanding the stability properties of the essential pseudospectrum under polynomially controlled perturbations.

Definition 2.1. An operator $S \in \mathcal{L}(X, Y)$ is to be strictly singular if for every infinite dimensional subspace M of X, the restriction of S to M is not a homeomorphism.

Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X into Y. Note that $\mathcal{S}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$. In general, strictly singular operators are not compact (see [12, 13]) and if $X = Y, \mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $\mathcal{K}(X) = \mathcal{S}(X)$. For basic properties of strictly singular operators we refer to [13, 18].

Definition 2.2. An minimal polynomial P is the unitary polynomial of smaller degree which cancels an endomorphism, that is to say a linear application of a vector space in itself.

In the following, we define the set of polynomially strict singular operators will denote by $\mathcal{P}_\mathcal{S}$ as follow:

$\mathcal{P}_\mathcal{S} = \{A \in \mathcal{L}(X)$, such that there exists a nonzero complex polynomial $P(z) := \sum_{k=0}^{p} a_k z^k$, satisfying $P\left(\frac{1}{n}\right) \neq 0, \forall n \in \mathbb{Z}^*$ and $P(A) \in \mathcal{S}(X)\}$.

In the following $\mathcal{E}_{\mathcal{PS}}(X)$ is a subset of $\mathcal{P}_{\mathcal{S}}$ as follow:

$$\mathcal{E}_{\mathcal{PS}}(X) := \{A \in \mathcal{P}_{\mathcal{S}}(X) \text{ such that the minimal polynomial } p(.) \text{ of } A \text{ satisfies } p(-1) \neq 0\}.$$

Let us recall the following results which are fundamental for the proofs of the main results.

Definition 2.3. Let T be a closed linear operator on a Banach space X. We define the class of *polynomially non-strictly singular perturbations* associated with T as the set of bounded perturbations $R \in \mathcal{C}(X)$ satisfying either of the following conditions:

(1) The *type* $\mathcal{A}_T$ perturbations:

$$\mathcal{A}_T(X) := \left\{R \in \mathcal{C}(X) : R \text{ is } T\text{-bounded}, \quad \text{and} \quad R(\lambda I - T - R)^{-1} \in \mathcal{PS}, \quad \forall \lambda \in \rho(T+R)\right\},$$

where $\mathcal{PS}$ denotes the class of *polynomially strictly singular operators*, and $\rho(T+R)$ is the resolvent set of $T+R$.

(2) The *type* $\mathcal{B}_T$ perturbations:

$$\mathcal{B}_T(X) := \begin{Bmatrix} R \in \mathcal{B}(X) : R \text{ is } T\text{-bounded}, \\ \exists Q \in \mathbb{C}[X] \setminus \{0\}, \quad Q(-1) \neq 0, \\ \text{such that} \quad \|Q\|_g\big(R(\lambda I - T - R)^{-1}\big) < |Q(-1)|, \quad \forall \lambda \in \rho(T+R) \end{Bmatrix},$$

where $\|Q\|_g(\cdot)$ is a polynomially controlled norm or measure defined on operators.

The class of polynomially non-strictly singular perturbations for T is then defined by the union

$$\mathcal{PNSS}_T(X) := \mathcal{A}_T(X) \cup \mathcal{B}_T(X).$$

Proposition 2.4. [4, Proposition 3.1]
Let $A \in \mathcal{PNSS}_T(X)$, Then $\lambda - A$ is a Fredholm operator of index zero.

3. Stability of essential pseudospectra by means of polynomially non-strict singular perturbations operators

The following theorem provides a practical criterion for the stability of some essential pseudospectra for perturbed linear operators.

Theorem 3.1. *Let $\varepsilon > 0$ and consider $A, B \in \mathcal{C}(X)$. Assume that there are $A_0,\ B_0 \in \mathcal{L}(X)$ and $S_1,\ S_2 \in \mathcal{PNSS}_T(X)$ such that*

$$AA_0 = I - S_1, \tag{3.4}$$

$$BB_0 = I - S_2. \tag{3.5}$$

(i) If $0 \in \Phi_A \cap \Phi_B, A_0 - B_0 \in \mathcal{F}_+(X)$ and $i(A) = i(B)$ then

$$\sigma_{\mathrm{eap},\varepsilon}(A) = \sigma_{\mathrm{eap},\varepsilon}(B). \tag{3.6}$$

(ii) If $0 \in \Phi_A \cap \Phi_B, A_0 - B_0 \in \mathcal{F}_-(X)$ and $i(A) = i(B)$ then

$$\sigma_{e\delta,\varepsilon}(A) = \sigma_{e\delta,\varepsilon}(B). \tag{3.7}$$

(iii) If $A_0 - B_0 \in \mathcal{F}(X)$, then

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(B). \tag{3.8}$$

If, further, $0 \in \Phi_A \cap \Phi_B$ such that $i(A) = i(B)$, then

$$\sigma_{w,\varepsilon}(A) = \sigma_{w,\varepsilon}(B). \tag{3.9}$$

Proof. Let λ be a complex number, Equations (3.4) and (3.5) imply

$$(\lambda - A - D)A_0 - (\lambda - B - D)B_0 = S_1 - S_2 + (\lambda - D)(A_0 - B_0). \tag{3.10}$$

