Methods of Functional Analysis and Topology
Vol. 31 (2025), no. 4, pp. 261–309
doi.org/10.31392/MFAT-npu26_4.2025.01

METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

MATRICES INDUCED BY SCALED HYPERCOMPLEX NUMBERS OVER THE REAL FIELD $\mathbb{R}$

DANIEL ALPAY AND ILWOO CHO

Abstract. In this paper, we construct, and study a certain type of definite, or indefinite inner product spaces over the real field $\mathbb{R}$, induced by the scaled hypercomplex numbers $\mathbb{H}_t$ for a fixed scale $t \in \mathbb{R}$, and some bounded operators acting on such vector spaces. In particular, we are interested in the vector spaces $\mathbb{H}_t^N$ consisting of all N-tuples of scaled hypercomplex numbers of $\mathbb{H}_t$, and the $(N \times N)$-matrices acting on $\mathbb{H}_t^N$ whose entries are from $\mathbb{H}_t$, i.e., $\mathbb{H}_t$-matrices, for all $N \in \mathbb{N}$. For an arbitrarily fixed $N \in \mathbb{N}$, we define $\mathbb{H}_t^N$ as a subspace of a certain functional vector space $\mathbf{H}_{t:2}$ equipped with a well-defined definite (if $t < 0$), or indefinite (if $t \geq 0$) inner product introduced in [6, 7, 8]. So, one can check immediately that our subspace $\mathbb{H}_t^N$ becomes a restricted definite, or indefinite inner product Banach space. Operator-theoretic, operator-algebraic and free-probabilistic properties of $\mathbb{H}_t$-matrices are considered and characterized on $\mathbb{H}_t^N$.

Contents

1. Introduction

For a fixed scale $t \in \mathbb{R}$, a t-scaled hypercomplex number is a pair $(a, b) \in \mathbb{C}^2$ of complex numbers $a, b \in \mathbb{C}$, contained in a noncommutative ring,

$$\mathbb{H}_t \overset{\text{denote}}{=} \left(\mathbb{C}^2,\ +,\ \cdot_t\right),$$

with the identity $(0, 0)$ and the unity $(1, 0)$, where $(+)$ the usual vector addition on $\mathbb{C}^2$, and $(\cdot_t)$ is the t-scaled vector multiplication,

$$(a_1, b_1) \cdot_t (a_2, b_2) = \left(a_1 a_2 + t b_1 \overline{b_2},\ a_1 b_2 + b_1 \overline{a_2}\right), \tag{1.1}$$

for all $(a_l, b_l) \in \mathbb{C}^2$, for $l = 1, 2$, where $\overline{z}$ are the conjugates of $z \in \mathbb{C}$ (see [1, 2, 3, 4]). By the canonical representation $\left(\mathbb{C}^2, \pi_t\right)$ of $\mathbb{H}_t$ of [1], every hypercomplex number $(a, b) \in \mathbb{H}_t$

2020 *Mathematics Subject Classification.* 20G20, 46S10, 47S10.
Keywords. Scaled Hypercomplexes $\mathbb{H}_t$, Matrices over $\mathbb{H}_t$.

is realized to be a (2×2)-matrix,

$$\pi_t((a,b)) \overset{\text{denote}}{=} [(a,b)]_t \overset{\text{def}}{=} \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

where $M_2(\mathbb{C})$ is the matrix algebra acting on $\mathbb{C}^2$. The definition of $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ is motivated by the well-known quaternions (e.g., [10, 11, 14, 17, 20, 21, 23, 26]), and the split-quaternions (e.g., [9, 14, 19]). Indeed, $\mathbb{H}_{-1}$ is the noncommutative field $\mathbb{H}$ of all quaternions, and $\mathbb{H}_1$ is the noncommutative unital ring of all split-quaternions (e.g., [1, 2, 3]). Algebra, analysis, and certain free-probabilistic models on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ are studied in [1, 2, 3, 4, 8]. In particular, analysis and operator theory on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ is considered by defining symmetric bilinear forms $\{\langle,\rangle_t\}_{t\in\mathbb{R}}$ on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ in [3]. In such a case, the pairs $\{(\mathbb{H}_t, \langle,\rangle_t)\}_{t<0}$ form Hilbert spaces over $\mathbb{R}$ (in short, $\mathbb{R}$-Hilbert spaces), meanwhile, the pairs $\{(\mathbb{H}_t, \langle,\rangle_t)\}_{t\geq 0}$ become indefinite semi-inner product spaces over $\mathbb{R}$ (in short, $\mathbb{R}$-ISIPSs), under the semi-norms,

$$\|(a,b)\|_t = \sqrt{|a|^2 + |t|\,|b|^2}, \ \ \forall\,(a,b)\in\mathbb{H}_t, \ \forall t\in\mathbb{R},$$

