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INERTIAL KM-TYPE ALGORITHM FOR SOLVING SPLIT MONOTONE VARIATIONAL INCLUSION PROBLEM AND HIERARCHICAL FIXED POINT PROBLEM

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Abstract. The primary objective of this paper is to present and investigate an inertial Krasnoselski-Mann (KM) type iterative method for approximating a common solution to a split monotone variational inclusion problem and a hierarchical fixed point problem for a finite family of l-strictly pseudocontractive non-self mappings. Additionally, we demonstrate that the iterative sequences provided by the proposed method converge weakly to a common solution to these problems. The methodology and conclusions described in this work extend and unify previously published findings in this domain. Finally, a numerical example is presented to demonstrate the suggested iterative method's convergence analysis of the sequences obtained. We also carried out a justification how the inertial term is useful.

1. Introduction

Let Ξ_1 and Ξ_2 be two real Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $\mathcal{C} \subset \Xi_1$ and $\mathcal{D} \subset \Xi_2$ be two nonempty, closed and convex sets. A non-self mapping $\mathcal{A} : \mathcal{C} \to \Xi_1$ is referred as l-strictly pseudocontractive if there exists a constant $l \in [0,1)$ in such a way that

$$\|\mathcal{A}p - \mathcal{A}q\|^2 \leq \|p - q\|^2 + l\|(\mathcal{I} - \mathcal{A})p - (\mathcal{I} - \mathcal{A})q\|^2, \quad \forall p, q \in \mathcal{C}.$$

$\mathcal{A}$ is nonexpansive nonself-mapping, if $l = 0$.
In a fixed point problem (FPP) one needs to find an element $p \in \mathcal{C}$ in such a way that

$$\mathcal{A}p = p, \tag{1.1}$$

where $\mathcal{A} : \mathcal{C} \to \Xi_1$ is a mapping. We represent the solution set of FPP (1.1) by $\mathrm{F}(\mathcal{A}) = \{p \in \mathcal{C} : \mathcal{A}p = p\}$.
We consider the hierarchical fixed point problem (HFPP) as follows: find $q^* \in \bigcap_{i=1}^N \mathrm{F}(\mathcal{A}_i)$ in such a way that

$$\langle q^* - p, q^* - \mathcal{S}q^* \rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^N \mathrm{F}(\mathcal{A}_i), \tag{1.2}$$

where $\{\mathcal{A}_i\}_{i=1}^N : \mathcal{C} \to \Xi_1$ is a finite family of l-strictly pseudocontractive mappings such that $\bigcap_{i=1}^N \mathrm{F}(\mathcal{A}_i) \neq \phi$ and $\mathcal{S} : \mathcal{C} \to \mathcal{C}$ is a nonexpansive mapping. We represent the solution set of HFPP (1.2) by Φ.
If we take $\mathcal{A}_i = \mathcal{A}$, then HFPP (1.2) reduces to the following HFPP: Find $q^* \in \mathrm{F}(\mathcal{A})$ in such a way that

$$\langle q^* - p, q^* - \mathcal{S}q^* \rangle \leq 0, \quad \forall p \in \mathrm{F}(\mathcal{A}), \tag{1.3}$$

for a nonexpansive mapping $\mathcal{A}$ with regard to another nonexpansive mapping $\mathcal{S}$, which was proposed in 2006 by Moudafi and Mainge [12].

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It can be easily seen that solving HFPP (1.3) is similar to solve the following FPP: Find $q^* \in \mathcal{C}$ in such a way that

$$q^* = (\mathcal{P}_{\mathrm{F}(\mathcal{A})} \circ \mathcal{S})q^*,$$

where $\mathcal{P}_{\mathrm{F}(\mathcal{A})}$ stands for the metric projection of Ξ_1 onto $\mathrm{F}(\mathcal{A})$.
Assume $\Phi_1 := \{q^* \in \mathcal{C} : (\mathcal{P}_{\mathrm{F}(\mathcal{A})} \circ \mathcal{S})q^* = q^*\}$ indicates the solution set of HFPP (1.3). It is obvious that the HFPP (1.3) is similar to the following variational inclusion

$$0 \in N_{\mathrm{F}(\mathcal{A})}q^* + (\mathcal{I} - \mathcal{S})q^*,$$

where $N_{\mathrm{F}(\mathcal{A})}$ represents the normal cone to $\mathrm{F}(\mathcal{A})$, which is described as follows:

$$N_{\mathrm{F}(\mathcal{A})} = \begin{cases} \{y \in \Xi_1 : \langle q - p, y\rangle \leq 0, \quad \forall q \in \mathrm{F}(\mathcal{A})\}, & \text{if } p \in \mathrm{F}(\mathcal{A}), \\ \phi, & \textit{else}. \end{cases}$$

If $\mathcal{S}$ becomes $\mathcal{I}$, the solution set of HFPP (1.3) becomes simply $\mathrm{F}(\mathcal{A})$. The HFPP (1.3) covers hierarchical minimization problems, minimization problems over equilibrium constraints, monotone variational inequality on fixed point sets etc. If we set $\mathcal{S} = \mathcal{I} - \gamma_1 h_1$, where l-strongly monotone and h_1 is η-Lipschitzian continuous with $\gamma_1 \in (0, 2l/\eta^2)$, then the HFPP (1.3) is converted to the standard variational inequality problem over $\mathrm{F}(\mathcal{A})$, known as hierarchical variational inequality problem (HVIP): Find $q^* \in \mathrm{F}(\mathcal{A})$ in such a way that

$$\langle h_1(q^*), p - q^*\rangle \geq 0, \quad \forall p \in \mathrm{F}(\mathcal{A}),$$

which was studied by Yamada and Ogura [23]. Moudafi [18] proposed the Krasnoselski-Mann type algorithm to solve the HFPP (1.3) as follows: For a given $x_0 \in \mathcal{C}$, the sequence $\{x_m\}$ generated by

$$x_{m+1} = (1 - \alpha_m)x_m + \alpha_m(\tau_m \mathcal{S}x_m + (1 - \tau_m)\mathcal{A}x_m), \tag{1.4}$$

where $\{\alpha_m\}$ and $\{\tau_m\}$ are two sequences in (0,1). The Krasnoselski-Mann type algorithm can be used to represent some signal processing and image reconstruction techniques, and the fundamental property of its underlying convergence theorems provides a standard framework for examining numerous specific algorithms (see [19, 21]).
On the other hand, Moudafi [1] proposed the split monotone variational inclusion problem (SMVIP) as follows: Find $q^* \in \Xi_1$ in such a way that

$$0 \in h_1(q^*) + M(q^*), \tag{1.5}$$

and $v^* = \mathcal{B}q^* \in \Xi_2$ solves

$$0 \in h_2(v^*) + N(v^*), \tag{1.6}$$

where $M : \Xi_1 \to 2^{\Xi_1}$ and $N : \Xi_2 \to 2^{\Xi_2}$ are multi-valued maximal monotone mappings, $h_1 : \Xi_1 \to \Xi_1$ and $h_2 : \Xi_2 \to \Xi_2$ are two mappings and $\mathcal{B} : \Xi_1 \to \Xi_2$ is a bounded linear operator. The solution set of SMVIP (1.5)-(1.6) is indicated by $\Omega = \{q^* \in \Xi_1 : q^* \in$ (MVIP (1.5)) and $\mathcal{B}q^* \in$ (MVIP (1.6))$\}$.
Moudafi [1] proposed an iterative method and studied the weak convergence theorem for SMVIP (1.5)-(1.6) as follows: For a given $x_0 \in \Xi_1$, compute iterative sequence $\{x_m\}$ generated by

$$x_{m+1} = U(x_m + \gamma_1 \mathcal{B}^*(W - \mathcal{I})\mathcal{B}x_m), \quad \text{for } \lambda_1 > 0,$$

where $U = J^M_{\lambda_1}(\mathcal{I} - \lambda_1 h_1)$, $W = J^N_{\lambda_1}(\mathcal{I} - \lambda_1 h_2)$ and $\gamma_1 \in (0, \frac{1}{\mathcal{Q}})$ with $\mathcal{Q}$ being the spectral radius of the operator $\mathcal{B}^*\mathcal{B}$.
The split feasibility problem, split zero problem, split fixed point problem, split variational inequality (see [1, 6]) are special cases of SMVIP (1.5)-(1.6). They've been studied by a variety of researchers and are used to tackle real-world problems like inverse problem modeling, sensor systems in computerized tomography, data compression, and radiation therapy; for detail, (see [5, 8]).