(i) Let $\lambda \notin \sigma_{\text{eap},\varepsilon}(B)$, then $\lambda \in \Phi^{\varepsilon}_{+B}$ such that $i(\lambda - B - D) \leq 0$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since $B + D$ is closed and $\mathcal{D}(B + D) = \mathcal{D}(B)$ endowed with the graph norm is a Banach space denoted by X_{B+D}. We can regard $B + D$ an operator from X_{B+D} into X. This will be denoted by $\widehat{B+D}$. Using Equation (1.1) we can show that

$$\lambda - \widehat{B+D} \in \Phi^b_+(X_B, X) \text{ and } i(\lambda - \widehat{B+D}) \leq 0.$$

Moreover, since $S_2 \in \mathcal{E}_{\mathcal{PS}}(X)$, applying Proposition 2.4, we obtain $I - S_2 \in \Phi(X)$. Applying [[25], Theorem 2.7, p.171] and Equation (3.5), we get $B_0 \in \Phi^b(X, X_B)$. That is $(\lambda - \widehat{B+D})B_0 \in \Phi^b_+(X)$. Remembering that $A_0 - B_0 \in \mathcal{F}_+(X)$ and taking into account Equation (3.10), asserts that $(\lambda - \widehat{A+D})A_0 \in \Phi^b_+(X)$ and

$$i((\lambda - \widehat{A+D})A_0) = i((\lambda - \widehat{B+D})B_0). \tag{3.11}$$

A similar reasoning as before combining Equations (1.1) and (3.4), Proposition 2.4 and [[25], Corollary 1.6, p. 166], [[25], Theorem 2.6, p. 170] shows that $A_0 \in \Phi^b(X, X_A)$ where $X_A := (\mathcal{D}(A), \|.\|_A)$. By [[25], Theorem 1.4, p. 108] one sees that

$$A_0 S = I - F \quad on \ X_A,$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{K}(X_A)$, by Equation (3.5) we have

$$(\lambda - \widehat{B+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F.$$

Combining the fact that $S \in \Phi^b(X_A, X)$ with [[25], Theorem 6.6, p. 129], we show that $(\lambda - \widehat{A+D})A_0 S \in \Phi^b_+(X_A, X)$. Following [[25], Theorem 6.3, p. 128], we derive $(\lambda - \widehat{A+D}) \in \Phi^b_+(X_A, X)$. Thus, Equation (1.1) asserts that

$$(\lambda - A - D) \in \Phi_+(X). \tag{3.12}$$

On the other hand, the assumptions $S_1, S_2 \in \mathcal{PNSS}_T(X)$, Equations (3.4), (3.5) and Proposition 1.1, [[25], Theorem 2.3 , p. 111] reveals that

$$i(A) + i(A_0) = i(I - S_1) = 0 \ \text{ and } \ i(B) + i(B_0) = i(I - S_2) = 0,$$

since $i(A) = i(B)$. That is $i(A_0) = i(B_0)$.

Using Equation (3.11) and [[22], Theorem 2.3, p. 111], we can write

$$i(\lambda - A - D) + i(A_0) = i(\lambda - B - D) + i(B_0).$$

Therefore

$$i(\lambda - A - D) \leq 0, \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{3.13}$$

Using Equations (3.12) and (3.13), we conclude that

$$\lambda \notin \sigma_{eap,\varepsilon}(A).$$

Therefore we prove the inclusion

$$\sigma_{eap,\varepsilon}(A) \subset \sigma_{eap,\varepsilon}(B).$$

The opposite inclusion follows from symmetry and we obtain Equation (3.6).

(ii) The proof of Equation (3.7) may be checked in a similar way to that in (i). It suffices to replace $\sigma_{eap,\varepsilon}(.)$, $\Phi_+(.)$, $i(.) \leq 0$, [[25], Theorem 6.6, p. 129], [[25], Theorem 6.3, p. 128] by $\sigma_{e\delta,\varepsilon}(.)$, $\Phi_-(.)$, $i(.) \geq 0$, [[22], Theorem 5 (i), p. 150], [[25], Theorem 6.7, p. 129] respectively. The details are therefore omitted.

(iii) If $\lambda \notin \sigma_{e,\varepsilon}(B)$, then $\lambda - B - D \in \Phi(X)$. Since B is closed, its domain $\mathcal{D}(B)$ becomes a Banach space X_B for the graph norm $\|.\|_B$. The use of Equation (1.1) leads to $\lambda - \widehat{B+D} \in \Phi^b(X_B, X)$. Moreover, Equation (3.5), Proposition 1.1 and [[25], Theorem 5.13] reveals that $B_0 \in \Phi^b(X, X_B)$ and consequently $(\lambda - \widehat{B+D})B_0 \in \Phi^b(X)$. Following with the assumption, Equation (3.10) and [[25], Theorem 5.13], leads to estimate $(\lambda - \widehat{A+D})A_0 \in \Phi^b(X)$ with

$$i[(\lambda - \widehat{A+D})A_0] = i[(\lambda - \widehat{B+D})B_0]. \tag{3.14}$$

Since $A \in \mathcal{C}(X)$, proceeding as above, Equation (3.4) implies that $A_0 \in \Phi^b(X, X_A)$. By [[25], Theorem 5.4] we can write

$$A_0 S = I - F \quad on\ X_A, \tag{3.15}$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{F}(X_A)$. Taking into account Equation (3.15) we infer that

$$(\lambda - \widehat{A+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F.$$

Therefore, since $S \in \Phi^b(X_A, X)$, the use of [[25], Theorem 6.6] amounts to

$$(\lambda - \widehat{A+D})A_0 S \in \Phi^b(X_A, X).$$

Applying [[25],Theorem 6.3], we prove that $(\lambda - \widehat{A+D}) \in \Phi^b(X_A, X)$ and consequently

$$(\lambda - A - D) \in \Phi(X).$$

Thus $\lambda \notin \sigma_{e,\varepsilon}(A)$. This implies that $\sigma_{e,\varepsilon}(A) \subset \sigma_{e,\varepsilon}(B)$. Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(A)$, we can easily derive the opposite inclusion.