where $|a|, |b|$ are the moduli on $\mathbb{C}$, and $|t|$ is the absolute value on $\mathbb{R}$. Also, it is shown that $\mathbb{H}_t$ is (isomorphic to) a complete semi-normed $\mathbb{R}$-$*$-algebra,

$$\mathcal{M}_t = \{m_h \in B_{\mathbb{R}}(\mathbb{H}_t) : h \in \mathbb{H}_t\},$$

over $\mathbb{R}$, operator-algebraically. (e.g., see [3, 4]). Especially, all elements of $\mathcal{M}_t$ are adjointable over $\mathbb{R}$ (in short, $\mathbb{R}$-adjointable) with the adjoint $m_h^{\circledast} \overset{\text{def}}{=} m_{h^{\circledast}} \in \mathcal{M}_t$, for all $h\in\mathbb{H}_t$, where $(\circledast)$ is the hypercomplex-conjugate on $\mathbb{H}_t$,

$$(a,b)^{\circledast} \overset{\text{def}}{=} (\overline{a}, -b), \quad \forall\,(a,b)\in\mathbb{H}_t.$$

(Remark that, in [1, 2, 3, 4], we denoted $h^{\circledast}$ by $h^{\dagger}$.)

Meanwhile, different from [1, 2, 3, 4], we introduced-and-studied a new $\mathbb{R}$-adjoint, denoted by $[*]$, on the t-scaled hypercomplexes $\mathbb{H}_t$,

$$(a,b)^{[*]} \overset{\text{def}}{=} \left(a, \overline{b}\right), \qquad \forall\,(a,b)\in\mathbb{H}_t,$$

in [6]. Under this new $\mathbb{R}$-adjoint $[*]$, our t-scaled hypercomplexes $\mathbb{H}_t$ becomes a Pontryagin space over $\mathbb{R}$, for all "non-zero" scales $t\in\mathbb{R}\setminus\{0\}$, different from the case where we have the $\mathbb{R}$-adjoint $(\circledast)$, the hypercomplex-conjugate. On such a Pontryagin space $\mathbb{H}_t$, we constructed a Hardy-like vector space $\mathbf{H}_{t:2}[[q]]$ in $\mathbb{H}_t$ whose vectors are functions acting on the open unit ball $\mathbb{U}_t$ of $\mathbb{H}_t$, and defined-and-considered block-Toeplitz-like operators acting on $\mathbf{H}_{t:2}[[q]]$. The general constructions and approaches of [5] motivate those of [6].

The similar version of [6] up to the $\mathbb{R}$-adjoint $(\circledast)$ is considered in [7]. Readers can realize that the constructions and approaches of [7] are similar to those of [6], but the structures we handled therein are "not" equivalent at all. i.e., the main results of [7] and those of [6] provide non-equivalent analyses and operator theories. In this paper, we follow the settings of [7], because the $\mathbb{R}$-adjoint $(\circledast)$ gives a natural (Clifford-algebra-theoretic) extension from the initial inclusion $\mathbb{R}\subset\mathbb{C}$, compared with the $\mathbb{R}$-adjoint $[*]$ of [6]. However, it is true that the $\mathbb{R}$-adjoint $[*]$ gives interesting unified (Krein-space-)operator-theoretic backgrounds on the vector spaces over $\mathbb{R}$ (in short, $\mathbb{R}$-vector spaces) induced by $\mathbb{H}_t$. In this paper, we focus on $(\circledast)$-depending structures.

In Section 2, we review definitions and basic results of scaled hypercomplex numbers. And then, in Section 3, we re-considered the Hardy-like $\mathbb{R}$-vector space $\mathbf{H}_{t:2}[[q]]$, called the $\mathbb{H}_t$-Hardy space, introduced in [7] (which is not equivalent to that of [6]) to understand our analytic structures of this paper. Note that, just like in, but different from, the usual operator theory, our $\mathbb{H}_t$-Hardy space forms a complete semi-normed, definite or indefinite semi-inner-product space over $\mathbb{R}$. If $t\neq 0$, then it is a complete normed, definite, or indefinite inner-product space over $\mathbb{R}$.

In Section 4, we define and study finite-dimensional $\mathbb{R}$-vector space $\mathbb{H}_t^N$ "over $\mathbb{H}_t$," as $\mathbb{R}$-vector subspaces of $\mathbf{H}_{t:2}\,[[q]]$, equipped with an inherited definite, or indefinite semi-inner product, and a restricted complete semi-norm, for $N \in \mathbb{N}$. In Section 5, some operators acting on $\mathbb{H}_t^N$ are introduced and considered. Especially, matrices with $\mathbb{H}_t$-entries acting on $\mathbb{H}_t^N$ are studied, as "$\mathbb{R}$-linear" transformations.

