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MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

ON m-QUASI-n-POWER-TOTALLY-(α,β)-NORMAL OPERATORS

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Abstract. In this paper, we introduce the notion of m-quasi-n-power-totally-(α,β)-normal operators on a Hilbert space $\mathscr{H}$ as : An operator $\mathcal{L}$ is called m-quasi-n-power-totally-(α,β)-normal $(0\le\alpha\le 1\le\beta)$ if

$$\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m \le \mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m \le \beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m$$

for natural numbers m and n and for all $\lambda\in\mathbb{C}$. This paper aims to study several properties of m-quasi-n-power-totally-(α,β)-normal operators.

1. Introduction

Let $\mathscr{H}$ be a non zero complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Let m and n be natual numbers.

Definition 1.1. Let $\mathcal{L}\in\mathscr{B}(\mathscr{H})$.

(1) An operator $\mathcal{L}$ is called (α,β)-normal [9, 16] $(0\le\alpha\le 1\le\beta)$ if
$$\alpha^2\mathcal{L}^*\mathcal{L}\le\mathcal{L}\mathcal{L}^*\le\beta^2\mathcal{L}^*\mathcal{L}.$$

(2) An operator $\mathcal{L}$ is called quasi-(α,β)-normal [19] $(0\le\alpha\le 1\le\beta)$ if
$$\alpha^2\mathcal{L}^{*2}\mathcal{L}^2\le\mathcal{L}^*\mathcal{L}\mathcal{L}^*\mathcal{L}\le\beta^2\mathcal{L}^{*2}\mathcal{L}^2.$$

(3) An operator $\mathcal{L}$ is called m-quasi-(α,β)-normal [19] $(0\le\alpha\le 1\le\beta)$ if
$$\alpha^2\mathcal{L}^{(m+1)*}\mathcal{L}^{m+1}\le\mathcal{L}^{m*}(\mathcal{L}\mathcal{L}^*)\mathcal{L}^m\le\beta^2\mathcal{L}^{(m+1)*}\mathcal{L}^{m+1}$$
for a natural number m.

(4) An operator $\mathcal{L}$ is called m-quasi-totally-(α,β)-normal [19] $(0\le\alpha\le 1\le\beta)$ if
$$\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)\mathcal{L}^m\le\mathcal{L}^{m*}(\mathcal{L}-\lambda)(\mathcal{L}-\lambda)^*\mathcal{L}^m$$
$$\le\beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)\mathcal{L}^m$$
for a natural number m and for all $\lambda\in\mathbb{C}$.

In general the following implications holds:
$$(\alpha,\beta)-\text{ normal }\subseteq \text{quasi}-(\alpha,\beta)-\text{ normal}$$
$$\subseteq m-\text{quasi}-(\alpha,\beta)-\text{normal}\subseteq m-\text{quasi}-\text{totally}-(\alpha,\beta)-\text{normal}.$$

Many authors have studied various generalizations of normal operators in [1, 2, 3, 6, 8, 14, 17, 20] and hyponormal operators in [4, 13, 15, 18]. The concept of n-normal operators extends the concept of normal operators and has been investigated by A.A. Jibril[14] and S.A. Alzuraiqi et al.[3]. An operator $\mathcal{L}$ is n-normal if it satisfies the equation $\mathcal{L}^n\mathcal{L}^*=\mathcal{L}^*\mathcal{L}^n$. In general normal $\subseteq n-$ normal. Recently, M. Guesba [11] established sufficient conditions under which the sum and product of two hyponormal operators are hyponormal. Some results on numerical radius inequalities for (α,β)-normal and normal operators are given in [9, 10].

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This paper focuses on a class of operators on Hilbert spaces that generalizes the concepts of n-normal, (α,β)-normal and m-quasi-(α,β)-normal operators. Specifically, we introduce the class of m-quasi-n-power-totally-(α,β)-normal operators as:
An operator $\mathcal{L}$ is called m-quasi-n-power-totally-(α,β)-normal $(0 \le \alpha \le 1 \le \beta)$ if

$$\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m \le \mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m$$
$$\le \beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m$$

for natural numbers m, n and for all $\lambda \in \mathbb{C}$.

Remark 1.2. (1) If $m = 0$, then m-quasi-n-power-totally-(α,β)-normal is called n-power-totally-(α,β)-normal.
(2) If $n = 1$, then m-quasi-n-power-totally-(α,β)-normal is called m-quasi-totally-(α,β)-normal.
(3) If $m = 0, n = 1$, then m-quasi-n-power-totally-(α,β)-normal is called totally-(α,β)-normal.

Example 1.3. Given the operator $\mathcal{L} = diag(2,4)$ in $\mathscr{B}(\mathbb{C}^2)$ is 2-quasi-3-power-(α,β)-normal for $\alpha = 0.5$ and $\beta = 2$.

Example 1.4. The following operator $\mathcal{L}$ in $\mathscr{B}(\mathbb{C}^2)$ is 2-quasi-2-power-(α,β)-normal for $\alpha = 0.6$ and $\beta = 1.4$.

$$\mathcal{L} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

2. Main Results

We begin with the following theorem.

Theorem 2.1. *Let $\mathcal{L} \in \mathscr{B}(\mathscr{H})$ be an m-quasi-n-power-totally-(α,β)-normal if and only if it is n-power-totally-(α,β)-normal on $\overline{ran(\mathcal{L}^m)}$.*

Proof. $\mathcal{L} \in \mathscr{B}(\mathscr{H})$ is m-quasi-n-power-totally-(α,β)-normal operator

$$\Leftrightarrow \quad \alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m \le \mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m$$
$$\le \beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m$$
$$\Leftrightarrow \quad \langle\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x, x\rangle \le \langle\mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m x, x\rangle$$
$$\le \langle\beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x, x\rangle, \quad \text{for every } x \in \mathscr{H}$$
$$\Leftrightarrow \quad \langle\alpha^2(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x, \mathcal{L}^m x\rangle \le \langle(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m x, \mathcal{L}^m x\rangle$$
$$\le \langle\beta^2(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x, \mathcal{L}^m x\rangle, \quad \text{for every } x \in \mathscr{H}$$
$$\Leftrightarrow \quad \alpha^2(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n \le (\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^* \le \beta^2(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n, \quad \text{on } \overline{ran(\mathcal{L}^m)}.$$

□

Theorem 2.2. *Let $\mathcal{L} \in \mathscr{B}(\mathscr{H})$ be m-quasi-n-power-totally-(α,β)-normal and let $\mathcal{M}$ be a closed subsapce of $\mathscr{H}$ which reduces $\mathcal{L}$. Then $\mathcal{L}|\mathcal{M}$ is m-quasi-n-power-totally-(α,β)-normal.*

Proof. Let $\mathcal{M}$ be a reducing subspace of $\mathcal{L}$. Then

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix} \text{ on } \mathscr{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

From the fact that $\mathcal{L}$ is m-quasi-n-power-totally-(α,β)-normal, we have

$$\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m \le \mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m$$
$$\le \beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m.$$

Therefore

$$\begin{aligned}&\alpha^2\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}^{m*}\begin{pmatrix}\mathcal{L}_1-\lambda & 0\\ 0 & \mathcal{L}_2-\lambda\end{pmatrix}^{*}\begin{pmatrix}\mathcal{L}_1-\lambda & 0\\ 0 & \mathcal{L}_2-\lambda\end{pmatrix}^{n}\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}^{m}\\ &\le\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}^{m*}\begin{pmatrix}\mathcal{L}_1-\lambda & 0\\ 0 & \mathcal{L}_2-\lambda\end{pmatrix}^{n}\begin{pmatrix}\mathcal{L}_1-\lambda & 0\\ 0 & \mathcal{L}_2-\lambda\end{pmatrix}^{*}\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}^{m}\\ &\le\beta^2\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}^{m*}\begin{pmatrix}\mathcal{L}_1-\lambda & 0\\ 0 & \mathcal{L}_2-\lambda\end{pmatrix}^{*}\begin{pmatrix}\mathcal{L}_1-\lambda & 0\\ 0 & \mathcal{L}_2-\lambda\end{pmatrix}^{n}\begin{pmatrix}\mathcal{L}_1 & 0\\ 0 & \mathcal{L}_2\end{pmatrix}^{m}.\end{aligned}$$