If we set $h_1 \equiv 0, h_2 \equiv 0$, then the SMVIP (1.5)-(1.6) becomes the following split variational inclusion problem (SVIP): Find $q^* \in \Xi_1$

$$0 \in M(q^*), \tag{1.7}$$

and $v^* = \mathcal{B}q^* \in \Xi_2$ solves

$$0 \in N(v^*). \tag{1.8}$$

In 2012, Byrne et al. [4] proposed an iterative algorithm and provided the weak and strong convergence theorems for solving SVIP (1.7)-(1.8) as follows: For given $x_0 \in \Xi_1$, compute iterative sequence $\{x_m\}$ generated by

$$x_{m+1} = J^M_{\lambda_1}(x_m + \gamma_1 \mathcal{B}^*(J^N_{\lambda_1} - \mathcal{I})\mathcal{B}x_m), \qquad \lambda_1 > 0.$$

Kazmi et al. [21] proposed an iterative method as follows: For given $x_0 \in \mathcal{C}$, compute iterative sequence $\{x_m\}$ generated by

$$\begin{aligned} y_m &= (1-\alpha_m)x_m + \alpha_m(\tau_m \mathcal{S}x_m + (1-\tau_m)\mathcal{A}x_m), \\ x_{m+1} &= U(y_m + \gamma_1 \mathcal{B}^*(W - \mathcal{I})\mathcal{B}y_m), \quad \forall m \geq 0, \end{aligned} \tag{1.9}$$

where $U = J^M_{\lambda_1}(\mathcal{I} - \lambda_1 h_1)$, $W = J^N_{\lambda_1}(\mathcal{I} - \lambda_1 h_2)$ and $\mathcal{S}, \mathcal{A}$ are nonexpansive mappings on $\mathcal{C}$ and step size $\gamma_1 \in (0, \frac{1}{\mathcal{Q}})$, $\mathcal{Q}$ is the spectral radius of the operator $\mathcal{B}^*\mathcal{B}$ and $\mathcal{B}^*$ is the adjoint of the bounded linear operator $\mathcal{B}$. Under some certain conditions, the sequence $\{x_m\}$ generated by (1.9) converges weakly to the common solution of HFPP (1.3) and SMVIP (1.5)-(1.6).

In general, the Krasnoselski-Mann type iterative approach has a slow convergence rate. The term $\vartheta_m(x_m - x_{m+1})$, also known as the inertial extrapolation term, was presented in particular as a valuable tool for speeding up the convergence rate of iterative methods, and many authors have researched and improved the inertial type algorithm in many ways (see [22, 24, 26, 27, 28, 29]).

Motivated and inspired by the work of Moudafi [18] and Kazmi et al. ([21, 25]) we propose and analyze an inertial Krasnoleski-Mann type iterative method with the help of averaged mappings for finding a common solution of SMVIP (1.5)-(1.6) and HFPP (1.2) for a finite family of l-strictly pseudocontractive nonself-mappings in the setting of real Hilbert space. Furthermore, we establish that the sequences developed by our proposed iterative technique converge weakly to a common solution to these problems. The iterative approach and results mentioned in this article are original and can be regarded as a generalisation and refinement of previously published work in this field.

2. Preliminaries

In this section, we need to review some basic definitions and lemmas that are required to prove our main convergence result.

A mapping $\mathcal{A} : \Xi_1 \to \Xi_1$ is said to be

(i) monotone if

$$\langle \mathcal{A}p - \mathcal{A}q, p - q\rangle \geq 0, \ \ \forall\, p, q \in \Xi_1.$$

(ii) α-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle \mathcal{A}p - \mathcal{A}q, p - q\rangle \geq \alpha\|\mathcal{A}p - \mathcal{A}q\|^2, \ \ \forall\, p, q \in \Xi_1.$$

(iii) nonexpansive if

$$\|\mathcal{A}p - \mathcal{A}q\| \leq \|p - q\|, \quad \forall\, p, q \in \Xi_1.$$

(iv) firmly nonexpansive if

$$\langle \mathcal{A}p - \mathcal{A}q, p - q\rangle \geq \|\mathcal{A}p - \mathcal{A}q\|^2, \ \ \forall\, p, q \in \Xi_1.$$

The metric projection $\mathcal{P}_{\mathcal{C}} : \Xi_1 \to \mathcal{C}$ is defined, for every point $p \in \Xi_1$, as the unique nearest point in $\mathcal{C}$ denoted by $\mathcal{P}_{\mathcal{C}}(p)$ in such a way that

$$\|p - \mathcal{P}_{\mathcal{C}}(p)\| \le \|p - q\|, \quad \forall q \in \mathcal{C}.$$

$\mathcal{P}_{\mathcal{C}}$ is well known to be nonexpansive, and it is firmly nonexpansive. Furthermore, $\mathcal{P}_{\mathcal{C}}$ is characterized by the following property:

$$\langle p - \mathcal{P}_{\mathcal{C}}(p), q - \mathcal{P}_{\mathcal{C}}(p)\rangle \le 0, \quad \forall p \in \Xi_1,\ q \in \mathcal{C}. \tag{2.10}$$

A multi-valued mapping $M : \Xi_1 \to 2^{\Xi_1}$ is called monotone if for all $p_1, p_2 \in \Xi_1$, $q_1 \in Mp_1$ and $q_2 \in Mp_2$ such that

$$\langle p_1 - p_2, q_1 - q_2\rangle \ge 0.$$

A monotone mapping M is maximal if $G(M)$, the graph of M defined as

$$G(M) = \{(p_1, q_1) : q_1 \in Mp_1\},$$

is not adequately included in the graph of any other monotone mapping.
Remark: It is commonly known that a monotone mapping M is maximal iff for $(p_1, q_1) \in \Xi_1 \times \Xi_1, \langle p_1 - p_2, q_1 - q_2\rangle \ge 0$ for each $(p_2, q_2) \in G(M)$ implies that $q_1 \in Mp_1$.
Let $M : \Xi_1 \to 2^{\Xi_1}$ be a multi-valued maximal monotone mapping. Then the resolvent operator $J^M_{\lambda_1} : \Xi_1 \to \Xi_1$ is defined by

$$J^M_{\lambda_1}(p_1) = (\mathcal{I} + \lambda_1 M)^{-1}(p_1), \quad \forall p_1 \in \Xi_1.$$

for some $\lambda_1 > 0$, where $\mathcal{I}$ denotes the identity operator on Ξ_1. We notice that for all $\lambda_1 > 0$ the resolvent operator $J^M_{\lambda_1}$ is single-valued, nonexpansive and firmly nonexpansive.

Definition 2.1. [12] *A sequence $\{M_m\}$ of maximal monotone mappings defined on Ξ_1 is said to be graph convergent to M if $\{\text{graph}(M_m)\}$ converges to graph(M) in the Kuratowski-Painleve sense, i.e.,*

$$\limsup_{m\to\infty} \text{graph}(M_m) \subset \text{graph}(M) \subset \liminf_{m\to\infty} \text{graph}(M_m).$$

Definition 2.2. [9] *A mapping $\mathcal{A} : \Xi_1 \to \Xi_1$ is said to be averaged mapping if there exists some number $\alpha \in (0\ 1)$ such that $\mathcal{A} = (1-\alpha)\mathcal{I} + \alpha\mathcal{S}$, where $\mathcal{I} : \Xi_1 \to \Xi_1$ is the identity mapping and $\mathcal{S} : \Xi_1 \to \Xi_1$ is a nonexpansive mapping. An averaged mapping is also a nonexpansive mapping and F($\mathcal{S}$)=F($\mathcal{A}$).*

Lemma 2.3. [13] Assume that $\mathcal{S}$ is a l-strictly pseudocontractive mapping on a Hilbert space Ξ_1. Define a mapping $\mathcal{A}$ by $\mathcal{A}p = \alpha p + (1-\alpha)\mathcal{S}p$ for all $p \in \Xi_1$, where $\alpha \in [l, 1)$. Then $\mathcal{A}$ is nonexpansive mapping with F($\mathcal{A}$)=F($\mathcal{S}$).