Now, we prove Equation (3.9). If $\lambda \notin \sigma_{w,\varepsilon}(B)$, then, $\lambda \in \Phi_B^{\varepsilon}$ and $i(\lambda - B - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. On the other hand, since $S_1, S_2 \in \mathcal{E}_{\mathcal{PS}}(X)$ and $i(A) = i(B) = 0$, using the Atkinson theorem, we obtain $i(A_0) = i(B_0) = 0$. This together with Equation (3.14) gives $i(\lambda - \widehat{A+D}) = i(\lambda - \widehat{B+D})$. Consequently $i(\lambda - A - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$.. Hence $\lambda \notin \sigma_{w,\varepsilon}(A)$, which proves the inclusion $\sigma_{w,\varepsilon}(A) \subset \sigma_{w,\varepsilon}(B)$. The opposite inclusion follows by symmetry. □

In the following theorems we give some perturbation results of the pseudo left, pseudo right Fredholm and pseudo left, pseudo right Weyl spectra for bounded linear operator in Banach space.

Theorem 3.2. *Let A and B be two operators in $\mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold:*

(i) *Assume that $\lambda - A \in \Phi_l(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_l \in \mathcal{I}nv^F_{\lambda - A - D, l}(X)$ such that $BA_l \in \mathcal{PNSS}_T(X)$, then*

$$\sigma^l_{e,\varepsilon}(A + B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) *Assume that $\lambda - A \in \Phi_r(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_r \in \mathcal{I}nv^F_{\lambda - A - D, r}(X)$ such that $A_r B \in \mathcal{PNSS}_T(X)$, then*

$$\sigma^r_{e,\varepsilon}(A + B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof.

(i) Let $\lambda \notin \sigma^{\text{left}}_{\text{e},\varepsilon}(A)$, $\lambda - A - D \in \Phi^{\varepsilon}_l(X)$. As A_l is a left Fredholm inverse of $\lambda - A - D$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. then by Lemma 1.3 there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$A_l(\lambda - A - D) + K = I.$$

Then, we can write

$$\lambda - A - B - D = (I - BA_l)(\lambda - A - D) - BK. \tag{3.16}$$

Using the fact that $BA_l \in \mathcal{PNSS}_T(X)$ and according to Proposition 2.4, we have $I - BA_l \in \Phi(X)$. Consequently, by Lemma 1.5 we get

$$(I - BA_l)(\lambda - A - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Thus, combining the fact that $BK \in \mathcal{K}(X)$ with the use of Equation 3.16 and Lemma 1.4, we have $\lambda - A - B - D \in \Phi_l(X)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Therefore, $\lambda \notin \sigma^l_{\mathrm{e},\varepsilon}(A + B)$ as required.

(ii) Let $\lambda \notin \sigma^r_{\mathrm{e},\varepsilon}(A)$, then $\lambda - A - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since A_{r} is a right Fredholm inverse of $\lambda - A - D$. From Lemma 1.3 we infer there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$(\lambda - A - D)A_{\mathrm{r}} = I - K\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Then, we can write $\lambda - A - B - D$ with the following form

$$\lambda - A - B - D = (\lambda - A - D)(I - A_{\mathrm{r}}B) - KB,\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon. \tag{3.17}$$

Since $A_{\mathrm{r}}B \in \mathcal{PNSS}_T(X)$ then, according to Proposition 2.4, we have $I - A_{\mathrm{r}}B \in \Phi(X)$. Consequently, by Lemma 1.5, we get

$$(\lambda - A - D)(I - A_{\mathrm{r}}B) \in \Phi_{\mathrm{r}}(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

On the other hand, from Equation 3.17 and Lemma 1.4 and the fact $BK \in \mathcal{K}(X)$ we show that $\lambda - A - B - D \in \Phi_{\mathrm{r}}(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. We deduce that, $\lambda \notin \sigma^r_{\mathrm{e},\varepsilon}(A + B)$.

□

Theorem 3.3. *Let A and B be two operators in $\mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold:*

(i) *Assume that $\lambda - A \in \Phi_l(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_l \in \mathcal{I}nv^F_{\lambda - A - D,l}(X)$ such that $BA_l \in \mathcal{PNSS}_T(X)$, then*

$$\sigma^l_{e,\varepsilon}(A + B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) *Assume that $\lambda - A \in \Phi_r(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_r \in \mathcal{I}nv^F_{\lambda - A - D,r}(X)$ such that $A_rB \in \mathcal{PNSS}_T(X)$, then*

$$\sigma^r_{e,\varepsilon}(A + B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof.