In Section 6, as in classical free probability theory (over $\mathbb{C}$), we define and study certain statistical-analytic structures acting on $\mathbb{H}_t^N$ over $\mathbb{R}$.

In Section 7, a representation $\left(\mathbb{C}^{2N}, \Pi_t\right)$ of $\mathbb{H}_t$-matrices of Section 6 is introduced. Our matrices of Section 6 are realized as $(2N \times 2N)$-matrices over the complex field $\mathbb{C}$, acting on $\mathbb{C}^{2N}$ as $\mathbb{R}$-linear transformations. As application, in Section 8, invariant subspaces (as a $\mathbb{R}$-vector space) of our $\mathbb{H}_t$-matrices in $\mathbb{H}_t^N$ are constructed, similar to, but different from, the usual spectral theory (over $\mathbb{C}$).

2. Scaled Hypercomplex Numbers

Let $t \in \mathbb{R}$ be an arbitrary scale, and let

$$\mathbb{H}_t = \text{span}_{\mathbb{R}} \{1, i, j_t, k_t\} \tag{2.1}$$

be the $\mathbb{R}$-vector space spanned by $\{1, i, j_t, k_t\}$, where $i = \sqrt{-1}$ in $\mathbb{C}$, and j_t and k_t are additional t-depending imaginary numbers satisfying the relation:

$$i^2 = -1, \;\; j_t^2 = t = k_t^2,$$

$$ij_t = k_t, \;\; j_t k_t = -ti, \;\; k_t i = j_t,$$

and (2.2)

$$ik_t = -j_t, \;\; k_t j_t = ti, \;\; j_t i = -k_t.$$

Then this $\mathbb{R}$-vector space $\mathbb{H}_t$ of (2.1) is well-defined under the relation (2.2) on its $\mathbb{R}$-basis elements $\{1, i, j_t, k_t\}$. i.e., every element $h \in \mathbb{H}_t$ is expressed by

$$h = x + yi + uj_t + vk_t, \text{ with } x, y, u, v \in \mathbb{R}.$$

Note that, by the relation (2.2), the vector-multiplication on this $\mathbb{R}$-vector space $\mathbb{H}_t$ is well-defined to be

$$\begin{aligned} h_1 h_2 = &\left(x_1 x_2 - y_1 y_2 + t u_1 u_2 + t v_1 v_2\right) + \left(x_1 y_2 + y_1 x_2 - t u_1 v_2 + t v_1 u_2\right) i \\ &\left(x_1 u_2 - y_1 v_2 + u_1 x_2 + v_1 y_2\right) j_t + \left(x_1 v_2 + y_1 u_2 - u_1 y_2 + v_1 x_2\right) k_t, \end{aligned} \tag{2.3}$$

for all $h_l = x_l + y_l i + u_l j_t + v_l k_t \in \mathbb{H}_t$ for all $l = 1, 2$, by (2.2). Remark that, up to the representation of [1, 2], this vector-multiplication (2.3) is equivalent to the t-scaled multiplication $(\cdot_t)$ of (1.1) on $\mathbb{H}_t$ (e.g., see [3, 4]).

By the well-defined vector multiplication (2.3) on $\mathbb{H}_t$, this $\mathbb{R}$-vector space $\mathbb{H}_t$ forms an algebra over $\mathbb{R}$ (in short, a $\mathbb{R}$-algebra) (e.g., [1, 2, 3, 4]). On this $\mathbb{R}$-algebra $\mathbb{H}_t$, one can define a unary operation $\circledast : \mathbb{H}_t \to \mathbb{H}_t$ by

$$\left(x + yi + uj_t + vk_t\right)^{\circledast} = x - yi - uj_t - vk_t. \tag{2.4}$$

Then this satisfies that

$$h^{\circledast\circledast} = h, \;\text{ and }\; (rh)^{\circledast} = rh^{\circledast},$$

for all $h \in \mathbb{H}_t$, and $r \in \mathbb{R}$, and

$$\left(h_1 + h_2\right)^{\circledast} = h_1^{\circledast} + h_2^{\circledast}, \text{ and } \left(h_1 h_2\right)^{\circledast} = h_2^{\circledast} h_1^{\circledast},$$

for all $h_1, h_2 \in \mathbb{H}_t$. i.e., this operation $(\circledast)$ of (2.4) becomes an adjoint (or, an involution) on $\mathbb{H}_t$ over $\mathbb{R}$ (in short, a $\mathbb{R}$-adjoint on $\mathbb{H}_t$). It says that the $\mathbb{R}$-algebra $\mathbb{H}_t$ forms a $*$-algebra over $\mathbb{R}$ (in short, $\mathbb{R}$-$*$-algebra) equipped with its $\mathbb{R}$-adjoint $(\circledast)$ of (2.4) (e.g., see [1, 2, 3, 4, 8]) for details).