Lemma 2.4. [10] If the mapping $\{\mathcal{A}_i\}_{i=1}^N$ are averaged and have a common fixed point, then

$$\cap_{i=1}^n \mathrm{F}(\mathcal{A}_i) = \mathrm{F}(\mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_N).$$

In particular, for $N = 2$, $\mathrm{F}(\mathcal{A}_1) \cap \mathrm{F}(\mathcal{A}_2) = \mathrm{F}(\mathcal{A}_1\mathcal{A}_2) = \mathrm{F}(\mathcal{A}_2\mathcal{A}_1)$.

Lemma 2.5. [13] Let $\mathcal{A} : \mathcal{C} \to \Xi_1$ be a l-strictly pseudocontractive mapping with $\mathrm{F}(\mathcal{A}) \ne \phi$. Then $\mathrm{F}(\mathcal{P}_{\mathcal{C}}\mathcal{A}) = \mathrm{F}(\mathcal{A})$.

Lemma 2.6 (Demiclosedness Principle). [11] Assume that $\mathcal{A}$ is nonexpansive self mapping of a closed convex subset $\mathcal{C}$ of a Hilbert space Ξ_1. If $\mathcal{A}$ has a fixed point, then $\mathcal{I} - \mathcal{A}$ is demiclosed, i.e., whenever $\{x_m\}$ is a sequence in $\mathcal{C}$ weakly converge to some $p \in \mathcal{C}$ and $\{(\mathcal{I} - \mathcal{A})x_m\}$ converges strongly to some $p \in \mathcal{C}$, then $(\mathcal{I} - \mathcal{A})p = p$.

Lemma 2.7. [15] Let $\{\epsilon_m\}, \{\eta_m\}$ and $\{\alpha_m\}$ be sequences in $[0, \infty)$ satisfying $\epsilon_{m+1} \le \epsilon_m + \alpha_m(\epsilon_m - \epsilon_{m-1}) + \eta_m$ for all $m \ge 1$ provided $\sum_{m=1}^{\infty} \eta_m < +\infty$ and with $0 \le \alpha_m \le \alpha < 1$ for all $m \ge 1$. Then the following hold:

(i) $\sum_{m\geq 1}[\epsilon_m - \epsilon_{m-1}]_+ < +\infty$, where $[s]_+ = \max\{s, 0\}$;

(ii) There exists $\epsilon^* \in [0, +\infty)$ such that $\lim_{m\to\infty} \epsilon_m = \epsilon^*$.

Lemma 2.8. [17] We have the following assertions:

(a) Let M be a maximal monotone mapping on Ξ_1. Then $\{t_m^{-1}M\}$ is graph convergent to $N_{M^{-1}0}$ as $t_m \to 0$ provided that $M^{-1}0 \neq \phi$.

(b) Let $\{M_m\}$ be a sequence of maximal monotone mappings on Ξ_1 which is graph convergent to a mapping M defined on Ξ_1. If B is a Lipschitz maximal monotone mapping on Ξ_1, then $\{B + M_m\}$ is graph convergent to $B + M$ and $B + M$ is maximal monotone.

3. Main result

In this section, the proposed iterative approach to approximate a common solution of HFPP (1.2) and SMVIP (1.5)-(1.6) of a finite family of l-strictly pseudocontractive nonself-mappings in real Hilbert space is used to prove a weak convergence theorem.

Theorem 3.1. *Let Ξ_1 and Ξ_2 be two real Hilbert spaces and let $\mathcal{C}$ and $\mathcal{D}$ be nonempty, closed and convex subsets of Ξ_1 and Ξ_2, respectively. Let $\mathcal{B} : \Xi_1 \to \Xi_2$ be a bounded linear operator with its adjoint operator $\mathcal{B}^*$. Assume that $M : \Xi_1 \to 2^{\Xi_1}$ and $N : \Xi_2 \to 2^{\Xi_2}$ be two multivalued maximal monotone mappings. Let $h_1 : \mathcal{C} \to \Xi_1$ and $h_2 : \mathcal{D} \to \Xi_2$ be η_1- and η_2-inverse strongly monotone mappings, respectively. Let $\mathcal{S} : \Xi_1 \to \Xi_1$ be a nonexpansive self-mappings and $\{\mathcal{A}_i\}_{i=1}^N : \mathcal{C} \to \Xi_1$ be l_i-strictly pseudocontractive nonself-mappings. Suppose that $\mathcal{J} = \Omega \cap \Phi \neq \phi$. Define a sequence $\{x_m\}$ as follows: $x_0,\ x_1 \in \Xi_1$,*

$$\begin{cases} w_m = x_m + \vartheta_m(x_m - x_{m-1}), \\ y_m = (1-\alpha_m)w_m + \alpha_m(\tau_m \mathcal{S} w_m + (1-\tau_m)\mathcal{A}_N^m \mathcal{A}_{N-1}^m \ldots \mathcal{A}_1^m w_m), \\ x_{m+1} = U(y_m + \gamma_1 \mathcal{B}^*(W - \mathcal{I})\mathcal{B} y_m), \quad \forall m \geq 0, \end{cases} \tag{3.11}$$

where $U = J_{\lambda_1}^M(\mathcal{I} - \lambda_1 h_1)$, $W = J_{\lambda_1}^N(\mathcal{I} - \lambda_1 h_2)$ and $\gamma_1 \in (0, \frac{1}{\|\mathcal{B}\|^2})$, $\mathcal{A}_i^m = (1-\delta_m^i)\mathcal{I} + \delta_m^i \mathcal{P}_{\mathcal{C}}(\xi_i \mathcal{I} + (1-\xi_i)\mathcal{A}_i)$, $0 \leq l_i \leq \xi_i < 1$, $\delta_m^i \in (0,1)$ for $i = 1, 2, \ldots N$. Let $\{\alpha_m\}, \{\tau_m\}$ be two real sequences in (0,1), $\{\vartheta_m\} \subset [0, \vartheta]$ for some $\vartheta \in [0,1)$ and $\lambda_1 \subset (0, \alpha)$, where $\alpha = 2\min\{\eta_1, \eta_2\}$. Also, let the following conditions hold:

(i) $\sum_{m=1}^{\infty} \vartheta_m \|x_m - x_{m-1}\| < \infty$;

(ii) $\sum_{m=0}^{\infty} \tau_m < \infty$;

(iii) $\lim_{m\to\infty} \frac{\|y_m - w_m\|}{\alpha_m \tau_m} = 0$.

Then the sequence $\{x_m\}$ converges weakly to $q^* \in \mathcal{J}$.

Proof. We initiate by proving the theorem for $N = 2$. The technique is easily adaptable to the general case.
Since $h_1 : \mathcal{C} \to \Xi_1$ is η_1-inverse strongly monotone mapping then for any $p, q \in \mathcal{C}$, we have

$$\begin{aligned} \|(\mathcal{I} - \lambda_1 h_1)p - (\mathcal{I} - \lambda_1 h_1)q\|^2 &= \|(p-q) - \lambda_1(h_1 p - h_1 q)\|^2 \\ &\leq \|p-q\|^2 - \lambda_1(2\eta_1 - \lambda_1)\|h_1 p - h_1 q\|^2 \\ &\leq \|p-q\|^2, \end{aligned}$$

which proves that $(\mathcal{I} - \lambda_1 h_1)$ is nonexpansive. Similarly, we can also prove that $(\mathcal{I} - \lambda_1 h_2)$ is nonexpansive mapping. Hence U and W are also nonexpansive mappings. Let $q^* \in \mathcal{J}$.