Hence

$$\begin{aligned}\alpha^2\begin{pmatrix}\mathcal{L}_1^{m*}(\mathcal{L}_1-\lambda)^*(\mathcal{L}_1-\lambda)^n\mathcal{L}_1^m & 0\\ 0 & \mathcal{V}\end{pmatrix}&\le\begin{pmatrix}\mathcal{L}_1^{m*}(\mathcal{L}_1-\lambda)^n(\mathcal{L}_1-\lambda)^*\mathcal{L}_1^m & 0\\ 0 & \mathcal{V}\end{pmatrix}\\ &\le\beta^2\begin{pmatrix}\mathcal{L}_1^{m*}(\mathcal{L}_1-\lambda)^*(\mathcal{L}_1-\lambda)^n\mathcal{L}_1^m & 0\\ 0 & \mathcal{V}\end{pmatrix}.\end{aligned}$$

for some operator $\mathcal{V}$. This indicates that

$$\begin{aligned}\alpha^2\mathcal{L}_1^{m*}(\mathcal{L}_1-\lambda)^*(\mathcal{L}_1-\lambda)^n\mathcal{L}_1^m&\le\mathcal{L}_1^{m*}(\mathcal{L}_1-\lambda)^n(\mathcal{L}_1-\lambda)^*\mathcal{L}_1^m\\ &\le\beta^2\mathcal{L}_1^{m*}(\mathcal{L}_1-\lambda)^*(\mathcal{L}_1-\lambda)^n\mathcal{L}_1^m.\end{aligned}$$

So, $\mathcal{L}_1=\mathcal{L}|\mathcal{M}$ is m-quasi-n-power-totally-(α,β)-normal. □

Theorem 2.3. [19] *Let $\mathcal{L}\in\mathscr{B}(\mathscr{H})$ such that $\mathcal{L}^m$ does not have a dense range, then the following statements are equivalent.*

(1) *$\mathcal{L}$ is a m-quasi-totally-(α,β)-normal operator.*

(2) $\mathcal{L}=\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ 0 & \mathcal{C}\end{pmatrix}$ *on* $\mathscr{H}=\overline{ran(\mathcal{L}^m)}\oplus ker(\mathcal{L}^{*m})$, *where* $\mathcal{A}=\mathcal{L}_{|\overline{ran(\mathcal{L})}}$ *is a totally (α,β)-normal operator, $\mathcal{B}\mathcal{B}^*=0$ and $\mathcal{C}^m=0$. Furthermore $\sigma(\mathcal{L})=\sigma(\mathcal{A})\cup\{0\}$.*

The next theorem give an equivalent condition for $\mathcal{L}$ to be a m-quasi-n-power-totally-(α,β)-normal operator.

Theorem 2.4. *Let $\mathcal{L}\in\mathscr{B}(\mathscr{H})$ such that $\mathcal{L}^m$ does not have a dense range, then the following are equivalent.*

(1) *$\mathcal{L}$ is a m-quasi-n-power-totally-(α,β)-normal operator.*

(2) $\mathcal{L}=\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ 0 & \mathcal{C}\end{pmatrix}$ *on* $\mathscr{H}=\overline{ran(\mathcal{L}^m)}\oplus ker(\mathcal{L}^{m*})$, *where* $\mathcal{A}=\mathcal{L}_{|\overline{ran(\mathcal{L}^m)}}$ *satisfies*

$$\begin{aligned}\alpha^2(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n&\le(\mathcal{A}-\lambda)^n(\mathcal{A}-\lambda)^*+\left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right)\mathcal{B}^*\\ &\le\beta^2(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n,\end{aligned}$$

for all $\lambda\in\mathbb{C}$ and $\mathcal{C}^m=0$. Furthermore $\sigma(\mathcal{L})=\sigma(\mathcal{A})\cup\{0\}$.

Proof. (1) $\Rightarrow$ (2). Consider the matrix representation of $\mathcal{L}$ with respect to the decomposition $\mathscr{H}=\overline{ran(\mathcal{L}^m)}\oplus ker(\mathcal{L}^{*m})$: $\mathcal{L}=\begin{pmatrix}\mathcal{A} & \mathcal{B}\\ 0 & \mathcal{C}\end{pmatrix}$. Let $\mathcal{P}$ be the projection onto $\overline{ran(\mathcal{L}^m)}$. Then $\begin{pmatrix}\mathcal{A} & 0\\ 0 & 0\end{pmatrix}=\mathcal{L}\mathcal{P}=\mathcal{P}\mathcal{L}\mathcal{P}$. Since $\mathcal{L}$ is m-quasi-n-power-totally-(α,β)-normal operator, we have then

$$\alpha^2\mathcal{P}\Big(\mathcal{L}^{*m}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m\Big)\mathcal{P}\le\mathcal{P}\Big(\mathcal{L}^{*m}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m\Big)\mathcal{P}$$

$$\leq \beta^2 \mathcal{P}\Big(\mathcal{L}^{*m}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m\Big)\mathcal{P}$$

That is

$$\begin{aligned}\alpha^2(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n \leq (\mathcal{A}-\lambda)^n(\mathcal{A}-\lambda)^* + \left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right)\mathcal{B}^*\\ \leq \beta^2(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n,\end{aligned}$$

for all $\lambda \in \mathbb{C}$.

However, let $x = x_1 + x_2 \in \mathscr{H} = \overline{ran(\mathcal{L}^m)} \oplus ker(\mathcal{L}^{*m})$. A simple computation shows that

$$\begin{aligned}\langle \mathcal{C}^m x_2, x_2\rangle &= \langle \mathcal{L}^m(I-\mathcal{P})x, (I-\mathcal{P})x\rangle\\ &= \langle (I-\mathcal{P})x, \mathcal{L}^{*m}(I-\mathcal{P})x\rangle = 0.\end{aligned}$$

So, $\mathcal{C}^m = 0$.

Since $\sigma(\mathcal{L}) \cup \mathcal{T} = \sigma(\mathcal{A}) \cup \sigma(\mathcal{C})$, where $\mathcal{T}$ is the union of the holes in $\sigma(\mathcal{L})$ which are, in fact, a subset of $\sigma(\mathcal{A}) \cap \sigma(\mathcal{C})$ by Corollary 7 of [12], and $\sigma(\mathcal{A}) \cap \sigma(\mathcal{C})$ has no interior point and $\mathcal{C}$ is nilpotent, we have $\sigma(\mathcal{L}) = \sigma(\mathcal{A}) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $\mathcal{L} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{C}\end{pmatrix}$ onto $\mathscr{H} = \overline{ran(\mathcal{L}^m)} \oplus ker(\mathcal{L}^{*m})$, with

$$\alpha^2(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n \leq (\mathcal{A}-\lambda)^n(\mathcal{A}-\lambda)^* + \left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right)\mathcal{B}^*$$

$$\leq \beta^2(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n,$$

for all $\lambda \in \mathbb{C}$ and $\mathcal{C}^m = 0$.