(i) Assume that $\lambda \notin \sigma^l_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_l(X)$ and $i(\lambda - A - D) \le 0$. A similar reasoning as above gives $\lambda - A - B - D \in \Phi_l(X)$ and it suffices to prove that $i(\lambda - A - B - D) \le 0$. Since $BK \in \mathcal{K}(X)$ then, Using Equation 3.16 together with Lemmas 1.4 and 1.5, we obtain that

$$i(\lambda - A - B - D) = i(I - BA_l) + i(\lambda - A - D).$$

Now, Since $BA_l \in \mathcal{PNSS}_T(X)$, we get by Proposition 2.4, that $i(I - BA_l) = 0$. We deduce that

$$i(\lambda - A - B - D) = i(\lambda - A - D) \le 0.$$

Finally, we conclude that $\lambda - A - B - D \in \mathcal{W}_l(X)$, which entails that $\lambda \notin \sigma^{\mathrm{l}}_{\mathrm{w},\varepsilon}(A + B)$.

(ii) with the same reasoning of (i). Let $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_r(X)$ and $i(\lambda - A - D) \geq 0$. Proceeding as the proof above, we establish that $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \geq 0$. Therefore, $\lambda - A - B - D \in \mathcal{W}_r(X)$ and we deduce that $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A + B)$.

□

Remark 3.4. The results of Theorems 3.1, 3.2 and 3.3 is an extension and an improvement of the results of in [1, 2, 3, 5, 6] to a large class of polynomially strict singular operators. ♢

4. Characterization essential spectrum of two linares bounded operators

The aim of this section is to carry out a new criterions allowing to investigate some spectral analysis of sum of two linear bounded operators. We beginning by give the following lemma when we need in the sequel.

Lemma 4.1. [7, Lemma 4.1] *Let $A \in \mathcal{L}(X)$.*
(i) If $C\sigma^l_e(A)$ is connected, then

$$\sigma^l_e(A) = \sigma^l_w(A).$$

(ii) If $C\sigma^r_e(A)$ is connected, then

$$\sigma^r_e(A) = \sigma^r_w(A).$$

Theorem 4.2.
Let A, $B \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:

(i) *Assume that the subsets $C\sigma^l_e(A)$ and $C\sigma^l_e(B)$ are connected, and $-\lambda^{-1}ABQ_l \in \mathcal{PNSS}_T(X)$, $-\lambda^{-1}BAQ_l \in \mathcal{PNSS}_T(X)$, for every $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, then we have:*

$$\left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma^l_{w,\varepsilon}(A+B) \setminus \{0\}.$$

(ii) *Assume that the subsets $C\sigma^r_e(A)$ and $C\sigma^r_e(B)$ are connected, and $-\lambda^{-1}Q_rAB \in \mathcal{PNSS}_T(X)$, $-\lambda^{-1}Q_rBA \in \mathcal{PNSS}_T(X)$, for every $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, then we have:*

$$\left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma^r_{w,\varepsilon}(A+B) \setminus \{0\}.$$

(iii) *Assume that the subsets $C\sigma^l_e(A)$, $C\sigma^l_e(B)$, $C\sigma^r_e(A)$ and $C\sigma^r_e(B)$ are connected, and $-\lambda^{-1}ABQ_l \in \mathcal{PNSS}_T(X)$, $-\lambda^{-1}BAQ_l \in \mathcal{PNSS}_T(X)$, $-\lambda^{-1}Q_rAB \in \mathcal{PNSS}_T(X)$ and $-\lambda^{-1}Q_rBA \in \mathcal{PNSS}_T(X)$, for $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$ and $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, then we have:*

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}. \quad ♢$$

Proof. Firstly we note two equality which is used repeatedly

$$(\lambda - A)(\lambda - B - D) = A(B + D) + \lambda(\lambda - A - B - D). \tag{4.18}$$

$$(\lambda - B - D)(\lambda - A) = (B + D)A + \lambda(\lambda - A - B - D). \tag{4.19}$$

(i) Let $\lambda \notin \sigma^l_{w,\varepsilon}(A+B) \cup \{0\}$ so we have $\lambda - A - B - D \in \Phi_l(X)$ and $i(\lambda - A - B - D) \leq 0$. Then following to the Lemma 1.3 there exist $Q_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$.

So when we use Equation (4.18) we obtain

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= A(B + D) + \lambda(\lambda - A - B - D).\\ &= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D).\\ &= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK.\\ &= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK,\end{aligned}$$

Since $\lambda[\lambda^{-1}ABQ_l + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$ it follows from Proposition 1.1 that $\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$, this implies by the use of Lemma 1.4 that

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABQ_lK \in \Phi_l(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_l(X)$ and as a direct consequence of Lemma 1.6 we obtain

$$\lambda - B - D \in \Phi_l(X), \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{4.20}$$

In the other hand, when we use the Equation (4.19) we have

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= BA[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\ &= [BAQ_l + \lambda I](\lambda - A - B - D) + BAK,\\ &= \lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK.\end{aligned}$$

Since $\lambda[\lambda^{-1}BAQ_l + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$ it follows from Proposition 1.1 that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) \in \Phi_l(X).$$

Obviously, since $BAK \in \mathcal{K}(X)$ and applying Lemma 1.4, we find that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK \in \Phi_l(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore using Lemma 1.6 we obtain

$$\lambda - A \in \Phi_l(X). \tag{4.21}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \le 0.$$

Case1: If $i(\lambda - A) \le 0$

Using Lemma 4.1 the index $i(\lambda - B - D)$ must be negative.Therefore adding this condition to Equations (4.20) and (4.21) we obtain

$$\lambda \notin \left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case2: If $i(\lambda - B - D) \le 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ must be negative.