Then $q^* \in \Phi$ and $q^* \in \Omega$, we have

$$\begin{aligned}\|w_m - q^*\| &= \|x_m + \vartheta_m(x_m - x_{m-1}) - q^*\| \\ &= \|(x_m - q^*) + \vartheta_m(x_m - x_{m-1})\| \\ &\le \|x_m - q^*\| + \vartheta_m\|x_m - x_{m-1}\|. \end{aligned} \tag{3.12}$$

From Lemma 2.3, Lemma 2.4 and Lemma 2.5 we get $q^* = \mathcal{A}_2^m\mathcal{A}_1^m q^*$. Hence, we have

$$\begin{aligned}\|y_m - q^*\| &= \|(1-\alpha_m)w_m + \alpha_m(\tau_m\mathcal{S}w_m + (1-\tau_m)\mathcal{A}_2^m\mathcal{A}_1^m w_m) - q^*\| \\ &\le (1-\alpha_m)\|w_m - q^*\| + \alpha_m[\tau_m\|\mathcal{S}w_m - q^*\| + (1-\tau_m)\|\mathcal{A}_2^m\mathcal{A}_1^m w_m - q^*\|] \\ &\le (1-\alpha_m)\|w_m - q^*\| + \alpha_m[\tau_m\|w_m - q^*\| + (1-\tau_m)\|w_m - q^*\|] \\ &+ \alpha_m\tau_m\|\mathcal{S}q^* - q^*\| = \|w_m - q^*\| + \alpha_m\tau_m\|\mathcal{S}q^* - q^*\| \\ &\le \|x_m - q^*\| + \vartheta_m\|x_m - x_{m-1}\| + \alpha_m\tau_m\|\mathcal{S}q^* - q^*\|. \end{aligned} \tag{3.13}$$

Also, since $q^* \in \mathcal{J}$, we have $Uq^* = q^*$ and $W\mathcal{B}q^* = \mathcal{B}q^*$. Let $z_m = y_m + \gamma_1\mathcal{B}^*(W - \mathcal{I})\mathcal{B}y_m$. Then we have

$$\begin{aligned}\|z_m - q^*\|^2 &= \|y_m + \gamma_1\mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m - q^*\|^2 \\ &= \|y_m - q^*\|^2 + \gamma_1^2\|\mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m\|^2 + 2\gamma_1\langle y_m - q^*, \mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m\rangle \\ &= \|y_m - q^*\|^2 + \gamma_1^2\|\mathcal{B}^*\|\|(W-\mathcal{I})\mathcal{B}y_m\|^2 + 2\gamma_1\langle y_m - q^*, \mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m\rangle. \end{aligned} \tag{3.14}$$

Further, we have

$$\begin{aligned}&2\gamma_1\langle y_m - q^*, \mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m\rangle \\ &= 2\gamma_1\langle \mathcal{B}y_m - \mathcal{B}q^*, (W-\mathcal{I})\mathcal{B}y_m\rangle \\ &= 2\gamma_1\langle \mathcal{B}y_m - \mathcal{B}q^* + (W-\mathcal{I})\mathcal{B}y_m - (W-\mathcal{I})\mathcal{B}y_m, (W-\mathcal{I})\mathcal{B}y_m\rangle \\ &= 2\gamma_1\{\langle W\mathcal{B}y_m - \mathcal{B}q^*, (W-\mathcal{I})\mathcal{B}y_m\rangle - \|(W-\mathcal{I})\mathcal{B}y_m\|^2\} \\ &= \gamma_1\{\|W\mathcal{B}y_m - \mathcal{B}q^*\|^2 + \|(W-\mathcal{I})\mathcal{B}y_m\|^2 - \|\mathcal{B}y_m - \mathcal{B}q^*\|^2 \\ &- 2\|(W-\mathcal{I})\mathcal{B}y_m\|^2\} \\ &\le \gamma_1\{\|\mathcal{B}y_m - \mathcal{B}q^*\|^2 - \|\mathcal{B}y_m - \mathcal{B}q^*\|^2 - \|(W-\mathcal{I})\mathcal{B}y_m\|^2\} \\ &= -\gamma_1\|(W-\mathcal{I})\mathcal{B}y_m\|^2. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we have

$$\|z_m - q^*\|^2 \le \|y_m - q^*\|^2 - \gamma_1(1 - \gamma_1\|\mathcal{B}^2\|)\|(W-\mathcal{I})\mathcal{B}y_m\|^2. \tag{3.16}$$

Since $\gamma_1 \in \left(0, \frac{1}{\|\mathcal{B}\|^2}\right)$, (3.6) implies

$$\|z_m - q^*\|^2 \le \|y_m - q^*\|^2. \tag{3.17}$$

Next, using (3.12), (3.13), (3.16) and (3.17)

$$\begin{aligned}\|x_{m+1} - q^*\|^2 &= \|U(y_m + \gamma_1\mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m) - q^*\|^2 \\ &\le \|(y_m + \gamma_1\mathcal{B}^*(W-\mathcal{I})\mathcal{B}y_m) - q^*\|^2 \\ &\le \|y_m - q^*\|^2 - \gamma_1(1 - \gamma_1\|\mathcal{B}\|^2)||(W-\mathcal{I})\mathcal{B}y_m\|^2 \\ &\le (\|x_m - q^*\| + \vartheta_m\|x_m - x_{m-1}\| + \alpha_m\tau_m\|\mathcal{S}q^* - q^*\|)^2 \\ &- \gamma_1(1 - \gamma_1\|\mathcal{B}\|^2)||(W-\mathcal{I})\mathcal{B}y_m\|^2 \\ &\le (\|x_m - q^*\| + \vartheta_m\|x_m - x_{m-1}\| + \alpha_m\tau_m\|\mathcal{S}q^* - q^*\|)^2, \end{aligned} \tag{3.18}$$

which implies that

$$\|x_{m+1} - q^*\| \le \|x_m - q^*\| + \vartheta_m\|x_m - x_{m-1}\| + \alpha_m\tau_m\|\mathcal{S}q^* - q^*\|. \tag{3.19}$$

Since $\sum_{m=1}^{\infty} \vartheta_m \|x_m - x_{m-1}\| < \infty$ and $\sum_{m=0}^{\infty} \tau_m < \infty$, we have $\sum_{m=0}^{\infty} \alpha_m \tau_m < \infty$ and using Lemma 2.7 to (3.19), we deduce that $\lim_{m\to\infty} \|x_m - q^*\|$ exist and finite. Therefore $\{x_m\}$ is bounded. Furthermore, it follows from (3.12), (3.13) and (3.17) that the sequences $\{w_m\}$, $\{y_m\}$ and $\{z_m\}$ are bounded.