Since $\mathcal{L}^m = \begin{pmatrix} \mathcal{A}^m & \mathcal{Y} \\ 0 & 0\end{pmatrix}$, $\mathcal{Y} = \sum_{j=0}^{m-1}\mathcal{A}^j\mathcal{B}\mathcal{C}^{m-1-j}$,

$(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n$

$$= \begin{pmatrix} (\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n & (\mathcal{A}-\lambda)^*\left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right) \\ \mathcal{B}^*(\mathcal{A}-\lambda)^n & \mathcal{B}^*\left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right) + (\mathcal{C}-\lambda)^*(\mathcal{C}-\lambda)^n\end{pmatrix}$$

and

$(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*$

$$= \begin{pmatrix} (\mathcal{A}-\lambda)^n(\mathcal{A}-\lambda)^* + \left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right)\mathcal{B}^* & \left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right)(\mathcal{C}-\lambda)^* \\ (\mathcal{C}-\lambda)^n\mathcal{B}^* & (\mathcal{C}-\lambda)^n(\mathcal{C}-\lambda)^*\end{pmatrix}.$$

Further

$$\begin{aligned}\mathcal{L}^m\mathcal{L}^{*m} &= \begin{pmatrix} \mathcal{A}^m\mathcal{A}^{*m} + \mathcal{Y}\mathcal{Y}^* & 0 \\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix} \mathcal{D} & 0 \\ 0 & 0\end{pmatrix},\end{aligned}$$

where $\mathcal{D} = \mathcal{A}^m\mathcal{A}^{*m} + \mathcal{Y}\mathcal{Y}^* = \mathcal{D}^*$.

Hence for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
&\alpha^2\mathcal{L}^m\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)\mathcal{L}^m\mathcal{L}^{*m}\\
=&\begin{pmatrix} \alpha^2\mathcal{D}(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n\mathcal{D} & 0\\ \\ 0 & 0\end{pmatrix}\\
\leq&\begin{pmatrix} \mathcal{D}\left((\mathcal{A}-\lambda)^n(\mathcal{A}-\lambda)^* + \left(\sum_{j=0}^{n-1}(\mathcal{A}-\lambda)^j\mathcal{B}(\mathcal{C}-\lambda)^{n-1-j}\right)\mathcal{B}^*\right)\mathcal{D} & 0\\ 0 & 0\end{pmatrix}\\
=&\mathcal{L}^m\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\big)\mathcal{L}^m\mathcal{L}^{*m}\\
\leq&\begin{pmatrix} \beta^2\mathcal{D}(\mathcal{A}-\lambda)^*(\mathcal{A}-\lambda)^n\mathcal{D} & 0\\ 0 & 0\end{pmatrix}\\
=&\beta^2\mathcal{L}^m\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)\mathcal{L}^m\mathcal{L}^{*m}.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\alpha^2\mathcal{L}^m\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)\mathcal{L}^m\mathcal{L}^{*m}\\
&\leq \mathcal{L}^m\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\big)\mathcal{L}^m\mathcal{L}^{*m}\\
&\leq \beta^2\mathcal{L}^m\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)\mathcal{L}^m\mathcal{L}^{*m}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\alpha^2\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)\mathcal{L}^m &\leq \mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\big)\mathcal{L}^m\\
&\leq \beta^2\mathcal{L}^{*m}\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)\mathcal{L}^m,
\end{aligned}$$

on $\mathscr{H} = \overline{ran(\mathcal{L}^{*m})} \oplus ker(\mathcal{L}^m)$.
This leads to the conclusion that, $\mathcal{L}$ is m-quasi-n-power-totally-(α,β)-normal. □

Theorem 2.5. *Let* $\mathcal{L} = \begin{pmatrix} \mathcal{A} & \mathcal{B}\\ 0 & \mathcal{C}\end{pmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$. *If* $\mathcal{A}$ *is a surjective* m*-quasi-*n*-power-*(α,β)*-normal operator and* $\mathcal{C}^m = 0$ *for some integer* m*, then* $\mathcal{L}$ *is similar to a* m*-quasi-*n*-power-*(α,β)*-normal operator.*

Proof. Under the condition $\mathcal{A}$ is surjective and $\mathcal{C}$ is nilpotent, we have $\sigma(\mathcal{A})\cap\sigma(\mathcal{C}) = \phi$. Then there exists an operator X such that $\mathcal{A}X - X\mathcal{C} = \mathcal{B}$. Since

$$\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}\begin{pmatrix} \mathcal{A} & \mathcal{B}\\ 0 & \mathcal{C}\end{pmatrix} = \begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}\begin{pmatrix} I & X\\ 0 & I\end{pmatrix},$$

it is clear that $\mathcal{L}$ is similar to $\mathcal{T} = \begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}$. Let $\mathcal{A}$ be m-quasi-n-power-(α,β)-normal operator and $\mathcal{C}^m = 0$. Then

$$\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m \leq \mathcal{T}^{m*}\mathcal{T}^n(\mathcal{T}-\lambda)^*\mathcal{T}^m \leq \beta^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m.$$

$$\begin{aligned}
&\alpha^2\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{m*}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{*}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{n}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{m}\\
&\leq \begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{m*}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{n}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{*}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{m}\\
&\leq \beta^2\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{m*}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{*}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{n}\begin{pmatrix} \mathcal{A} & 0\\ 0 & \mathcal{C}\end{pmatrix}^{m}.
\end{aligned}$$

We have

$$\alpha^2\begin{pmatrix} \mathcal{A}^{m*}\mathcal{A}^*\mathcal{A}^n\mathcal{A}^m & 0\\ 0 & 0\end{pmatrix} \leq \begin{pmatrix} \mathcal{A}^{m*}\mathcal{A}^n\mathcal{A}^*\mathcal{A}^m & 0\\ 0 & 0\end{pmatrix}$$

$$\leq \beta^2 \begin{pmatrix} \mathcal{A}^{m*}\mathcal{A}^*\mathcal{A}^n\mathcal{A}^m & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\mathcal{L}$ is similar to a m-quasi-n-power-totally-(α,β)-normal operator □

In the following results, direct sum and tensor product for two m-quasi-n-power-(α,β)-normal operator are studied.

Theorem 2.6. *If $\mathcal{L},\mathcal{T}$ are m-quasi-n-power-(α,β)-normal operator, then $\mathcal{L}\oplus\mathcal{T}$ is m-quasi-n-power-(α,β)-normal operator.*

Proof. Consider,

$$\begin{aligned}
&\big[(\mathcal{L}\oplus\mathcal{T})^{m*}(\mathcal{L}\oplus\mathcal{T})^{n}(\mathcal{L}\oplus\mathcal{T})^{*}(\mathcal{L}\oplus\mathcal{T})^{m}\big](x_1\oplus x_2)\\
&\qquad= \big[(\mathcal{L}^{m*}\oplus\mathcal{T}^{m*})(\mathcal{L}^{n}\oplus\mathcal{T}^{n})(\mathcal{L}^{*}\oplus\mathcal{T}^{*})(\mathcal{L}^{m}\oplus\mathcal{T}^{m})\big](x_1\oplus x_2)\\
&\qquad= (\mathcal{L}^{m*}\mathcal{L}^{n}\mathcal{L}^{*}\mathcal{L}^{m}\oplus\mathcal{T}^{m*}\mathcal{T}^{n}\mathcal{T}^{*}\mathcal{T}^{m})(x_1\oplus x_2)\\
&\qquad= (\mathcal{L}^{m*}\mathcal{L}^{n}\mathcal{L}^{*}\mathcal{L}^{m}x_1)\oplus(\mathcal{T}^{m*}\mathcal{T}^{n}\mathcal{T}^{*}\mathcal{T}^{m}x_2)\\
&\qquad\geq (\alpha^2\mathcal{L}^{m*}\mathcal{L}^{*}\mathcal{L}^{n}\mathcal{L}^{m}x_1)\oplus(\alpha^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}^{n}\mathcal{T}^{m}x_2)\\
&\qquad= \alpha^2\big[(\mathcal{L}^{m*}\mathcal{L}^{*}\mathcal{L}^{n}\mathcal{L}^{m}\oplus\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}^{n}\mathcal{T}^{m})\big](x_1\oplus x_2)\\
&\qquad= \big[\alpha^2(\mathcal{L}\oplus\mathcal{T})^{m*}(\mathcal{L}\oplus\mathcal{L})^{*}(\mathcal{L}\oplus\mathcal{T})^{n}(\mathcal{L}\oplus\mathcal{T})^{m}\big](x_1\oplus x_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\big[(\mathcal{L}\oplus\mathcal{T})^{m*}(\mathcal{L}\oplus\mathcal{T})^{n}(\mathcal{L}\oplus\mathcal{T})^{*}(\mathcal{L}\oplus\mathcal{T})^{m}(x_1\oplus x_2)\big]\\
&\qquad\leq \big[\beta^2(\mathcal{L}\oplus\mathcal{T})^{m*}(\mathcal{L}\oplus\mathcal{T})^{*}(\mathcal{L}\oplus\mathcal{T})^{n}(\mathcal{L}\oplus\mathcal{T})^{m}\big](x_1\oplus x_2).
\end{aligned}$$