Then adding this condition to Equations (4.20) and (4.21) we assert

$$\lambda \notin \left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case3: If $i(\lambda - A) > 0$.

Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be positive which contradicts the fact that $i(\lambda - A - B - D) \le 0$.

Case4: If $i(\lambda - B - D) > 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positive which contradicts the fact that $i(\lambda - A - B - D) \le 0$.

(ii) Let $\lambda \notin \sigma^r_{w,\varepsilon}(A+B)\cup\{0\}$ then $\lambda-A-B-D \in \Phi_r(X)$ and $i(\lambda-A-B-D) \leq 0$. So by Lemma 1.3 there exist $Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $(\lambda - A - B - D)Q_r = I - K$ So following to the Equation (4.18) we have

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)[Q_rAB + \lambda I] + ABK,\\ &= \lambda(\lambda - A - B - D)[\lambda^{-1}Q_rAB + I] + ABK.\end{aligned}$$

Since $\lambda[\lambda^{-1}Q_rAB + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ it follows by Proposition 1.1 that

$$\lambda[\lambda^{-1}Q_rAB + I](\lambda - A - B - D) \in \Phi_r(X).$$

Since $ABK \in \mathcal{K}(X)$ then

$$\lambda[\lambda^{-1}Q_rAB + I](\lambda - A - B - D) + ABK \in \Phi_r(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$, following to Lemma 1.6 we infer that

$$\lambda - A \in \Phi_r(X). \tag{4.22}$$

In the other hand, the use of Equation (4.19) assert

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= BA[(\lambda - A - B - D)Q_r + K]BA + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)[Q_rBA + \lambda I] + KBA,\\ &= \lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA.\end{aligned}$$

Since by hypothesis $[\lambda^{-1}Q_rBA + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ we have by Proposition 1.1

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] \in \Phi_r(X).$$

Since $KBA \in \mathcal{K}(X)$ we obtain

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA \in \Phi_r(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$ then the use of Lemma 1.6 infer that

$$\lambda - B - D \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.23}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \geq 0.$$

Case 1: If $i(\lambda - A) \geq 0$

Using Lemma 4.1 the index $i(\lambda-B-D)$ must be positive. Therefore adding this condition to Equations (4.22) and (4.23) we get

$$\lambda \notin \left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case 2: If $i(\lambda - B - D) \geq 0$.

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positive.

Then adding this condition to Equations (4.20) and (4.21) we obtain

$$\lambda \notin \left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case 3: If $i(\lambda - A) < 0$

Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.

Case 4: If $i(\lambda - B - D) < 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.

(iii) Let $\lambda \notin \sigma_{w,\varepsilon}(A+B) \cup \{0\}$ therefore $\lambda - A - B - D \in \Phi(X)$ and $i(\lambda - A - B - D) = 0$ then there exist $Q_l, Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$ and $(\lambda - A - B - D)Q_r = I - K$.

Now, according to items (i) and (ii) we get

$$\big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}. \qquad \square$$

Theorem 4.3.

Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{PNSS}_T(X)$ then*

$$\sigma^l_{e,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma^l_e(A) \cup \sigma^l_{e,\varepsilon}(B)\big] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{PNSS}_T(X)$ then*

$$\sigma^r_{e,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma^r_e(A) \cup \sigma^r_{e,\varepsilon}(B)\big] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{PNSS}_T(X)$ and $-\lambda^{-1}ABQ \in \mathcal{PNSS}_T(X)$ then*

$$\sigma_{e,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big] \setminus \{0\}. \qquad \diamondsuit$$

Proof. (i) Let $\lambda \notin \sigma^l_{e,\varepsilon}(A+B) \cup \{0\}$, then $\lambda - A - B - D \in \Phi_l(X)$. We assume there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, thus, using Equation (4.18) we have

$$\begin{aligned}
(\lambda - A)(\lambda - B - D) &= A(B+D) + \lambda(\lambda - A - B - D),\\
&= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\
&= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\
&= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.
\end{aligned}$$

Obviously, $-\lambda^{-1}ABQ_l \in \mathcal{PNSS}_T(X)$ then by Proposition 2.4 we infer that $\lambda^{-1}ABQ_l + I \in \Phi(X)$. Therefore, by Lemma 1.5 we obtain $[\lambda^{-1}ABQ_l + \lambda I](\lambda - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$ and by applying Lemma 1.4 we obtain

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK \in \Phi_l(X).$$

We conclude that

$$(\lambda - A)(\lambda - B - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon.$$

Hence, by Lemma 1.6 we deduce that

$$(\lambda - B - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.24}$$

On the other hand, using the fact that $AB = BA$ and according to the Equation (4.19) we observe that

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= AB + \lambda(\lambda - A - B - D),\\ &= AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D),\\ &= [ABQ_l + \lambda I](\lambda - A - B - D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore, by Lemma 1.6 we deduce that

$$(\lambda - A) \in \Phi_l(X). \tag{4.25}$$

Finally, the two Equations (4.24) and (4.25) imply that $\lambda \notin \left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \cup \{0\}$. So, we obtain

$$\left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^l(A + B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [7, Theorem 4.3].

(ii) Let $\lambda \notin \sigma_{e,\varepsilon}^r(A + B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$.