Since $\gamma_1(1 - \gamma_1\|\mathcal{B}\|^2) > 0$, it follows from $\lim_{m\to\infty} \tau_m = 0$, and (i), then (3.18) implies that

$$\lim_{m\to\infty} \|(\mathcal{I} - W)\mathcal{B}y_m\| = 0. \tag{3.20}$$

Since $z_m = y_m + \gamma_1 \mathcal{B}^*(W - \mathcal{I})\mathcal{B}y_m$, we have

$$\|z_m - y_m\| = \gamma_1 \|\mathcal{B}^*(\mathcal{I} - W)\mathcal{B}y_m\|. \tag{3.21}$$

From (3.20) and (3.21), we have

$$\lim_{m\to\infty} \|z_m - y_m\| = 0. \tag{3.22}$$

Now, we estimate

$$\begin{aligned}
&\|x_{m+1} - x_m\|^2 \\
&= \|x_{m+1} - q^* - x_m + q^*\|^2 \\
&= \|x_{m+1} - q^*\|^2 - \|x_m - q^*\|^2 - 2\langle x_{m+1} - x_m, x_m - q^*\rangle \\
&= \|x_{m+1} - q^*\|^2 - \|x_m - q^*\|^2 - 2\langle x_{m+1} - \tilde{x}, x_m - q^*\rangle + 2\langle x_m - \tilde{x}, x_m - q^*\rangle,
\end{aligned} \tag{3.23}$$

where $\tilde{x}$ is a weak cluster point of $\{x_m\}$. Since $\lim_{m\to\infty} \|x_m - q^*\|$ exists, we get

$$\lim_{m\to\infty} \|x_{m+1} - x_m\| = 0. \tag{3.24}$$

Now,

$$\begin{aligned}
\|x_{m+1} - q^*\|^2 &= \|Uz_m - Uq^*\|^2 \\
&= \|J_{\lambda_1}^M(\mathcal{I} - \lambda_1 h_1)z_m - J_{\lambda_1}^M(\mathcal{I} - \lambda_1 h_1)q^*\|^2 \\
&\le \|(\mathcal{I} - \lambda_1 h_1)z_m - (\mathcal{I} - \lambda_1 h_1)q^*\|^2 \\
&\le \|z_m - q^*\|^2 - \lambda_1(2\eta_1 - \lambda_1)\|h_1 z_m - h_1 q^*\|^2.
\end{aligned} \tag{3.25}$$

which implies that

$$\begin{aligned}
&\lambda_1(2\eta_1 - \lambda_1)\|h_1 z_m - h_1 q^*\|^2 \\
&\le \|z_m - q^*\|^2 - \|x_{m+1} - q^*\|^2 \\
&\le \|y_m - q^*\|^2 - \|x_{m+1} - q^*\|^2 \\
&\le \|w_m - q^*\|^2 + \alpha_m^2 \tau_m^2 \|\mathcal{S}q^* - q^*\|^2 \\
&+ 2\alpha_m \tau_m \|w_m - q^*\| \|\mathcal{S}q^* - q^*\| - \|x_{m+1} - q^*\|^2 \\
&\le \|x_m - q^*\|^2 - \|x_{m+1} - q^*\|^2 + \vartheta_m^2 \|x_m - x_{m-1}\|^2 \\
&+ 2\|x_m - q^*\| \vartheta_m \|x_m - x_{m-1}\| + \alpha_m^2 \tau_m^2 \|\mathcal{S}q^* - q^*\|^2 \\
&+ 2\alpha_m \tau_m \|\mathcal{S}q^* - q^*\| (\|x_m - q^*\| + \vartheta_m \|x_m - x_{m-1}\|) \\
&\le \|x_m - x_{m+1}\| (\|x_m - q^*\| + \|x_{m+1} - q^*\|) + \vartheta_m^2 \|x_m - x_{m-1}\|^2 \\
&+ 2\|x_m - q^*\| \vartheta_m \|x_m - x_{m-1}\| + \alpha_m^2 \tau_m^2 \|\mathcal{S}q^* - q^*\|^2 \\
&+ 2\alpha_m \tau_m \|\mathcal{S}q^* - q^*\| (\|x_m - q^*\| + \vartheta_m \|x_m - x_{m-1}\|).
\end{aligned} \tag{3.26}$$

Since, $\lambda_1(2\eta_1 - \lambda_1) > 0$, $\|x_m - q^*\|$ is bounded, $\lim_{m\to\infty} \vartheta_m \|x_m - x_{m-1}\| = 0$, and $\lim_{m\to\infty} \tau_m = 0$, we have

$$\lim_{m\to\infty} \|h_1 z_m - h_1 q^*\| = 0. \tag{3.27}$$

Since $J_{\lambda_1}^M$ is firmly nonexpansive, we have

$$\begin{aligned}
\|x_{m+1}-q^*\|^2 &= \|J_{\lambda_1}^M(\mathcal{I}-\lambda_1 h_1)z_m - J_{\lambda_1}^M(\mathcal{I}-\lambda_1 h_1)q^*\|^2 \\
&\leq \langle(\mathcal{I}-\lambda_1 h_1)z_m-(\mathcal{I}-\lambda_1 h_1)q^*, x_{m+1}-q^*\rangle \\
&= \frac{1}{2}[\|(\mathcal{I}-\lambda_1 h_1)z_m-(\mathcal{I}-\lambda_1 h_1)q^*\|^2+\|x_{m+1}-q^*\|^2 \\
&- \|z_m-x_{m+1}-\lambda_1(h_1 z_m-h_1 q^*)\|^2] \\
&\leq \frac{1}{2}[\|z_m-q^*\|^2+\|x_{m+1}-q^*\|^2-\|z_m-x_{m+1}\|^2 \\
&+ 2\lambda_1\langle z_m-x_{m+1}, h_1 x_m-h_1 q^*\rangle-\lambda_1^2\|h_1 z_m-h_1 q^*\|^2] \\
&\leq \frac{1}{2}[\|z_m-q^*\|^2+\|x_{m+1}-q^*\|^2-\|z_m-x_{m+1}\|^2 \\
&+ 2\lambda_1\|z_m-x_{m+1}\|\|h_1 z_m-h_1 q^*\|], \qquad (3.28)
\end{aligned}$$

which in turns yields

$$\|x_{m+1}-q^*\|^2 \leq \|z_m-q^*\|^2-\|z_m-x_{m+1}\|^2+2\lambda_1\|z_m-x_{m+1}\|\|h_1 z_m-h_1 q^*\|, \quad (3.29)$$

from (3.12), (3.13) and (3.17), implies that

$$\begin{aligned}
\|z_m-x_{m+1}\|^2 &\leq \|z_m-q^*\|^2-\|x_{m+1}-q^*\|^2+2\lambda_1\|z_m-x_{m+1}\|\|h_1 z_m-h_1 q^*\|, \\
&\leq \|y_m-q^*\|^2-\|x_{m+1}-q^*\|^2+2\lambda_1\|z_m-x_{m+1}\|\|h_1 z_m-h_1 q^*\|, \\
&\leq \|w_m-q^*\|^2+\alpha_m^2\tau_m^2\|\mathcal{S}q^*-q^*\|^2+2\alpha_m\tau_m\|w_m-q^*\|\|\mathcal{S}q^*-q^*\| \\
&- \|x_{m+1}-q^*\|^2+2\lambda_1\|z_m-x_{m+1}\|\|h_1 z_m-h_1 q^*\|, \\
&\leq \|x_m-x_{m+1}\|(\|x_m-q^*\|+\|x_{m+1}-q^*\|)+\vartheta_m^2\|x_m-x_{m-1}\|^2 \\
&+ 2\|x_m-q^*\|\vartheta_m\|x_m-x_{m-1}\|+\alpha_m^2\tau_m^2\|\mathcal{S}q^*-q^*\|^2 \\
&+ 2\alpha_m\tau_m\|\mathcal{S}q^*-q^*\|(\|x_m-q^*\| \\
&+ \vartheta_m\|x_m-x_{m-1}\|)+2\lambda_1\|z_m-x_{m+1}\|\|h_1 z_m-h_1 q^*\|, \qquad (3.30)
\end{aligned}$$

Using (i), (ii), (3.24) and (3.27) in (3.30), we have

$$\lim_{m\to\infty}\|z_m-x_{m+1}\|=0. \qquad (3.31)$$

Since

$$\|z_m-x_m\| \leq \|z_m-x_{m+1}\|+\|x_{m+1}-x_m\|,$$

from (3.24) and (3.31), implies that

$$\lim_{m\to\infty}\|z_m-x_m\|=0. \qquad (3.32)$$

Since

$$\|x_m-y_m\| \leq \|x_m-z_m\|+\|z_m-y_m\|,$$

from (3.22) and (3.32), implies that

$$\lim_{m\to\infty}\|x_m-y_m\|=0. \qquad (3.33)$$

using (i) and we observe that

$$\lim_{m\to\infty}\|w_m-x_m\|=\lim_{m\to\infty}\vartheta_m\|x_m-x_{m+1}\|=0. \qquad (3.34)$$