Hence $\mathcal{L}\oplus\mathcal{T}$ is m-quasi-n-power-(α,β)-normal operator. □

Theorem 2.7. *Let $\mathcal{L},\mathcal{T}\in\mathcal{B}(\mathscr{H})$ and let $(0\leq\alpha_1,\alpha_2\leq 1\leq\beta_1,\beta_2)$. The following conditions hold.*

1. *If $\mathcal{L}$ is m-quasi-n-power-(α_1,β_1)-normal operator and $\mathcal{T}$ is m-quasi-n-power-(α_2,β_2)-normal operator, then $\mathcal{L}\otimes\mathcal{T}$ is m-quasi-n-power-$(\alpha_1\alpha_2,\beta_1\beta_2)$-normal operator.*
2. *If $\mathcal{L}\otimes\mathcal{T}$ is m-quasi-n-power-(α,β)-normal operator, then there exists two constants $c_1\geq 0$ and $c_2\geq 0$ such that $\mathcal{L}$ is m-quasi-n-power-$\big(\alpha\sqrt{\frac{1}{c_2}},\beta\sqrt{c_1}\big)$-normal operator and $\mathcal{T}$ is m-quasi-n-power-$\big(\sqrt{c_2},\sqrt{\frac{1}{c_1}}\big)$-normal operator.*

Proof. (i) Since $\mathcal{L}$ is m-quasi-n-power-(α_1,β_1)-normal operator and $\mathcal{T}$ is m-quasi-n-power-(α_2,β_2)-normal operator, we have

$$\alpha_1^2\mathcal{L}^{m*}\mathcal{L}^{*}\mathcal{L}^{n}\mathcal{L}^{m}\leq\mathcal{L}^{m*}\mathcal{L}^{n}\mathcal{L}^{*}\mathcal{L}^{m}\leq\beta_1^2\mathcal{L}^{m*}\mathcal{L}^{*}\mathcal{L}^{n}\mathcal{L}^{m}$$

and

$$\alpha_2^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}^{n}\mathcal{T}^{m}\leq\mathcal{T}^{m*}\mathcal{T}^{n}\mathcal{T}^{*}\mathcal{T}^{m}\leq\beta_2^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}^{n}\mathcal{T}^{m}.$$

Consider,

$$\begin{aligned}
&\big(\beta_1\beta_2\big)^2(\mathcal{L}\otimes\mathcal{T})^{*m}(\mathcal{L}\otimes\mathcal{T})^{*}(\mathcal{L}\otimes\mathcal{T})^{n}(\mathcal{L}\otimes\mathcal{T})^{m}\\
&\qquad= \beta_1^2\beta_2^2(\mathcal{L}^{*m}\otimes\mathcal{T}^{m*})(\mathcal{L}^{*}\otimes\mathcal{T}^{*})(\mathcal{L}^{n}\otimes\mathcal{T}^{n})(\mathcal{L}^{m}\otimes\mathcal{T}^{m})\\
&\qquad= \big(\beta_1\big)^2\mathcal{L}^{*m}\mathcal{L}^{*}\mathcal{L}^{n}\mathcal{L}^{m}\otimes\big(\beta_2\big)^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}^{n}\mathcal{T}^{m}\\
&\qquad\geq \mathcal{L}^{m*}\mathcal{L}^{n}\mathcal{L}^{*}\mathcal{L}^{m}\otimes\mathcal{T}^{m*}\mathcal{T}^{n}\mathcal{T}^{*}\mathcal{T}^{m}\\
&\qquad= (\mathcal{L}^{m*}\otimes\mathcal{T}^{m*})(\mathcal{L}^{n}\otimes\mathcal{T}^{n})(\mathcal{L}^{*}\otimes\mathcal{T}^{*})(\mathcal{L}^{m}\otimes\mathcal{T}^{m})\\
&\qquad= (\mathcal{L}\otimes\mathcal{T})^{m*}(\mathcal{L}\otimes\mathcal{T})^{n}(\mathcal{L}\otimes\mathcal{T})^{*}(\mathcal{L}\otimes\mathcal{T})^{m}.
\end{aligned}$$

Hence

$$\big(\beta_1\beta_2\big)^2(\mathcal{L}\otimes\mathcal{T})^{*m}(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^m \geq (\mathcal{L}\otimes\mathcal{T})^{m*}(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^m.$$

Secondly,

$$\begin{aligned}
(\mathcal{L}\otimes\mathcal{T})^{m*}(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^m \\
&= (\mathcal{L}^{m*}\otimes\mathcal{T}^{m*})(\mathcal{L}^n\otimes\mathcal{T}^n)(\mathcal{L}^*\otimes\mathcal{T}^*)(\mathcal{L}^m\otimes\mathcal{T}^m)\\
&= \mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m\otimes\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^m\\
&\geq \big(\alpha_1\big)^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\otimes\big(\alpha_2\big)^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m\\
&= \big(\alpha_1\big)^2\big(\alpha_2\big)^2(\mathcal{L}^{*m}\otimes\mathcal{T}^{m*})(\mathcal{L}^*\otimes\mathcal{T}^*)(\mathcal{L}^n\otimes\mathcal{T}^n)(\mathcal{L}^m\otimes\mathcal{T}^m)\\
&= \big(\alpha_1\alpha_2\big)^2(\mathcal{L}\otimes\mathcal{T})^{*m}(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^m.
\end{aligned}$$

Hence

$$(\mathcal{L}\otimes\mathcal{T})^{m*}(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^m \geq \big(\alpha_1\alpha_2\big)^2(\mathcal{L}\otimes\mathcal{T})^{*m}(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^m.$$

Therefore, $\mathcal{L}\otimes\mathcal{T}$ is m-quasi-n-power-$(\alpha_1\alpha_2,\beta_1\beta_2)$-normal operator.

(ii) We have

$$\begin{aligned}
\beta^2(\mathcal{L}\otimes\mathcal{T})^{*m}(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^m &\geq (\mathcal{L}\otimes\mathcal{T})^{m*}(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^m\\
&\geq \alpha^2(\mathcal{L}\otimes\mathcal{T})^{*m}(\mathcal{L}\otimes\mathcal{T})^*(\mathcal{L}\otimes\mathcal{T})^n(\mathcal{L}\otimes\mathcal{T})^m.
\end{aligned}$$

So,

$$\beta^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\otimes\mathcal{T}^{*m}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m \geq \mathcal{L}^{*m}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m\otimes\mathcal{T}^{*m}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^m$$

and

$$\mathcal{L}^{*m}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m\otimes\mathcal{T}^{*m}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^m \geq \alpha^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\otimes\mathcal{T}^{*m}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m.$$

So there exists a constant $c_1>0$ such that

$$c_1\beta^2\mathcal{L}^{*m+1}\mathcal{L}^{m+n} \geq \mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m$$

and

$$\frac{1}{c_1}\mathcal{T}^{*m+1}\mathcal{T}^{m+n} \geq \mathcal{T}^{*m}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m.$$

On the other hand we can find a constant $c_2>0$ satisfies

$$c_2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m \geq \alpha^2\mathcal{L}^{*m+1}\mathcal{L}^{m+n}$$

and

$$\frac{1}{c_2}\mathcal{T}^{*m}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m \geq \mathcal{T}^{*m+1}\mathcal{T}^{m+n}.$$

It is straightforward to see that

$$0 \;\leq\; \alpha\sqrt{\frac{1}{c_2}},\ \sqrt{c_2} \;\leq\; 1 \;\leq\; \beta\sqrt{c_1},\ \sqrt{\frac{1}{c_1}}.$$

Therefore, $\mathcal{L}$ is m-quasi-n-power-$\big(\alpha\sqrt{\frac{1}{c_2}},\beta\sqrt{c_1}\big)$-normal operator and $\mathcal{T}$ is m-quasi-n-power-$\big(\sqrt{c_2},\sqrt{\frac{1}{c_1}}\big)$-normal operator. □

In the following theorem, the stability of the sum of two m-quasi$-n$-power-totally-(α,β)-normal operators is preserved under the specific conditions.