Definition 2.1. The $\mathbb{R}$-$*$-algebra $\mathbb{H}_t$ of (2.1) equipped with its $\mathbb{R}$-adjoint ($\circledast$) of (2.4) is called the t-scaled hypercomplexes for a scale $t \in \mathbb{R}$. All elements of $\mathbb{H}_t$ are called t-scaled hypercomplex numbers.

Note that, each t-scaled hypercomplex number $h = x+yi+uj_t+vk_t \in \mathbb{H}_t$ is understood to be

$$h = (x + yi) + (u + vi)\, j_t \text{ in } \mathbb{H}_t,$$

by (2.2). If $x + yi$ and $u + vi$ are denoted by a respectively b in $\mathbb{C}$, then this t-scaled hypercomplex number h is expressed to be $a + bj_t$ in $\mathbb{H}_t$. i.e.,

$$\mathbb{H}_t = \{a + bj_t : a, b \in \mathbb{C}\}.$$

Then one can define an injection $\pi_t : \mathbb{H}_t \to M_2(\mathbb{C})$ by

$$\pi_t (a + bj_t) = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}), \quad \forall a + bj_t \in \mathbb{H}_t, \tag{2.5}$$

where $\overline{z}$ are the conjugate of $z \in \mathbb{C}$. Then the pair $\left(\mathbb{C}^2, \pi_t\right)$ forms a representation of $\mathbb{H}_t$, satisfying

$$\pi_t (h_1 + h_2) = \pi_t (h_1) + \pi_t (h_2),$$

and

$$\pi_t (h_1 h_2) = \pi_t (h_1)\, \pi_t (h_2), \quad \forall h_1, h_2 \in \mathbb{H}_t,$$

by (2.5), where the right-hand sides are the matrix addition, respectively, the matrix multiplication on $M_2(\mathbb{C})$. i.e., $\mathbb{H}_t$ has its realization,

$$\mathcal{H}_2^t \overset{\text{def}}{=} \pi_t (\mathbb{H}_t) = \{\pi_t (h) : h \in \mathbb{H}_t\}, \tag{2.6}$$

in $M_2(\mathbb{C})$. By (2.6), one can restrict the normalized trace $\tau = \frac{1}{2}tr$ on $M_2(\mathbb{C})$ to that on $\mathcal{H}_2^t$, i.e.,

$$\tau \left([h]_t\right) \overset{\text{def}}{=} \frac{1}{2} tr \left(\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \right) = \frac{a+\overline{a}}{2} = \mathfrak{R}(a), \tag{2.7}$$

by (2.6), where $\mathfrak{R}(a)$ is the real part of a complex number a in $\mathbb{C}$. However, note here that $\tau \mid_{\mathcal{H}_2^t}$ is on $\mathcal{H}_2^t$ "over $\mathbb{R}$," meanwhile τ is on $M_2(\mathbb{C})$ "over $\mathbb{C}$." So, this morphism τ of (2.7) is a well-defined trace on the t-scaled realization $\mathcal{H}_2^t$ of (2.5) "over $\mathbb{R}$," satisfying

$$\tau (T_1 T_2) = \tau (T_2 T_1), \quad \forall T_1, T_2 \in \mathcal{H}_2^t.$$

By (2.6) and (2.7), we define the $\mathbb{R}$-trace, also denoted by τ, directly on $\mathbb{H}_t$, by

$$\tau (h) = \operatorname{Re}(h), \quad \forall h \in \mathbb{H}_t,$$

where $\operatorname{Re}(\bullet)$ is the real part,

$$\operatorname{Re}(x + yi + uj_t + vk_t) = x,$$

and $\operatorname{Im}(\bullet)$ is the imaginary part, (2.8)

$$\operatorname{Im}(x + yi + uj_t + vk_t) = yi + uj_t + vk_t,$$

on $\mathbb{H}_t$, for all $x, y, u, v \in \mathbb{R}$. So, by (2.7) and (2.8), one can define a bilinear form,

$$[,]_t : \mathbb{H}_t \times \mathbb{H}_t \longrightarrow \mathbb{R},$$

by (2.9)

$$[h_1, h_2]_t \overset{\text{def}}{=} \tau \left(h_1 h_2^{\circledast}\right) = \operatorname{Re}\left(h_1 h_2^{\circledast}\right).$$