Since

$$\|w_m-y_m\| \leq \|w_m-x_m\|+\|x_m-y_m\|,$$

from (3.33) and (3.34) implies that

$$\lim_{m\to\infty}\|w_m-y_m\|=0. \qquad (3.35)$$

Now, we show that $q^* \in \mathcal{J}$. Since $\mathcal{A}_2^m\mathcal{A}_1^m$ is an averaged mapping and nonexpansive. Due to the boundedness of $\{w_m\}$ and nonexpansivity of $\mathcal{S}$, there exists $L > 0$ such that $\|\mathcal{S}w_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| \leq L$ for all $m \geq 0$. Now, we know that

$$\begin{aligned}\|y_m - \mathcal{A}_2^n\mathcal{A}_1^n w_m\| &= \|(1-\alpha_m)w_m + \alpha_m(\tau_m S w_m + (1-\tau_m)\mathcal{A}_2^m\mathcal{A}_1^m w_m) - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| \\ &= (1-\alpha_m)\|w_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| + \alpha_m\tau_m\|\mathcal{S}w_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| \\ &\leq (1-\alpha_m)\|w_m - y_m\| + (1-\alpha_m)\|y_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| \\ &+ \alpha_m\tau_m\|\mathcal{S}w_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| \\ &\leq (1-\alpha_m)\|w_m - y_m\| + (1-\alpha_m)\|y_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| + \alpha_m\tau_m L,\end{aligned}$$

which implies

$$\begin{aligned}\alpha_m\|y_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| &\leq (1-\alpha_m)\|w_m - y_m\| + \alpha_m\tau_m L \\ &\leq \|w_m - y_m\| + \alpha_m\tau_m L.\end{aligned} \tag{3.36}$$

Hence, we have

$$\|y_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| \leq \tau_m \frac{\|w_m - y_m\|}{\alpha_m\tau_m} + \tau_m L. \tag{3.37}$$

From condition (ii) and (iii),

$$\lim_{m\to\infty} \|y_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| = 0. \tag{3.38}$$

Since

$$\|w_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| = \|w_m - y_m\| + \|y_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\|,$$

from (3.35) and (3.38) implies that

$$\lim_{m\to\infty} \|w_m - \mathcal{A}_2^m\mathcal{A}_1^m w_m\| = 0. \tag{3.39}$$

Since $\{w_m\}$ is bounded, there exists a subsequence $\{w_{m_k}\}$ of $\{w_m\}$ that weakly converges to q^*. Noticing that $\{\delta_m^i\}$ is bounded for i=1,2, we can consider $\delta_{m_j}^i \to \delta_\infty^i$ as $j \to \infty$, where $0 < \delta_\infty^i < 1$ for $i = 1, 2$. Define, for i=1,2,

$$\mathcal{A}_i^\infty = (1-\delta_\infty^i)I + \delta_\infty^i\mathcal{P}_\mathcal{C}(\xi_i I + (1-\xi_i)\mathcal{A}_i).$$

From Lemma 2.3 and Lemma 2.5, $\mathrm{F}(\mathcal{P}_\mathcal{C}(\xi_i\mathcal{I} + (1-\xi_i)\mathcal{A}_i) = \mathrm{F}(\mathcal{A}_i)$. Again, since $\mathcal{P}_\mathcal{C}(\xi_i\mathcal{I} + (1-\xi_i)\mathcal{A}_i)$ is nonexpansive , $\mathcal{A}_i^\infty$ is averaged and $\mathrm{F}(\mathcal{A}_i^\infty) = \mathrm{F}(\mathcal{A}_i)$ for i=1,2. Moreover, since

$$\mathrm{F}(\mathcal{A}_1^\infty) \cap \mathrm{F}(\mathcal{A}_2^\infty) = \mathrm{F}(\mathcal{A}_1) \cap \mathrm{F}(\mathcal{A}_2) = \mathrm{F}(\Phi) \neq \phi,$$

by Lemma 2.4, we get

$$\mathrm{F}(\mathcal{A}_2^\infty\mathcal{A}_1^\infty) = \mathrm{F}(\mathcal{A}_1^\infty) \cap \mathrm{F}(\mathcal{A}_2^\infty) = \mathrm{F}(\Phi).$$

Notice that

$$\|\mathcal{A}_i^{m_j}s - \mathcal{A}_i^\infty s\| \leq |\delta_{m_j}^i - \delta_\infty^i|(\|s\| + \|\mathcal{A}_i s\|),$$

hence, we get

$$\lim_{j\to\infty} \sup_{s\in S} \|\mathcal{A}_i^{m_j}s - \mathcal{A}_i^\infty s\| = 0, \tag{3.40}$$

where S is an arbitrary bounded subset of Ξ_1. Also, we obtain

$$\begin{aligned}\|w_{m_j} - \mathcal{A}_2^\infty \mathcal{A}_1^\infty w_{m_j}\| &\le \|w_{m_j} - \mathcal{A}_2^{m_j} \mathcal{A}_1^{m_j} w_{m_j}\| + \|\mathcal{A}_2^{m_j} \mathcal{A}_1^{m_j} w_{m_j} - \mathcal{A}_2^\infty \mathcal{A}_1^{m_j} w_{m_j}\| \\ &+ \|\mathcal{A}_2^\infty \mathcal{A}_1^{m_j} w_{m_j} - \mathcal{A}_2^\infty \mathcal{A}_1^\infty w_{m_j}\| \\ &\le \|w_{m_j} - \mathcal{A}_2^{m_j} \mathcal{A}_1^{m_j} w_{m_j}\| + \|\mathcal{A}_2^{m_j} \mathcal{A}_1^{m_j} w_{m_j} - \mathcal{A}_2^\infty \mathcal{A}_1^{m_j} w_{m_j}\| \\ &+ \|\mathcal{A}_1^{m_j} w_{m_j} - \mathcal{A}_1^\infty w_{m_j}\| \\ &\le \|w_{m_j} - \mathcal{A}_2^{m_j} \mathcal{A}_1^{m_j} w_{m_j}\| \\ &+ \sup_{s\in S'} \|\mathcal{A}_2^{m_j} s - \mathcal{A}_2^\infty s\| + \sup_{s\in S''} \|\mathcal{A}_1^{m_j} s - \mathcal{S}_1^\infty s\|, \end{aligned} \tag{3.41}$$

where S' is bounded subset including $\{\mathcal{A}_1^{m_j} w_{m_j}\}$ and S'' is a bounded subset including $\{w_{m_j}\}$. It follows that from (3.39), (3.40) and (3.41) that $\lim_{j\to\infty} \|w_{m_j} - \mathcal{A}_2^\infty \mathcal{A}_1^\infty w_{m_j}\| = 0$. So, by Lemma 2.6, we have $q^* \in \mathrm{F}(\mathcal{A}_2^\infty \mathcal{A}_1^\infty) = \mathrm{F}(\mathcal{A}_1) \cap \mathrm{F}(\mathcal{A}_2)$.
Now, we show that $q^* \in \Phi$. It follows from (3.11), we have

$$y_m - w_m = \alpha_m \left(\tau_m(\mathcal{S}w_m - w_m) + (1-\tau_m)(\mathcal{A}_2^m \mathcal{A}_1^m w_m - w_m)\right),$$

and hence

$$\frac{w_m - y_m}{\alpha_m \tau_m} = \left((\mathcal{I}-\mathcal{S})w_m + \left(\frac{1-\tau_m}{\tau_m}\right)(\mathcal{I} - \mathcal{A}_2^m \mathcal{A}_1^m) w_m\right). \tag{3.42}$$