Theorem 2.8. *Let $\mathcal{L},\mathcal{T}\in\mathscr{B}(\mathscr{H})$. If $\mathcal{L},\mathcal{T}$ are m-quasi-n-power-totally-(α,β)-normal operator satisfies the following conditions;*

- $(\mathcal{L}-\lambda)\mathcal{T} = (\mathcal{T}-\lambda)\mathcal{L} = 0$
- $\mathcal{T}^*(\mathcal{L}-\lambda) = \mathcal{L}^*(\mathcal{T}-\lambda) = 0$

- $(\mathcal{L}-\lambda)(\mathcal{T}-\lambda)^* = (\mathcal{L}-\lambda)^*(\mathcal{T}-\lambda) = 0$
- $\mathcal{L}\mathcal{T} = \mathcal{T}\mathcal{L} = 0$
- $(\mathcal{L}-\lambda)(\mathcal{T}-\lambda) = (\mathcal{T}-\lambda)(\mathcal{L}-\lambda) = 0$

Then $\mathcal{L}+\mathcal{T}$ is m-quasi-n-power-totally-(α,β)-normal operator.

Proof. If $\mathcal{L},\mathcal{T}$ are m-quasi-n-power-totally-(α,β)-normal operator, then we have

$$\begin{aligned}\alpha^2\mathcal{L}^{*m}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m &\le \mathcal{L}^{*m}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m\\ &\le \beta^2\mathcal{L}^{*m}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m,\end{aligned}$$

$$\begin{aligned}\alpha^2(\mathcal{T}^{*m})(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)^n(\mathcal{T}^m) &\le (\mathcal{T}^{*m})(\mathcal{T}-\lambda)^n(\mathcal{T}-\lambda)^*(\mathcal{T}^m)\\ &\le \beta^2(\mathcal{T}^{*m})(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)^n(\mathcal{T}^m)\end{aligned}$$

for all $\lambda\in\mathbb{C}$.

To show that $\mathcal{L}+\mathcal{T}$ is m-quasi-n-power-totally-(α,β)-normal operator.
First we have,
$\big((\mathcal{L}+\mathcal{T})^m\big)^*\big[\alpha^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)$
$-\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big]\big((\mathcal{L}+\mathcal{T})^m\big)$

$$\begin{aligned}
&= (\mathcal{L}^{m*}+\mathcal{T}^{m*})\big[\alpha^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\quad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big](\mathcal{L}^m+\mathcal{T}^m)\\
&= \mathcal{L}^{m*}\big[\alpha^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\quad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big]\mathcal{L}^m\\
&\quad +\mathcal{L}^{m*}\big[\alpha^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\quad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big](\mathcal{T}^m)\\
&\quad +(\mathcal{T}^{m*})\big[\alpha^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\quad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big]\mathcal{L}^m\\
&\quad +(\mathcal{T}^{m*})\big[\alpha^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\quad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big](\mathcal{T}^m)\\
&= \mathcal{L}^{m*}\big[\alpha^2\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)-\big((\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\big)\big]\mathcal{L}^m\\
&\quad +(\mathcal{T}^{m*})\big[\alpha^2\big((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)^n\big)-\big((\mathcal{T}-\lambda)^n(\mathcal{T}-\lambda)^*\big)\big](\mathcal{T}^m)\\
&\le 0.
\end{aligned}$$

Secondly,
$\big((\mathcal{L}+\mathcal{T})^m\big)^*\big[\beta^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)$

$$
\begin{aligned}
&-\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*)\big]\big((\mathcal{L}+\mathcal{T})^m\big)\\
&\quad= (\mathcal{L}^{m*}+\mathcal{T}^{m*})\big[\beta^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\qquad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*)\big](\mathcal{L}^m+\mathcal{T}^m)\\
&\quad= \mathcal{L}^{m*}\big[\beta^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\qquad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*)\big]\mathcal{L}^m\\
&\qquad +\mathcal{L}^{m*}\big[\beta^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\qquad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*)\big](\mathcal{T}^m)\\
&\qquad +(\mathcal{T}^{m*})\big[\beta^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\qquad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*)\big]\mathcal{L}^m\\
&\qquad +(\mathcal{T}^{m*})\big[\beta^2\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*\big)\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\\
&\qquad -\big((\mathcal{L}-\lambda)^n+(\mathcal{T}-\lambda)^n\big)\big((\mathcal{L}-\lambda)^*+(\mathcal{T}-\lambda)^*)\big](\mathcal{T}^m)\\
&\quad= \mathcal{L}^{m*}\big[\beta^2\big((\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\big)-\big((\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*)\big]\mathcal{L}^m\\
&\qquad +(\mathcal{T}^{m*})\big[\beta^2\big((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)^n\big)-\big((\mathcal{T}-\lambda)^n(\mathcal{T}-\lambda)^*)\big](\mathcal{T}^m)\\
&\quad\geq 0.
\end{aligned}
$$

Therefore $\mathcal{L}+\mathcal{T}$ is m-quasi-n-power-totally-(α, β)-normal operator. □

Theorem 2.9. *Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathscr{B}(\mathscr{H})$ be doubly commuting. If $\mathcal{L}_1$ is an m-quasi-n-power-(α, β)-normal and $\mathcal{L}_2$ is an m-quasi-n-power-(α', β')-normal, then $\mathcal{L}_1\mathcal{L}_2$ is an m-quasi-n-power-$(\alpha\alpha', \beta\beta')$-normal*

Proof. Consider,

$$
\begin{aligned}
(\alpha\alpha')^2(\mathcal{L}_1\mathcal{L}_2)^{m*}(\mathcal{L}_1\mathcal{L}_2)^*(\mathcal{L}_1\mathcal{L}_2)^n(\mathcal{L}_1\mathcal{L}_2)^m &= \alpha^2\alpha'^2\mathcal{L}_1^{m*}\mathcal{L}_1^*\mathcal{L}_1^n\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^*\mathcal{L}_2^n\mathcal{L}_2^m\\
&\leq \alpha^2\mathcal{L}_1^{m*}\mathcal{L}_1^*\mathcal{L}_1^n\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^n\mathcal{L}_2^*\mathcal{L}_2^m\\
&\leq \mathcal{L}_1^{m*}\mathcal{L}_1^n\mathcal{L}_1^*\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^n\mathcal{L}_2^*\mathcal{L}_2^m\\
&= (\mathcal{L}_1\mathcal{L}_2)^{m*}(\mathcal{L}_1\mathcal{L}_2)^n(\mathcal{L}_1\mathcal{L}_2)^*(\mathcal{L}_1\mathcal{L}_2)^m
\end{aligned}
$$

,

and

$$
\begin{aligned}
(\mathcal{L}_1\mathcal{L}_2)^{m*}(\mathcal{L}_1\mathcal{L}_2)^n(\mathcal{L}_1\mathcal{L}_2)^*(\mathcal{L}_1\mathcal{L}_2)^m &= \mathcal{L}_1^{m*}\mathcal{L}_1^n\mathcal{L}_1^*\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^n\mathcal{L}_2^*\mathcal{L}_2^m\\
&\leq \beta^2\mathcal{L}_1^{m*}\mathcal{L}_1^*\mathcal{L}_1^n\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^n\mathcal{L}_2^*\mathcal{L}_2^m\\
&\leq \beta^2\beta'^2\mathcal{L}_1^{m*}\mathcal{L}_1^*\mathcal{L}_1^n\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^*\mathcal{L}_2^n\mathcal{L}_2^m\\
&= \beta^2\beta'^2\mathcal{L}_1^{m*}\mathcal{L}_1^*\mathcal{L}_1^n\mathcal{L}_1^m\mathcal{L}_2^{m*}\mathcal{L}_2^*\mathcal{L}_2^n\mathcal{L}_2^m\\
&= (\beta\beta')^2(\mathcal{L}_1\mathcal{L}_2)^{m*}(\mathcal{L}_1\mathcal{L}_2)^*(\mathcal{L}_1\mathcal{L}_2)^n(\mathcal{L}_1\mathcal{L}_2)^m.
\end{aligned}
$$