Then this bilinear form (2.9) satisfies that:

$$[h, h]_t \geq 0, \quad \forall h \in \mathbb{H}_t, \quad \text{if} \quad t < 0,$$

$$[h, h]_t \in \mathbb{R}, \quad \forall h \in \mathbb{H}_t, \quad \text{if} \quad t \geq 0,$$

$$[h_1, h_2]_t = [h_2, h_1]_t, \quad \forall h_1, h_2 \in \mathbb{H}_t, \ \forall t \in \mathbb{R}$$

and (2.10)

$$[h, h]_t = 0 \Longleftrightarrow |a|^2 = t\,|b|^2, \text{ if } h = a + bj_t \in \mathbb{H}_t, \ a, b \in \mathbb{C},$$

for all $t \in \mathbb{R}$, where $|.|$ is the modulus on $\mathbb{C}$. Thus, if $t < 0$, then it forms a $\mathbb{R}$-inner product on $\mathbb{H}_t$, meanwhile, if $t \geq 0$, then it forms an $\mathbb{R}$-indefinite semi-inner product on $\mathbb{H}_t$ (e.g., see [3, 4, 8] for details). More precisely,

$$[h, q]_t = 0, \text{ for all } q \in \mathbb{H}_t \Longrightarrow h = 0 \in \mathbb{H}_t, \ \ \forall t \in \mathbb{R} \setminus \{0\},$$

which says that $[,]_t$ is non-degenerated on $\mathbb{H}_t$, for all $t \in \mathbb{R} \setminus \{0\}$. Meanwhile, if $t = 0$, then

$$[h, q]_0 = 0, \ \forall q \in \mathbb{H}_0 \Longrightarrow h = 0 + 0i + uj_0 + vk_0 \in \mathbb{H}_0, \tag{2.11}$$

for any $u, v \in \mathbb{R}$, which implies that $[,]_0$ is "not" non-degenerated on $\mathbb{H}_0$.

Proposition 2.2. *Let $\mathbb{H}_t$ be the t-scaled hypercomplexes, and $[,]_t$, the bilinear form* (2.9) *on $\mathbb{H}_t$, for all $t \in \mathbb{R}$. Then*
(1) *If $t < 0$, then $(\mathbb{H}_t, [,]_t)$ is a $\mathbb{R}$-inner product space.*
(2) *If $t > 0$, then $(\mathbb{H}_t, [,]_t)$ is a $\mathbb{R}$-indefinite inner product space.*
(3) *If $t = 0$, then $(\mathbb{H}_0, [,]_0)$ is a $\mathbb{R}$-indefinite semi-inner product space in the sense of* [3, 4, 6, 8]*. More precisely, the form $[,]_0$ is a positive semidefinite and degenerated.*

Proof. The proof is done by (2.10) and (2.11). □

By the above proposition, for any scale $t \in \mathbb{R}$, the pair $(\mathbb{H}_t, [,]_t)$ becomes a definite, or indefinite semi-inner-product $\mathbb{R}$-vector space in general. Thus, one can define a function,

$$\|.\|_t : \mathbb{H}_t \longrightarrow \mathbb{R},$$

by (2.12)

$$\|a + bj_t\|_t = \sqrt{|a|^2 + |t|\,|b|^2}, \ \ \forall a + bj_t \in \mathbb{H}_t,$$

where $a, b \in \mathbb{C}$, and $|.|$ in (2.12) is the absolute value on $\mathbb{R}$.

Proposition 2.3. *Let $\mathbb{H}_t$ be the t-scaled hypercomplexes, and $\|.\|_t$, the function* (2.12)*, for all $t \in \mathbb{R}$. Then*
(1) *If $t < 0$, then $(\mathbb{H}_t, \|.\|_t)$ is a $\mathbb{R}$-Hilbert space.*
(2) *If $t > 0$, then $(\mathbb{H}_t, \|.\|_t)$ is a $\mathbb{R}$-Pontryagin space (i.e., $\mathbb{R}$-Krein space with the finite-dimensional anti-Hilbert space).*
(3) *If $t = 0$, then $(\mathbb{H}_0, \|.\|_0)$ is a complete $\mathbb{R}$-semi-normed space, where the completeness means that all Cauchy sequences are convergent in $\mathbb{H}_0$.*

Proof. See [8] for details. □

Let $\mathbb{H}_t$ be the t-scaled hypercomplexes as a $\mathbb{R}$-$*$-algebra with its $\mathbb{R}$-adjoint $(\circledast)$. Then this algebra $\mathbb{H}_t$ acts on the $\mathbb{R}$-vector space $(\mathbb{H}_t, [,]_t) = (\mathbb{H}_t, \|.\|_t)$ via an action $\mathbf{m}$,

$$\mathbf{m} : h \in \mathbb{H}_t \longmapsto \mathbf{m}_h \in B_{\mathbb{R}}(\mathbb{H}_t),$$

defined by (2.13)

$$\mathbf{m}_h(q) \overset{\text{def}}{=} hq \in \mathbb{H}_t, \quad \forall q \in \mathbb{H}_t, \ \ \forall h \in \mathbb{H}_t,$$

where $B_{\mathbb{R}}(Y)$ means the operator algebra of all "bounded" $\mathbb{R}$-linear operators on a semi-normed $\mathbb{R}$-vector space $Y = (Y, \|.\|_Y)$ with its operator semi-norm,

$$\|T\| = \sup \{\|Ty\|_Y : \|y\|_Y = 1\}, \ \ \forall T \in B_{\mathbb{R}}(Y).$$