We assume there exists $Q_r \in \mathcal{I}nv^F_{\lambda - A - B - D, r}(X)$ thus,

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB.\end{aligned}$$

Evidently, $-\lambda^{-1}Q_rAB \in \mathcal{PNSS}_T(X)$ and by applying Proposition 2.4 we deduce that $\lambda^{-1}Q_rAB + I \in \Phi(X)$. Since, KAB is compact, then by Lemma 1.4 we obtain

$$(\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB \in \Phi_l(X).$$

Consequently, we have $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$ and by Lemma 1.6 we infer that

$$(\lambda - A) \in \Phi_r(X). \tag{4.26}$$

Further, we have $AB = BA$ so,

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_rAB + I] + KAB.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$. Then, by Lemma 1.6 we deduce that

$$(\lambda - B - D) \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.27}$$

Finally, the two Equations (4.26) and (4.27) imply that

$$\lambda \notin \left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \cup \{0\}.$$

So, we obtain

$$\left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^r(A + B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [7, Theorem 4.3].

(iii) Let $\lambda \notin \sigma_{e,\varepsilon}(A+B) \cup \{0\}$. Then $\lambda - A - B - D \in \Phi(X)$ means that $\lambda - A - B - D \in \Phi_l(X) \cap \Phi_r(X)$.

Now, by the hypothesis there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, and by applying the results in statements (i) and (ii) we infer that $(\lambda - A - B - D) \in \Phi_r(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$, therefore $(\lambda - A - B - D) \in \Phi(X)$.

Also, using the hypothesis that $-\lambda^{-1}QAB \in \mathcal{PNSS}_T(X)$, $-\lambda^{-1}ABQ \in \mathcal{PNSS}_T(X)$ and $AB = BA$ we give us this two condition:

$$(\lambda - A)(\lambda - B - D) \in \Phi(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \Phi(X).$$

Therefore, following Theorem 1.2 we obtain $(\lambda - A) \in \Phi(X)$ and $(\lambda - B - D) \in \Phi(X)$ means that $\lambda \notin \big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big] \cup \{0\}$. Then we get the following inclusion

$$\big[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\big] \setminus \{0\} \subseteq \sigma_{e,\varepsilon}(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [7, Theorem 4.3]. □

The same reasoning of the above theorem, we allow to obtain the result of the following result.

Theorem 4.4. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{PNSS}_T(X)$ then*

$$\sigma^l_{w,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma^l_{w,\varepsilon}(A) \cup \sigma^l_{w,\varepsilon}(B)\big] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{PNSS}_T(X)$ then*

$$\sigma^r_{w,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma^r_{w}(A) \cup \sigma^r_{w,\varepsilon}(B)\big] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{PNSS}_T(X)$ and $-\lambda^{-1}ABQ \in \mathcal{PNSS}_T(X)$ then*

$$\sigma_{w,\varepsilon}(A+B) \setminus \{0\} = \big[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\big] \setminus \{0\}. \qquad \diamondsuit$$

5. Application to bounded 2 × 2 block operator matrices forms

The objective of this section is to utilize Theorem 4.3 from Section 4 in order to analyze the pseudo left (right)-Fredholm essential spectra of the given operator matrix.

Let X_1 and X_2 be two Banach spaces and consider the 2×2 block operator matrices defined on $X_1 \times X_2$ by:

$$\mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$, $C \in \mathcal{L}(X_2, X_1)$ and $D \in \mathcal{L}(X_1, X_2)$.

Next, we define the following matrix:

$$\mathfrak{D} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where $D_1 \in \mathcal{L}(X_1)$, $D_2 \in \mathcal{L}(X_2)$ and $\|\mathfrak{D}\| = \max\{\|D_1\|, \|D_2\|\}$.

In the following theorem, we seek the pseudo left (right)-Fredholm essential spectra of Matrix $\mathcal{M}_C$.

Theorem 5.1. *Let the 2×2 block operator matrix $\mathcal{M}_C$ and $\varepsilon > 0$. In all that follows we will make the following assumptions:*

$$\mathcal{H}:\begin{cases}\|\mathfrak{D}\|<\varepsilon,\\ AC=CB,\\ A\in\Phi(X),B\in\Phi(X),\\ CB\in\mathcal{S}(X_1\times X_2).\end{cases}$$

Then, we have that

(*i*) $\sigma_{e,\varepsilon}^{left}(\mathcal{M}_C)\backslash\{0\}\subseteq\left[\sigma_{e,\varepsilon}^{left}(A)\cup\sigma_{e,\varepsilon}^{left}(B)\right]\backslash\{0\}.$

(*ii*) $\sigma_{e,\varepsilon}^{right}(\mathcal{M}_C)\backslash\{0\}\subseteq\left[\sigma_{e,\varepsilon}^{right}(A)\cup\sigma_{e,\varepsilon}^{right}(B)\right]\backslash\{0\}.$

Proof. We begin by presenting the polynomial P in the specified format:

$$\begin{array}{rccc}P&:&\mathbb{R}^2&\to&\mathbb{R}\\ &&(x,y)&\mapsto&P(x,y)=x.y\end{array}$$

We can write

$$\begin{aligned}\mathcal{M}:=\begin{pmatrix}A&C\\0&B\end{pmatrix}&=\begin{pmatrix}0&C\\0&0\end{pmatrix}+\begin{pmatrix}A&0\\0&B\end{pmatrix}\\ &=\quad\mathcal{M}_{\mathcal{C}}\quad+\quad\mathcal{M}_{A,B}.\end{aligned}$$