□

Theorem 2.10. *Let $\mathcal{L}$ be an m-quasi-n-power-totally-(α, β)-normal operator. If $\mathcal{L}^m$ has dense range, then $\mathcal{L}$ is totally-n-power-(α, β)-normal.*

Proof. Since $\mathcal{L}^m$ has a dense range, it follows that $\overline{ran(\mathcal{L}^m)} = \mathscr{H}$. Let $y \in \mathscr{H}$. Then there exists a sequence (x_n) in $\mathscr{H}$ such that $\mathcal{L}^m(x_n) \to y$ as $n \to \infty$.
Since $\mathcal{L}$ is m-quasi-n-power-totally-(α, β)-normal operator, we have

$$\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m \leq \mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m \leq \beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m$$
$$\left\langle\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x,x\right\rangle \leq \left\langle\mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m x,x\right\rangle$$
$$\leq \left\langle\beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x,x\right\rangle, \text{ for every } x\in\mathscr{H}.$$

In particular,

$$\left\langle\alpha^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x_n,x_n\right\rangle \leq \left\langle\mathcal{L}^{m*}(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*\mathcal{L}^m x_n,x_n\right\rangle$$
$$\leq \left\langle\beta^2\mathcal{L}^{m*}(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n\mathcal{L}^m x_n,x_n\right\rangle, \quad \text{for every } \ x_n\in\mathscr{H}.$$

It follows that

$$\left\langle\alpha^2(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n y,y\right\rangle \leq \left\langle(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^* y,y\right\rangle \leq \left\langle\beta^2(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n y,y\right\rangle,$$

for all $y\in\mathscr{H}$ and for all $\lambda\in\mathbb{C}$. Therefore $\mathcal{L}$ is totally-n-power-$(\alpha,\beta)-$ normal operator. □

Corollary 2.11. *Let $\mathcal{L}$ be an m-quasi-n-power-totally-(α,β)-normal operator. If $\mathcal{L}^m\neq 0$ and if $\mathcal{L}$ has no nontrivial $\mathcal{L}^m$-invariant closed subspace, then $\mathcal{L}$ is totally-n-power-(α,β)-normal.*

Proof. Since $\mathcal{L}^m$ has no nontrivial invariant closed subspace, it has no nontrivial hyperinvariant subspace. But $ker(\mathcal{L}^m)$ and $\overline{ran(\mathcal{L}^m)}$ are hyperinvariant subspaces, and $\mathcal{L}^m\neq 0$, hence $ker(\mathcal{L}^m)=0$ and $\overline{ran(\mathcal{L}^m)}=\mathscr{H}$. Therefore $\mathcal{L}$ is totally-n-power(α,β)-normal operator. □

Corollary 2.12. *If $\mathcal{L}$ is such that $a+b\mathcal{L}$ is m-quasi-n-power-totally-(α,β)-normal operator for all scalars a and b, then $\mathcal{L}$ is totally-n-power-(α,β)-normal.*

Proof. If $\mathcal{L}$ is m-quasi-n-power-totally-(α,β)-normal operator but not totally-n-power-(α,β)-normal operator, then $\mathcal{L}^m$ is not invertible. It is possible to find scalars a and $b\neq 0$ such that $\mathcal{T}=a+b\mathcal{L}$ is invertible m-quasi-n-power-totally-(α,β)-normal operator. Therefore $\mathcal{T}$ is totally-n-power-(α,β)-normal operators.

$$\mathcal{T}=a+b\mathcal{L}\Rightarrow\mathcal{L}=\frac{1}{b}(\mathcal{T}-a).$$

Therefore $\mathcal{L}$ is also totally-n-power-(α,β)-normal. □

Proposition 2.13. *Let $\mathcal{L}$ be an m-quasi-n-power-totally-(α,β)-normal operator. If a,b are non-zero eigenvalues of $\mathcal{L}$ such that $a\neq b$, then $ker\,(\mathcal{L}-a)\perp ker\,(\mathcal{L}-b)$.*

Proof. Let $x\in ker(\mathcal{L}-a)$ and $y\in ker(\mathcal{L}-b)$. Then $\mathcal{L}x=ax$ and $\mathcal{L}y=by$. Therefore $a<x,y>=b<x,y>$, and so $(a-b)<x,y>=0$. Hence $ker\,(\mathcal{L}-a)\perp ker\,(\mathcal{L}-b)$. □

Theorem 2.14. *The set $\{\mathcal{L}\in\mathscr{B}(\mathscr{H}):\alpha^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\leq\mathcal{L}^{*m}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m\leq\beta^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m$ and $(0\leq\alpha\leq 1\leq\beta)\}$ is arcwise connected for $m,n\in\mathbb{N}$.*

Proof. It is enough to prove that $\zeta\mathcal{L}$ is m-quasi-n-power-(α,β)-normal operator for every non zero complex number ζ. Now for $x\in\mathscr{H}$,

$$\left\langle\alpha^2(\zeta\mathcal{L})^{*m}(\zeta\mathcal{L})^*(\zeta\mathcal{L})^n(\zeta\mathcal{L})^m x,x,\right\rangle \leq \langle(\zeta\mathcal{L})^{*m}(\zeta\mathcal{L})^n(\zeta\mathcal{L})^*(\zeta\mathcal{L})^m x,x\rangle$$
$$\leq \left\langle\beta^2(\zeta\mathcal{L})^{*m}(\zeta\mathcal{L})^*(\zeta\mathcal{L})^n(\zeta\mathcal{L})^m x,x\right\rangle.$$

Hence

$$\begin{aligned} |\zeta|^{2m}\overline{\zeta}\zeta^n \left\langle \alpha^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m x, x, \right\rangle &\leq |\zeta|^{2m}\zeta^n\overline{\zeta} \left\langle \mathcal{L}^{*m}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m x, x\right\rangle \\ &\leq |\zeta|^{2m}\overline{\zeta}\zeta^n \left\langle \beta^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m x, x\right\rangle. \end{aligned}$$

Therefore $\alpha^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m \leq \mathcal{L}^{*m}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m \leq \beta^2\mathcal{L}^{*m}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m$.
This implies that the class of m-quasi-n-power-(α, β)-normal operator is arcwise connected. □

Theorem 2.15. *Let $\mathcal{L}$ be an m-quasi-n-power-totally-(α, β)-normal operator. If k is a complex number, then ker $(\mathcal{L} - k) \subseteq$ ker $(\mathcal{L}^{*m}(\mathcal{L} - k)^n(\mathcal{L} - k)^*\mathcal{L}^m)$ for each $k \neq 0$.*

Proof. Suppose $\mathcal{L}x = kx$. Since $\mathcal{L}$ is m-quasi-n-power-totally-(α, β)-normal,

$$\begin{aligned} \left\langle \alpha^2\mathcal{L}^{*m}(\mathcal{L} - \lambda)^*(\mathcal{L} - \lambda)^n\mathcal{L}^m x, x, \right\rangle &\leq \left\langle \mathcal{L}^{*m}(\mathcal{L} - \lambda)^n(\mathcal{L} - \lambda)^*\mathcal{L}^m x, x\right\rangle \\ &\leq \left\langle \beta^2\mathcal{L}^{*m}(\mathcal{L} - \lambda)^*(\mathcal{L} - \lambda)^n\mathcal{L}^m x, x\right\rangle \end{aligned}$$

for all $x \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$. In particular,

$$\begin{aligned} \left\langle \alpha^2\mathcal{L}^{*m}(\mathcal{L} - k)^*(\mathcal{L} - k)^n\mathcal{L}^m x, x, \right\rangle &\leq \left\langle \mathcal{L}^{*m}(\mathcal{L} - k)^n(\mathcal{L} - k)^*\mathcal{L}^m x, x\right\rangle \\ &\leq \left\langle \beta^2\mathcal{L}^{*m}(\mathcal{L} - k)^*(\mathcal{L} - k)^n\mathcal{L}^m x, x\right\rangle. \end{aligned}$$