Lemma 2.8 (a), ensure that the operator sequence $\{(\frac{1-\tau_m}{\tau_m})(\mathcal{I} - \mathcal{A}_2^m \mathcal{A}_1^m)\}$ graph converges to $N_{\mathrm{F}(\mathcal{A}_1)\cap \mathrm{F}(\mathcal{A}_2)}$, and hence, from Lemma 2.8 (b), the operator sequence $\{(\mathcal{I}-\mathcal{S}) + \left(\frac{1-\tau_m}{\tau_m}\right)(\mathcal{I} - \mathcal{A}_2^m \mathcal{A}_1^m)\}$ is graph convergent to $(\mathcal{I}-\mathcal{S}) + N_{\mathrm{F}(\mathcal{A}_1)\cap \mathrm{F}(\mathcal{A}_2)}$. Now by replacing m with m_j and allowing the limit in (3.42) and evaluating the fact that $\lim_{m\to\infty} \frac{y_m - w_m}{\alpha_m \tau_m} = 0$ and the graph of $(\mathcal{I}-\mathcal{S}) + N_{\mathrm{F}(\mathcal{A}_1)\cap \mathrm{F}(\mathcal{A}_2)}$. is weakly-strongly closed, we get

$$0 \in (\mathcal{I}-\mathcal{S})q^* + N_{\mathrm{F}(\mathcal{A}_1)\cap \mathrm{F}(\mathcal{A}_2)} q^*,$$

so $q^* \in \Phi$. On the other hand, since $\{x_m\}$ is bounded, then there is a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ that weakly converges to $q^* \in \Xi_1$. Further, since $\{x_m\}, \{z_m\}, \{y_m\}$ and $\{w_m\}$ have the same asymptotic behaviour, then there are subsequences $\{z_{m_k}\}$ of $\{z_m\}$, $\{w_{m_k}\}$ of $\{w_m\}$ and $\{y_{m_k}\}$ of $\{y_m\}$ converge weakly to q^*. The third equation of algorithm (3.11) can be expressed as

$$\frac{(z_{m_k} - x_{m_k+1}) - \lambda_1 h_1(z_{m_k})}{\lambda_1} \in M x_{m_k+1}. \tag{3.43}$$

We get $0 \in h_1(q^*) + Mq^*$ by going to limit k $\to \infty$ in (3.43) and taking into consideration that h_1 is $\frac{1}{\eta}$-Lipschitz continuous and the graph of maximal monotone operator is weakly-strongly closed. $\{\mathcal{B}y_{m_k}\}$ weakly converges to $\mathcal{B}q^*$ because $\mathcal{B}$ is continuous. The fact that $J_{\lambda_1}^N(\mathcal{I} - \lambda_1 h_2)$ is nonexpansive, (3.20) and Lemma 2.6 leads to the conclusion that $0 \in h_2(Aq^*) + N(Aq^*)$. Consequently $q^* \in \Omega$. Thus $q^* \in \mathcal{J}$, which completes the proof. □

4. Consequence

In this section, we deduce a special case from our main convergence Theorem 3.1. If we set $\mathcal{A}_i = \mathcal{A}$ in Theorem 3.1, we have the following result to approximate a common solution of SMVIP (1.5)-(1.6) and HFPP (1.3).

Corollary 4.1: Let Ξ_1 and Ξ_2 be two real Hilbert spaces and let $\mathcal{C}$ and $\mathcal{D}$ be nonempty, closed and convex subsets of Ξ_1 and Ξ_2, respectively. Let $\mathcal{B} : \Xi_1 \to \Xi_2$ be a bounded linear operator with its adjoint operator $\mathcal{B}^*$. Assume that $M : \Xi_1 \to 2^{\Xi_1}$ and $N : \Xi_2 \to 2^{\Xi_2}$ be two multivalued maximal monotone mappings. Let $h_1 : \mathcal{C} \to \Xi_1$ and $h_2 : \mathcal{D} \to \Xi_2$

be η_1- and η_2-inverse strongly monotone mappings, respectively. Let $\mathcal{S}, \mathcal{A} : \Xi_1 \to \Xi_1$ be two nonexpansive mappings. Assume that $\mathcal{J} = \Omega \cap \Phi_1 \neq \phi$. Define a sequence $\{x_m\}$ as follows: $x_0,\ x_1 \in \Xi_1$,

$$\begin{cases} w_m = x_m + \vartheta_m(x_m - x_{m-1}), \\ y_m = (1-\alpha_m)w_m + \alpha_m(\tau_m \mathcal{S} w_m + (1-\tau_m)\mathcal{A} w_m), \\ x_{m+1} = U(y_m + \gamma_1 \mathcal{B}^*(W - \mathcal{I})\mathcal{B} y_m), \quad \forall m \geq 0, \end{cases} \tag{4.44}$$

where $U = J^M_{\lambda_1}(\mathcal{I} - \lambda_1 h_1)$, $W = J^N_{\lambda_1}(\mathcal{I} - \lambda_1 h_2)$ and $\gamma_1 \in (0, \frac{1}{\|\mathcal{B}\|^2})$. Let $\{\alpha_m\}, \{\tau_m\}$ be two real sequences in (0,1), $\{\vartheta_m\} \subset [0, \vartheta]$ for some $\vartheta \in [0, 1)$ and $\lambda_1 \subset (0, \alpha)$, where $\alpha = 2\min\{\eta_1, \eta_2\}$. Also, let the following conditions hold:

(i) $\sum\limits_{m=1}^{\infty} \vartheta_m \|x_m - x_{m-1}\| < \infty$;

(ii) $\sum\limits_{m=0}^{\infty} \tau_m < \infty$;

(iii) $\lim\limits_{m\to\infty} \frac{\|y_m - w_m\|}{\alpha_m \tau_m} = 0$.

Then the sequence $\{x_m\}$ converges weakly to $q^* \in \Omega \cap \Phi_1$. □

5. Numerical example

We give an example and numerical result which justify the Theorem 3.1.

Let $\Xi_1 = \Xi_2 = \mathcal{C} = \mathcal{D} = \mathbb{R}$, the set of all real numbers, let $M : \Xi_1 \to 2^{\Xi_1}$ defined as $M(p) = \{3p\}$, $\forall p \in \Xi_1$ and $N : \Xi_2 \to 2^{\Xi_2}$ defined as $N(p) = \{5p\}$, $\forall p \in \Xi_2$. Let $h_1 : \mathcal{C} \to \Xi_1$ defined as $h_1(p) = 2p$, $\forall p \in \mathcal{C}$ and $h_2 : \mathcal{D} \to \Xi_2$ defined as $h_2(p) = 4p$, $\forall p \in \mathcal{D}$. For $\lambda_1 = \frac{1}{3}$, we compute that

$$U = J^M_{\lambda_1}(\mathcal{I} - \lambda_1 h_1)(p) = \frac{p}{6},$$
$$W = J^N_{\lambda_1}(\mathcal{I} - \lambda_1 h_2)(p) = -\frac{p}{8}.$$

Let us define the mapping $\mathcal{B} : \Xi_1 \to \Xi_2$ be defined by $\mathcal{B}(p) = -\frac{9p}{4}$, $\forall p \in \Xi_1$ and $\mathcal{S} : \Xi_1 \to \Xi_1$ be defined by $\mathcal{S}(p) = \sin p$, $\forall p \in \Xi_1$. Let $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{C} \to \Xi_1$ defined by $\mathcal{A}_1(p) = \frac{p}{3}$ and $\mathcal{A}_2(p) = \frac{p}{5}, \forall p \in \mathcal{C}$. Then, it is simple to check $\mathcal{A}_1$ and $\mathcal{A}_2$ are 0-strictly pseudocontractive mappings, $\mathcal{S}$ is nonexpansive and $\mathcal{B}$ is bounded linear operator with adjoint operator $\mathcal{B}^*$ such that $\|\mathcal{B}\| = \|\mathcal{B}^*\| = \frac{9}{4}$. Now let us choose $\alpha_m = 0.5$, $\tau_m = \frac{1}{m^2}$, $\gamma_1 = 0.1$, $\vartheta = 0.9$, $\xi_1 = 0.3 > l_1 = 0, \xi_2 = 0.4 > l_2 = 0$, $\delta^1_m = \frac{m+1}{m+2}$ and $\delta^2_m = \frac{m+2}{m+3}$. It is clear that $\mathrm{F}(\mathcal{A}_1) = \mathrm{F}(\mathcal{A}_2) = \{0\}$ and $\Omega = \{0 \in \Xi_1 : 0 \in$ (MVIP (1.5)) and $\mathcal{B}(0) \in$ (MVIP (1.6))$\}$. Therefore, $\mathcal{J} = \Omega \cap \Phi = \{0\} \neq \phi$. The stopping criteria for our proposed iterative method is $E_m = \|x_{m+1} - x_m\| \leq 1 \times 10^{-6}$. Figure 2 shows the error graph of sequence $\{x_m\}$.