We have:

$$P(\mathcal{M}_{\mathcal{C}},\mathcal{M}_{A,B})=\mathcal{M}_{\mathcal{C}}.\mathcal{M}_{A,B}=\begin{pmatrix}0&CB\\0&0\end{pmatrix}.$$

it follows from the hypothesis (H) that:

$$P(\mathcal{M}_{\mathcal{C}},\mathcal{M}_{A,B})\in\mathcal{S}(X_1\times X_2),\text{and }\mathcal{M}_{\mathcal{C}}.\mathcal{M}_{A,B}\in\mathcal{PNSS}_T(X).$$

Moreover we have $A+B\in\Phi(X)$ then there exist $A_0\in\mathcal{L}(X)$ and $K\in\mathcal{K}(X)$ such that $A_0(A+B)=I-K$. Then

$$A_0(A+B+D)=I-K',\text{with } K'\in\mathcal{K}(X).$$

Using Theorem 4.3, we obtain that

(*i*) $\sigma_{e,\varepsilon}^{left}(\mathcal{M})\backslash\{0\}=\sigma_{e,\varepsilon}^{left}(\mathcal{M}_{\mathcal{C}}+\mathcal{M}_{A,B})\backslash\{0\}=\left[\sigma_{e}^{left}(\mathcal{M}_{\mathcal{C}})\cup\sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right]\backslash\{0\}.$

(*ii*) $\sigma_{e,\varepsilon}^{right}(\mathcal{M})\backslash\{0\}=\sigma_{e,\varepsilon}^{right}(\mathcal{M}_{\mathcal{C}}+\mathcal{M}_{A,B})\backslash\{0\}=\left[\sigma_{e}^{right}(\mathcal{M}_{\mathcal{C}})\cup\sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right]\backslash\{0\}.$

Furthermore, we can readily demonstrate $\sigma_{e}^{left}(\mathcal{M}_{\mathcal{C}})=\sigma_{e}^{right}(\mathcal{M}_{\mathcal{C}})=\{0\}$. Consequently, applying [[2], Theorem 4 (i)], we show that

$$\begin{aligned}\sigma_{e,\varepsilon}^{left}(\mathcal{M})\backslash\{0\}&=\left[\sigma_{e}^{left}(\mathcal{M}_{\mathcal{C}})\cup\sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right]\backslash\{0\}\\ &=\left[\{0\}\cup\sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right]\backslash\{0\}\\ &=\sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\\ &\subseteq\left[\sigma_{e,\varepsilon}^{left}(A)\cup\sigma_{e,\varepsilon}^{left}(B)\right]\backslash\{0\},\end{aligned}$$

and

$$\begin{aligned}\sigma_{e,\varepsilon}^{right}(\mathcal{M})\backslash\{0\}&=\left[\sigma_{e}^{right}(\mathcal{M}_{\mathcal{C}})\cup\sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right]\backslash\{0\}\\ &=\left[\{0\}\cup\sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right]\backslash\{0\}\\ &=\sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\end{aligned}$$

$$\subseteq \left[\sigma_{e,\varepsilon}^{right}(A) \cup \sigma_{e,\varepsilon}^{right}(B)\right] \backslash \{0\}. \qquad \square$$

Conclusion

In this article, we have extended the study of essential pseudospectra by introducing and analyzing polynomially strict singular operators, which generalize classical strict singular operators. We established new stability results for the essential pseudospectra of closed linear operators under perturbations by this broader class of operators in Banach spaces. Our investigation also detailed how these perturbations affect the left and right Weyl as well as Fredholm essential pseudospectra. Additionally, we characterized the essential pseudospectra of the sum of two bounded linear operators and applied these findings to the pseudo-left (right)-Fredholm spectra of 2×2 block operator matrices.

This work not only broadens the understanding of essential pseudospectra in operator theory but also opens several avenues for further research. For instance, how might these polynomially strict singular perturbations influence pseudospectral properties in other classes of operators or different functional settings? Can these results be extended or refined in the context of unbounded operators or non-Banach space? Moreover, what potential applications could arise in applied fields such as quantum mechanics, control theory, or numerical analysis from these generalized pseudospectral insights?

Compliance with ethical standards

Data Availability Statement No data were produced for this paper.

Conflit of interest: No potential conflit of interest was reported by the authors.

References

[1] F. Abdmouleh, A. Ammar and A. Jeribi, *A characterization of the pseudo-Browder essential spectra of linear operators and application to a transport equation*, J. Comput. Theor. Transp.44 (3) (2015),141–153, https://doi.org/10.1080/23324309.2015.1033222.

[2] F. Abdmouleh and B. Elgabeur, *Pseudo essential spectra in Banach space and application to operator matrices*, Acta Appl. Math. 178 (2022), 1–17. https://doi.org/10.1007/s10440-022-00527-5.

[3] F. Abdmouleh and B. Elgabeur, *On the pseudo semi-Browder essential spectra and application to* $2x2$ *block operator matrices*, Filomat. 37(19) (2023), 6373–6386. https://doi.org/10.2298/FIL2319373A.

[4] F. Abdmouleh and I. Walha, *Characterization and stability of the essential spectrum based on measures of polynomially non-strict singularity operators. Indagationes Mathematicae.* 26(3) (2015), 455–467 https://doi.org/10.1016/j.indag.2015.01.005.