This clearly forces $x \in$ *ker* $(\mathcal{L}^{*m}(\mathcal{L} - k)^n(\mathcal{L} - k)^*\mathcal{L}^m)$. □

Theorem 2.16. *Let $\mathcal{L} \in \mathscr{B}(\mathscr{H})$ and $\mathcal{N} \in \mathscr{B}(\mathscr{H})$ be an invertible operator such that $\mathcal{N}^*\mathcal{N}$ commutes with $\mathcal{L}$. Then $\mathcal{L}$ is m-quasi-totally-n-power-(α, β)-normal operator if and only if $\mathcal{N}\mathcal{L}\mathcal{N}^{-1}$ is m-quasi-n-power-totally-(α, β)-normal operator.*

Proof. Assume that $\mathcal{L}$ is m-quasi-n-power-totally-(α, β)-normal operator.
Consider,

$$\begin{aligned} &\alpha^2((\mathcal{N}\mathcal{L}\mathcal{N}^{-1})^m)^*(\mathcal{N}\mathcal{L}\mathcal{N}^{-1} - \lambda)^*(\mathcal{N}\mathcal{L}\mathcal{N}^{-1} - \lambda)^n((\mathcal{N}\mathcal{L}\mathcal{N}^{-1})^m) \\ &\leq ((\mathcal{N}\mathcal{L}\mathcal{N}^{-1})^m)^*(\mathcal{N}\mathcal{L}\mathcal{N}^{-1} - \lambda)^n(\mathcal{N}\mathcal{L}\mathcal{N}^{-1} - \lambda)^*((\mathcal{N}\mathcal{L}\mathcal{N}^{-1})^m) \\ &\leq \beta^2((\mathcal{N}\mathcal{L}\mathcal{N}^{-1})^m)^*(\mathcal{N}\mathcal{L}\mathcal{N}^{-1} - \lambda)^*(\mathcal{N}\mathcal{L}\mathcal{N}^{-1} - \lambda)^n((\mathcal{N}\mathcal{L}\mathcal{N}^{-1})^m). \end{aligned}$$

We have

$$\begin{aligned} &\alpha^2(\mathcal{N}^{-1})^*(\mathcal{L}^m)^*\mathcal{N}^*((\mathcal{N}^{-1})^*(\mathcal{L} - \lambda)^*\mathcal{N}^*)(\mathcal{N}(\mathcal{L} - \lambda)^n\mathcal{N}^{-1})\mathcal{N}(\mathcal{L}^m)\mathcal{N}^{-1} \\ &\leq (\mathcal{N}^{-1})^*(\mathcal{L}^m)^*\mathcal{N}^*(\mathcal{N}(\mathcal{L} - \lambda)^n\mathcal{N}^{-1})((\mathcal{N}^{-1})^*(\mathcal{L} - \lambda)^*\mathcal{N}^*)\mathcal{N}(\mathcal{L}^m)\mathcal{N}^{-1} \\ &\leq \beta^2(\mathcal{N}^{-1})^*(\mathcal{L}^m)^*\mathcal{N}^*((\mathcal{N}^{-1})^*(\mathcal{L} - \lambda)^*\mathcal{N}^*)(\mathcal{N}(\mathcal{L} - \lambda)^n\mathcal{N}^{-1})\mathcal{N}(\mathcal{L}^m)\mathcal{N}^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha^2(\mathcal{N}^{-1})^*(\mathcal{L}^m)^*(\mathcal{L} - \lambda)^*\mathcal{N}^*\mathcal{N}(\mathcal{L} - \lambda)^n(\mathcal{L}^m)\mathcal{N}^{-1} \\ &\leq (\mathcal{N}^{-1})^*(\mathcal{L}^m)^*(\mathcal{L} - \lambda)^n\mathcal{N}^*\mathcal{N}\mathcal{N}^{-1}(\mathcal{N}^{-1})^*(\mathcal{L} - \lambda)^*\mathcal{N}^*\mathcal{N}(\mathcal{L}^m)\mathcal{N}^{-1} \\ &\leq \beta^2(\mathcal{N}^{-1})^*(\mathcal{L}^m)^*(\mathcal{L} - \lambda)^*\mathcal{N}^*\mathcal{N}(\mathcal{L} - \lambda)^n(\mathcal{L}^m)\mathcal{N}^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} &\alpha^2\mathcal{N}(\mathcal{L}^m)^*(\mathcal{L} - \lambda)^*(\mathcal{L} - \lambda)^n(\mathcal{L}^m)\mathcal{N}^{-1} \\ &\leq \mathcal{N}(\mathcal{L}^m)^*(\mathcal{L} - \lambda)^n(\mathcal{L} - \lambda)^*(\mathcal{L}^m)\mathcal{N}^{-1} \\ &\leq \beta^2\mathcal{N}(\mathcal{L}^m)^*(\mathcal{L} - \lambda)^*(\mathcal{L} - \lambda)^n(\mathcal{L}^m)\mathcal{N}^{-1}. \end{aligned}$$

We have
$$\mathcal{N}\Big(\alpha^2(\mathcal{L}^m)^*(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n(\mathcal{L}^m) \leq (\mathcal{L}^m)^*(\mathcal{L}-\lambda)^n(\mathcal{L}-\lambda)^*(\mathcal{L}^m)$$
$$\leq \beta^2(\mathcal{L}^m)^*(\mathcal{L}-\lambda)^*(\mathcal{L}-\lambda)^n(\mathcal{L}^m)\Big)\mathcal{N}^{-1}.$$
Therefore, $\mathcal{N}\mathcal{L}\mathcal{N}^{-1}$ is m-quasi-n-power-totally-(α,β)-normal operator.
Conversely, assume that $\mathcal{N}\mathcal{L}\mathcal{N}^{-1}$ is m-quasi-n-power-totaly-(α,β)-normal.
Set $\mathcal{T}=\mathcal{N}\mathcal{L}\mathcal{N}^{-1}$. We observe that $\mathcal{T}$ commutes with $(\mathcal{N}^{-1})^*\mathcal{N}^{-1}$ and $\mathcal{N}^{-1}\mathcal{T}\mathcal{N}=\mathcal{L}$. By taking into account the preceding part of the theorem, we have $\mathcal{N}^{-1}\mathcal{T}\mathcal{N}$ is m-quasi-n-power-totaly-(α,β)-normal. □

For $\mathcal{T},\mathcal{L}\in\mathscr{B}(\mathscr{H})$ the operator $\Gamma_{\mathcal{T},\mathcal{L}}$ defined as $\Gamma_{\mathcal{T},\mathcal{L}}:\mathcal{C}_2(\mathscr{H})$ such that $X\to\mathcal{T}X\mathcal{L}\in\mathcal{C}_2(\mathscr{H})$ has been studied in [7].