It is not difficult to check that, in our case,

$$\left\|\mathbf{m}_h(q)\right\|_t = \|hq\|_t \leq \|h\|_t \|q\|_t, \quad \forall h, q \in \mathbb{H}_t,$$

implying that (2.14)

$$\|\mathbf{m}_h\| = \|h\|_t < \infty, \text{ in } B_{\mathbb{R}}(\mathbb{H}_t), \quad \forall h \in \mathbb{H}_t.$$

Also, it can be checked that

$$[\mathbf{m}_h(h_1), h_2]_t = [hh_1, h_2]_t = \left[h_1, h^{\circledast} h_2\right]_t = [h_1, \mathbf{m}_{h^{\circledast}}(h_2)]_t,$$

for all $h, h_1, h_2 \in \mathbb{H}_t$.

Theorem 2.4. *The t-scaled hypercomplexes $\mathbb{H}_t$ forms a complete $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra equipped with its $\mathbb{R}$-adjoint ($\circledast$) of* (2.4).

Proof. If we define a subset M of $B_{\mathbb{R}}(\mathbb{H}_t)$ by

$$\mathsf{M} \overset{\text{def}}{=} \{\mathbf{m}_h : h \in \mathbb{H}_t\},$$

where $\mathbf{m}$ is the action (2.13), then it forms a complete semi-normed $\mathbb{R}$-$*$-subalgebra of $B_{\mathbb{R}}(\mathbb{H}_t)$. It is easy to check that $\mathbb{H}_t$ and M are isometrically isomorphic by (2.13) and (2.14). Indeed, there exists an isometric $*$-isomorphism,

$$h \in \mathbb{H}_t \longmapsto m_h \in \mathsf{M}.$$

□

3. The $\mathbb{H}_t$-Hardy Space $\mathbf{H}_{t:2}[[q]]$

In this section, we define the$\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ in a $\mathbb{H}_t$-variable $q = z + wj_t$, with the $\mathbb{C}$-variables $z = x + yi$ and $w = u + vi$, where x, y, u, v are $\mathbb{R}$-variables, for an arbitrarily fixed scale $t \in \mathbb{R}$. Since the t-scaled hypercomplexes $\mathbb{H}_t$ is a $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra (and hence, it is a ring), one can construct the corresponding (pure-algebraic) formal-series ring $\mathbb{H}_t[[q]]$ (without considering topology),

$$\mathbb{H}_t[[q]] \overset{\text{def}}{=} \left\{\sum_{n=0}^{\infty} q^n h_n : h_n \in \mathbb{H}_t, \ \forall n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right\}, \tag{3.1}$$

having the functional addition $(+)$,

$$(f+g)(q) \overset{\text{def}}{=} f(q) + g(q) = \sum_{k=0}^{\infty} q^n (f_n + g_n),$$

and the Cauchy product $(\star)$,

$$(f \star g)(q) \overset{\text{def}}{=} \sum_{n=0}^{\infty} q^n \left(\sum_{n_1, n_2 \in \mathbb{N}_0, n_1 + n_2 = n} f_{n_1} g_{n_2}\right),$$

for all

$$f(q) = \sum_{n=0}^{\infty} q^n f_n, \quad g(q) = \sum_{n=0}^{\infty} q^n g_n \in \mathbb{H}_t[[q]],$$

equivalently,

$$(f \star g)(q) \overset{\text{def}}{=} \sum_{n=0}^{\infty} q^n (f(q) g_n), \ \forall f(q) \in \mathbb{H}_t[[q]].$$

This formal-series ring $\mathbb{H}_t[[q]] = (\mathbb{H}_t[[q]], +, \star)$ of (3.1) is well-defined as a $\mathbb{R}$-algebra pure-algebraically, because the $\mathbb{R}$-scalar product,

$$r\left(\sum_{n=0}^{\infty} q^n f_n\right) \overset{\text{def}}{=} \left(q^0 r + \sum_{n=1}^{\infty} q^n 0\right) \star \left(\sum_{n=0}^{\infty} q^n f_n\right) = \sum_{n=0}^{\infty} q^n (r f_n),$$

is well-defined on $\mathbb{H}_t[[q]]$, for all $r \in \mathbb{R}$.