[5] A. Ammar, B. Boukettaya and A. Jeribi, *A note on the essential pseudospectra and application*, Linear Multilinear Algebra 64(8) (2016), 1474–1483. https://doi.org/10.1080/03081087.2015.1091436.

[6] A. Ammar, A. Jeribi and K. Mahfoudhi, *Browder essential approximate pseudospectrum and defect pseudospectrum on a Banach space*, Extracta Math. 34(1) (2019), 29–40. https://doi.org/10.17398/2605-5686.34.1.29.

[7] S. Charfi, A. Elleuch and I. Walha, *Spectral theory involving the concept of quasi- compact perturbations*, Mediterr. J. Math. 17 (2020), Article 32. https://doi.org/10.1007/s00009-019-1468-x.

[8] E. B. Davies, *Spectral Theory and Differential Operators*, Cambridge University Press, Cambridge, vol.42 of Cambridge Studies in Advanced Mathematics, 1995.https://doi.org/10.1017/CBO9780511623721.

[9] A. Dehici and N. Boussetila, *Properties of polynomially Riesz operators on some Banach spaces*, Lobachevskii J. Math. 32(1) (2011), 39–47. https://doi.org/10.1134/S1995080211010021.

[10] B. Elgabeur, *A characterization of essential pseudospectra involving polynomially compact operators* Filomat. 38 (2024) 11675-11691. https://doi.org/10.2298/FIL2433675E

[11] B. Elgabeur, *A New Approach of Essential Pseudo Spectrum in Banach Space and Application to Transport Equation* Bol. Soc. Paran. Mat. 43 (2025) 1-14. DOI:10.5269/bspm.76015

[12] I. C. Gohberg, A. Markus and I. A. Feldman, *Normally solvable operators and ideals associated with them.* Amer. Math. Soc. Transl., 2 (61), 63–84, Amer. Math. Soc., Providence, (1967).

[13] S. Goldberg, *Unbounded linear operators: Theory and Applications.* McGraw-Hill New-York, (1966).

[14] M. González and M.O. Onieva, *On Atkinson operators in locally convex spaces.* Math. Z. 190 (1985), 505–517. https://doi.org/10.1007/BF01214750.

[15] K. Gustafson and J. Weidmann, *On the essential spectrum.* J. Math. Anal. Appl. 6 (25) (1969) 121–127.DOI : 10.1016/0022-247X(69)90217-0.

[16] D. Hinrichsen and A. J. Pritchard, *Robust stability of linear evolution operators on Banach spaces.* SIAM J. Control Optim. 32(6) (1994), 1503–1541. https://doi.org/10.1137/S0363012992230404.

[17] A. Jeribi, N. Moalla and S. Yengui, *Some results on perturbation theory of matrix operators, M-essential spectra and application to an example of transport operators.* J. Math. Appl, 44 (2021). DOI: 10.7862/rf.2021.3.

[18] T. Kato, *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, 1966. https://doi.org/10.1007/978-3-642-66282-9.

[19] H. J. Landau, *On Szego's eigenvalue distribution theorem and non-Hermitian kernels* . J. Anal. Math. 28 (1975), 335–357. https://doi.org/10.1007/BF02786820.

[20] K. Latrach and A. Dehici, *Fredholm, semi-Fredholm perturbations, and es- sential spectra.* J. Math. Anal. Appl. 259 (2001), no. 1, 277301.

[21] K. Latrach, J. M. Paoli and M. A. Taoudi, *A characterization of polynomially Riesz strongly continuous semigroups.* Comment. Math. Univ. Carolin. 47 (2006), no. 2, 275289.

[22] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras.* Operator Theory: Advances and Applications, vol.139, Birkhäuser, Basel, 2003. https://doi.org/10.1007/978-3-0348-7788-6

[23] V. Rakocevic, *Approximate point spectrum and commuting compact perturbations.* Glasg. Math. J. 28(2) (1986), 193–198. https://doi.org/10.1017/S0017089500006509.

[24] Schmoeger M. *The spectral mapping theorem for the essential approximate point spectrum.* Colloq. Math, (1967);74:167–176. DOI : 10.4064/cm-74-2.

[25] M. Schechter, *Principles of Functional Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 36, American Mathematical Society, Providence, RI, 2002. https://doi.org/10.1090/gsm/036.

[26] L. N. Trefethen, *Pseudospectra of linear operators*, SIAM Rev. 39(3) (1997), 383–406. https://doi.org/10.1137/S0036144595295284.

[27] J. M. Varah, *The computation of bounds for the invariant subspaces of a general matrix operator.* Ph.D. Thesis, Computer Science Department, Stanford University, 1967. https://dl.acm.org/doi/abs/10.5555/905073.

[28] S. C. Zivkovic-Zlatanovic, D. S. Djordjevic, R. E. Harte, *Left-right Browder and left-right Fredholm operators.* Integral Equations Operator Theory, (2011) 69, 347–363. DOI: 10.1007/s00020-010-1839-y.

[29] S. C. Zivkovic-Zlatanovic, D. S. Djordjevic, R. E. Harte, *On left and right Browder operators.* Journal of the Korean Mathematical Society, (2011), 48(5),1053–1063. DOI: 10.4134/JKMS.2011.48.5.1053.

Bilel Elgabeur: bilelelgabeur@gmail.com
University of Aix-Marseille, France

Received 16/04/2024; Revised 01/10/2025