TABLE 1. Numerical results for two different initial values of algorithm (3.11)

Number of iterations (m)	$\mathbf{x_m}$ $\mathbf{x_0 = 5}$ $\mathbf{x_1 = 2}$	$\mathbf{E_m = \|x_{m+1} - x_m\|}$ $\mathbf{x_0 = 5}$ $\mathbf{x_1 = 2}$	$\mathbf{x_m}$ $\mathbf{x_0 = -3}$ $\mathbf{x_1 = -5}$	$\mathbf{E_m = \|x_{m+1} - x_m\|}$ $\mathbf{x_0 = -3}$ $\mathbf{x_1 = -5}$
1	5.000000	3.000000	-3.000000	2.000000
2	2.000000	2.038429	-5.000000	4.678384
3	-0.038429	$5.5022e^{-02}$	-0.321616	$5.0504e^{-01}$
4	-0.093451	$8.6273e^{-02}$	0.183422	$1.5149e^{-01}$
5	-0.007178	$1.0651e^{-02}$	0.031937	$3.7083e^{-02}$
6	0.003473	$2.8370e^{-03}$	-0.005146	$3.2700e^{-03}$
7	0.000636	$7.2900e^{-04}$	-0.001876	$1.9270e^{-03}$
8	-0.000093	$5.7000e^{-05}$	0.000052	$3.4000e^{-05}$
9	-0.000036	$3.7000e^{-05}$	0.000086	$8.0000e^{-05}$
10	0.000001	$1.0000e^{-06}$	0.000006	$9.0000e^{-06}$
11	0.000002	$1.0000e^{-06}$	-0.000003	$3.0000e^{-06}$
12	0.000000	$0.0000e^{+00}$	-0.000001	$1.0000e^{-06}$
13	0.000000	$0.0000e^{+00}$	0.000000	$0.0000e^{+00}$

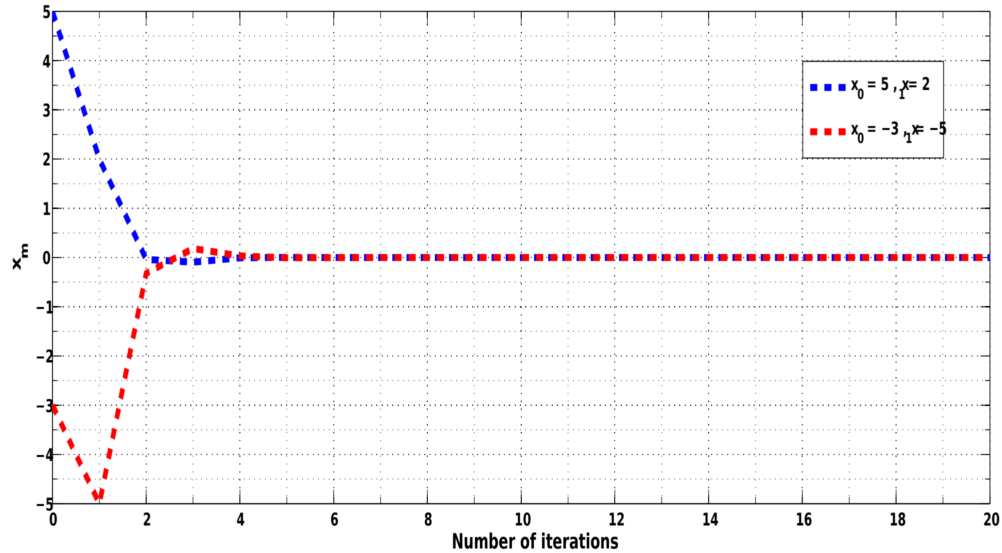

FIGURE 1. Convergence of $\{x_m\}$ with two different initial values x_0 and x_1.

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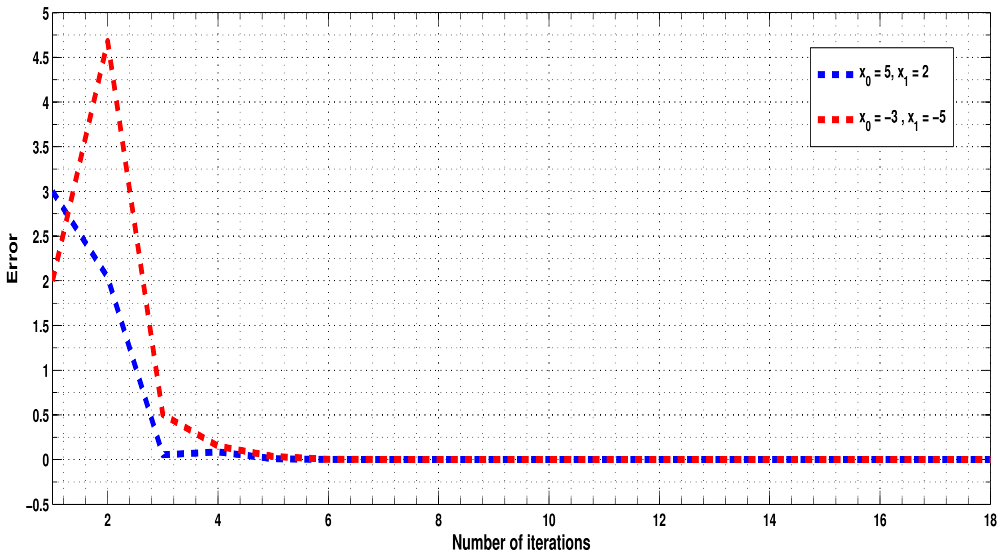

FIGURE 2. Error plotting of $\{x_m\}$ with two different initial values.

From Table 1 and Figure 1, we conclude that the sequence $\{x_m\}$ generated by proposed iterative method converges to 0.

TABLE 2. Numerical results after removing the inertial step in algorithm (3.11)

Number of iterations (m)	$\mathbf{x_m}$ $\mathbf{x_0 = 5}$	$\mathbf{x_m}$ $\mathbf{x_0 = -3}$
1	5.000000	-3.000000
5	4.189565	-2.860058
10	3.086497	-2.052167
15	3.002579	-2.005976
20	3.001178	-2.000037
25	3.000473	-1.755264
30	3.000036	-1.591005
35	2.568943	-1.359401
40	2.364907	-1.065945
45	2.001691	-1.008920
50	2.000202	-1.000894
55	1.649871	-0.915760
60	1.064894	-0.057913

Table 1 is a numerical interpretation of our proposed iterative algorithm (3.11) while Table 2 represents the numerical interpretation when we remove the inertial interpolation term from the algorithm (3.11). The convergence of our algorithm is faster than the algorithm obtained by removing the inertial term which shows the usefulness of the inertial interpolation term.

6. Conclusion

In this paper, we suggested and analyzed an inertial Krasnoselski-Mann type iterative method for approximating a common solution of a split monotone variational inclusion problem and a hierarchical fixed point problem for a finite family of k-strictly pseudocontractive nonself-mappings under the framework of real Hilbert spaces. We constructed an iterative method for the stated problems and proved a weak convergence theorem under some certain conditions. Further, we deduced a special case from our convergence result. Finally, a numerical example was presented to justify the convergence analysis of the proposed iterative method. We also carried out a justification how the inertial step is useful.

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Conflicts of Interest

The authors declare no conflict of interest.

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