Proposition 3.1. *The formal-series ring* $\mathbb{H}_t[[q]]$ *of* (3.1) *forms a* $\mathbb{R}$*-algebra.*

Proof. By definition, the family $\mathbb{H}_t[[q]]$ of (3.1) forms a ring having a well-defined $\mathbb{R}$-scalar product, introduced in the above paragraph, making $\mathbb{H}_t[[q]]$ be a $\mathbb{R}$-vector space. So, $\mathbb{H}_t[[q]]$ is both a ring and a $\mathbb{R}$-vector space, and hence, it forms a $\mathbb{R}$-algebra. □

Recall that $\|.\|_t$ be the $\mathbb{R}$-semi-norm (2.12) on $\mathbb{H}_t$ (i.e., it is a $\mathbb{R}$-norm if $t \neq 0$, while it is a $\mathbb{R}$-semi-norm if $t = 0$). Now, let $f(q) = \sum_{n=0}^{\infty} q^n f_n \in \mathbb{H}_t[[q]]$ with $f_n \in \mathbb{H}_t$, for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Observe that, for an arbitrary $q_o \in \mathbb{H}_t$, one may / can have

$$f(q_o) = \sum_{n=0}^{\infty} q_o^n f_n \in \mathbb{H}_t, \text{ or, undefined in } \mathbb{H}_t,$$

satisfying (3.2)

$$\limsup_{n\to\infty} \sqrt[n]{\|q_o^n f_n\|_t} \leq \|q_o\|_t \left(\limsup_{n\to\infty} \sqrt[n]{\|f_n\|_t}\right).$$

Proposition 3.2. *Let* $f(q) = \sum_{n=0}^{\infty} q^n f_n \in \mathbb{H}_t[[q]]$, *with* $f_n = a_n + b_n j_t \in \mathbb{H}_t$ *with* $a_n, b_n \in \mathbb{C}$, *for all* $n \in \mathbb{N}_0$. *If* $q_o \in \mathbb{H}_t$ *satisfies*

$$\|q_o\|_t < \left(\limsup_{n\to\infty} \sqrt[n]{\|f_n\|_t}\right)^{-1},$$

then $f(q_o)$ *is convergent in* $\mathbb{H}_t$ *in the sense that:* $f(q_o) \in \mathbb{H}_t \Longleftrightarrow \|f(q_o)\| < \infty$.

Proof. By (3.2) and the root test, if

$$\limsup_{n\to\infty} \sqrt[n]{\|q_o^n f_n\|_t} \leq \|q_o\|_t \left(\limsup_{n\to\infty} \sqrt[n]{\|f_n\|_t}\right) < 1,$$

equivalently, if

$$\|q_o\|_t < \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\|f_n\|_t}},$$

then $\|f(q_o)\|_t < \infty$, i.e., $f(q_o) \in \mathbb{H}_t$. □

Motivated by the above proposition, we consider the analyticity on the $\mathbb{R}$-algebra $\mathbb{H}_t[[q]]$.

Definition 3.3. Let U be an open subset of $\mathbb{H}_t$ under the $\|.\|_t$-semi-norm topology. Define the $\mathbb{H}_t$-analytic algebra $\mathscr{H}_t[[U]]$ by

$$\mathscr{H}_t[[U]] \overset{\text{def}}{=} \{f(q) \in \mathbb{H}_t[[q]] : f(q_0) \in \mathbb{H}_t, \ \forall q_0 \in U\}. \tag{3.3}$$

All elements $f(q)$ of $\mathscr{H}_t[[U]]$ are said to be $\mathbb{H}_t$-analytic functions on a domain U. If $U = \mathbb{H}_t$, then $\mathscr{H}_t[[\mathbb{H}_t]]$ is called the $\mathbb{H}_t$-entire algebra, and all elements of $\mathscr{H}_t[[\mathbb{H}_t]]$ are said to be $\mathbb{H}_t$-entire functions (on $\mathbb{H}_t$).

Observe that

$$f(q), g(q) \in \mathscr{H}_t[[U]] \Longrightarrow f(q) + g(q), \ f(q) \star g(q) \in \mathscr{H}_t[[U]],$$

and (3.4)

$$r \in \mathbb{R}, \ f(q) \in \mathscr{H}_2[[U]] \Longrightarrow rf(q) \in \mathscr{H}_t[[U]],$$

since, for any $q_o \in U$, one has

$$\|f(q_o) + g(q_o)\|_t \leq \|f(q_o)\|_t + \|g(q_o)\|_t < \infty,$$

$$\|f(q_o) \star g(q_o)\|_t \leq \|f(q_o)\|_t \|g(q_o)\|_t < \infty,$$

and

$$\|rf(q_o)\|_t = |r| \|f(q_o)\|_t < \infty, \ \forall r \in \mathbb{R}.$$

Thus, indeed, the $\mathbb{H}_t$-analytic algebra $\mathscr{H}_t[[U]]$ of a domain U forms a $\mathbb{R}$-algebra by (3.4). By (3.3), one can define a morphism,