The following results extends A. Bachir[5, Theorem 9]

Theorem 2.17. *If $\mathcal{T}\in\mathscr{B}(\mathscr{H})$ is m-quasi-n-power-(α,β)-normal operator and $\mathcal{L}$ is normal, then $\Gamma_{\mathcal{T},\mathcal{L}}$ is m-quasi-n-power-(α,β)-normal operator.*

Proof. Here,
$\Gamma_{\mathcal{T},\mathcal{L}}(X)=\mathcal{T}X\mathcal{L}$,
$\Gamma^*_{\mathcal{T},\mathcal{L}}(X)=\mathcal{T}^*X\mathcal{L}^*$,
$\Gamma^m_{\mathcal{T},\mathcal{L}}(X)=\mathcal{T}^mX\mathcal{L}^m$,
$\Gamma^{m*}_{\mathcal{T},\mathcal{L}}(X)=\mathcal{T}^{m*}X\mathcal{L}^{m*}$

First we have,
$\left(\Gamma^{m*}_{\mathcal{T},\mathcal{L}}\Gamma^n_{\mathcal{T},\mathcal{L}}\Gamma^*_{\mathcal{T},\mathcal{L}}\Gamma^m_{\mathcal{T},\mathcal{L}}-\alpha^2\Gamma^{m*}_{\mathcal{T},\mathcal{L}}\Gamma^*_{\mathcal{T},\mathcal{L}}\Gamma^n_{\mathcal{T},\mathcal{L}}\Gamma^m_{\mathcal{T},\mathcal{L}}\right)(X)$

$$\begin{aligned}
&= \mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}-\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m*}\\
&= \mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}-\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}\\
&\quad +\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}-\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m*}\\
&= \left(\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^m-\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m\right)X\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}\\
&\quad +\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\left(\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}-\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m*}\right)\\
&= \left(\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^m-\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m\right)X\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}\\
&\geq 0.
\end{aligned}$$

Secondly,
$\left(\beta^2\Gamma^{m*}_{\mathcal{T},\mathcal{L}}\Gamma^*_{\mathcal{T},\mathcal{L}}\Gamma^n_{\mathcal{T},\mathcal{L}}\Gamma^m_{\mathcal{T},\mathcal{L}}-\Gamma^{m*}_{\mathcal{T},\mathcal{L}}\Gamma^n_{\mathcal{T},\mathcal{L}}\Gamma^*_{\mathcal{T},\mathcal{L}}\Gamma^m_{\mathcal{T},\mathcal{L}}\right)(X)$

$$\begin{aligned}
&= \beta^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m*}-\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m*}\\
&= \beta^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m*}-\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m}\\
&\quad +\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\mathcal{L}^m\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m}-\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\mathcal{L}^{m*}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m}\\
&= \left(\beta^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m-\mathcal{T}^{m}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^{m*}\right)X\mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m}\\
&\quad +\mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^mX\left(\mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m}-\mathcal{L}^{m*}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^{m}\right)\\
&= \left(\beta^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m-\mathcal{T}^{m}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^{m*}\right)X\mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^{m}\\
&\geq 0.
\end{aligned}$$

Hence $\Gamma_{\mathcal{T},\mathcal{L}}$ is m-quasi-n-power-(α,β)-normal operator. □

Theorem 2.18. *If $\mathcal{L}$ m-quasi-n-power-(α,β)-normal operator and if $\mathcal{T}$ is unitary equivalent to $\mathcal{L}$, then $\mathcal{T}$ is m-quasi-n-power-(α,β)-normal operator for natural numbers m and n.*

Proof. Let $\mathcal{T}$ be an operator unitary equivalent to $\mathcal{L}$. Then $\mathcal{T}=\mathcal{U}^*\mathcal{L}\mathcal{U}$ for some unitary operator $\mathcal{U}$. Therefore for every $x\in\mathscr{H}$

$$\begin{aligned}
\alpha^2\big\langle \mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m x,x\big\rangle &= \big\langle (\mathcal{U}^*\mathcal{L}\mathcal{U})^{m*}(\mathcal{U}^*\mathcal{L}\mathcal{U})^*(\mathcal{U}^*\mathcal{L}\mathcal{U})^n(\mathcal{U}^*\mathcal{L}\mathcal{U})^m x,x\big\rangle\\
&= \alpha^2\big\langle (\mathcal{U}^*\mathcal{L}^{m*}\mathcal{U})(\mathcal{U}^*\mathcal{L}^*\mathcal{U})(\mathcal{U}^*\mathcal{L}^n\mathcal{U})(\mathcal{U}^*\mathcal{L}^m\mathcal{U})x,x\big\rangle\\
&= \alpha^2\big\langle (\mathcal{U}^*\mathcal{L}^{m*}\mathcal{U})(\mathcal{U}^*\mathcal{L}^*\mathcal{U})(\mathcal{U}^*\mathcal{L}^n\mathcal{U})(\mathcal{U}^*\mathcal{L}^m\mathcal{U})x,x\big\rangle\\
&= \alpha^2\big\langle \big(\mathcal{U}^*\mathcal{L}^{m*}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\mathcal{U}\big)x,x\big\rangle\\
&= \alpha^2\big\langle \big(\mathcal{L}^{m*}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\big)\mathcal{U}x,\mathcal{U}x\big\rangle\\
&\le \big\langle \big(\mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m\big)\mathcal{U}x,\mathcal{U}x\big\rangle\\
&= \big\langle \mathcal{T}^{m*}\mathcal{T}^n\mathcal{T}^*\mathcal{T}^m x,x\big\rangle\\
&= \big\langle \big(\mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m\big)\mathcal{U}x,\mathcal{U}x\big\rangle\\
&\le \beta^2\big\langle \big(\mathcal{L}^{m*}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m\big)\mathcal{U}x,\mathcal{U}x\big\rangle\\
&= \beta^2\big\langle \mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}^n\mathcal{T}^m x,x\big\rangle.
\end{aligned}$$

Hence $\mathcal{T}$ is m-quasi-n-power-(α,β)-normal operator for natural numbers m and n. □

In the next result, we study the necessary and sufficient condition to the weighted shift operator to be m-quasi-n-power-(α,β)-normal operator.

Theorem 2.19. *Let $\mathcal{L}$ be a weighted shift operator with nonzero weights $\{z_k)_{k=0}^{\infty}$. Then $\mathcal{L}$ is m-quasi-n-power-(α,β)-normal operator if and only if*

$$\alpha^2\big(z_k\cdots|z_{k+m+n-1}|^2\overline{z}_{k+m+n-2}\cdots\overline{z}_{k+n-1}\big)$$

$$\le\big(z_k\cdots|z_{k+m-1}|^2 z_{k+m-1}\cdots|z_{k+m+n-2}|^2\cdots\overline{z}_{k+n-1}\big)$$

$$\le\beta^2\big(z_k\cdots|z_{k+m+n-1}|^2\overline{z}_{k+m+n-2}\cdots\overline{z}_{k+n-1}\big).$$

Proof. Let $\{e_k)_{k=0}^{\infty}$ be an orthogonal basis of Hilbert space $\mathscr{H}$. Then
$\mathcal{L}(e_k)=z_ke_{k+1}$,
$\mathcal{L}^*(e_k)=\overline{z}_{k-1}e_{k-1}$,
$\mathcal{L}^m(e_k)=z_k\cdots z_{k+m-1}e_{k+m}$,
$\mathcal{L}^{m*}(e_k)=\overline{z}_{k-1}\cdots\overline{z}_{k-m}e_{k-m}$.

$$\mathcal{L}^{m*}\mathcal{L}^*\mathcal{L}^n\mathcal{L}^m(e_k)=\big(z_k\cdots|z_{k+m+n-1}|^2\overline{z}_{k+m+n-2}\cdots\overline{z}_{k+n-1}\big)e_{k+n-1},$$

$$\mathcal{L}^{m*}\mathcal{L}^n\mathcal{L}^*\mathcal{L}^m(e_k)=\big(z_k\cdots|z_{k+m-1}|^2 z_{k+m-1}\cdots|z_{k+m+n-2}|^2\cdots\overline{z}_{k+n-1}\big)e_{k+n-1}.$$

Thus

$$\alpha^2\big(z_k\cdots|z_{k+m+n-1}|^2\overline{z}_{k+m+n-2}\cdots\overline{z}_{k+n-1}\big)e_{k+n-1}$$

$$\le\big(z_k\cdots|z_{k+m-1}|^2 z_{k+m-1}\cdots|z_{k+m+n-2}|^2\cdots\overline{z}_{k+n-1}\big)e_{k+n-1}$$

$$\le\beta^2\big(z_k\cdots|z_{k+m+n-1}|^2\overline{z}_{k+m+n-2}\cdots\overline{z}_{k+n-1}\big)e_{k+n-1}.$$

□

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