$$\|.\|_{t,U} : \mathscr{H}_t[[U]] \to \mathbb{R},$$

by (3.5)

$$\|f(q)\|_{t,U} \stackrel{\text{def}}{=} \sup_{h \in U} \|f(h)\|_t,$$

for all $f(q) \in \mathscr{H}_t[[U]]$. Then it is a well-defined complete $\mathbb{R}$-semi-norm on $\mathscr{H}_t[[U]]$. More precisely, if $t \neq 0$, then $\|.\|_{t,U}$ of (3.5) is a complete $\mathbb{R}$-norm, meanwhile, if $t = 0$, then $\|.\|_{t,U}$ forms a complete $\mathbb{R}$-semi-norm on $\mathscr{H}_t[[U]]$, because $\|.\|_t$ is a complete $\mathbb{R}$-norm on $\mathbb{H}_t$ if $t \neq 0$, while, $\|.\|_0$ is a complete $\mathbb{R}$-semi-norm on $\mathbb{H}_0$.

Theorem 3.4. *The $\mathbb{H}_t$-analytic algebra $\mathscr{H}_t[[U]]$ on a domain $U \subseteq \mathbb{H}_t$ is a complete $\mathbb{R}$-semi-normed $\mathbb{R}$-algebra.*

Proof. By (3.4), the $\mathbb{H}_t$-analytic algebra $\mathscr{H}_t[[U]]$ of (3.3) on a domain U is a well-defined $\mathbb{R}$-algebra, equipped with the $\mathbb{R}$-semi-norm $\|.\|_{t,U}$ of (3.5). As we discussed in the above paragraph, this $\mathbb{R}$-semi-norm $\|.\|_{t,U}$ is complete on $\mathscr{H}_t[[U]]$. □

Now, we define a new $\mathbb{R}$-vector space $\mathbf{H}_{t:2}[[q]]$ in $\mathbb{H}_t$.

Definition 3.5. Let $\mathbb{U}_t = \{h \in \mathbb{H}_t : \|h\|_t < 1\}$ be the open unit ball of $\mathbb{H}_t$ up to the $\|.\|_t$-semi-norm topology. Define a $\mathbb{R}$-vector space $\mathbf{H}_{t:2}[[q]]$ by

$$\mathbf{H}_{t:2}[[q]] = \left\{ \sum_{n=0}^{\infty} q^n f_n : q \text{ acts on } \mathbb{U}_t, \ \sum_{n=0}^{\infty} \|f_n\|_t^2 < \infty \right\}, \tag{3.6}$$

where q is the $\mathbb{H}_t$-variable acting on $\mathbb{U}_t$ in $\mathbb{H}_t$. We call $\mathbf{H}_{t:2}[[q]]$, the $\mathbb{H}_t$-Hardy ($\mathbb{R}$-vector-)space.

Consider that if

$$f(q) = \sum_{n=0}^{\infty} q^n f_n, \ g(q) = \sum_{n=0}^{\infty} q^n g_n \in \mathbf{H}_{t:2}[[q]],$$

then (3.7)

$$\left(\sum_{n=0}^{\infty} \|f_n + g_n\|_t^2 \right)^{\frac{1}{2}} \leq \left(\sum_{n=0}^{\infty} \|f_n\|_t^2 \right)^{\frac{1}{2}} + \left(\sum_{n=0}^{\infty} \|g_n\|_t^2 \right)^{\frac{1}{2}},$$

by the Minkowski's inequality, implying that

$$f(q), \ g(q) \in \mathbf{H}_{t:2}[[q]] \Longrightarrow f(q) + g(q) \in \mathbf{H}_{t:2}[[q]], \tag{3.8}$$

by (3.7). Also, one has

$$r \in \mathbb{R}, \ f(q) \in \mathbf{H}_{t:2}[[q]] \Longrightarrow rf(q) \in \mathbf{H}_{t:2}[[q]], \tag{3.9}$$

since

$$\sum_{n=0}^{\infty} q^n f_n \in \mathbf{H}_{t:2}[[q]] \Longrightarrow \sum_{n=0}^{\infty} \|rf_n\|_t^2 = |r| \left(\sum_{n=0}^{\infty} \|f_n\|_t^2 \right) < \infty,$$

for all $r \in \mathbb{R}$. So, our $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ is a well-defined $\mathbb{R}$-vector space by (3.8) and (3.9).

Define now a form, φ_t on $\mathbf{H}_{t:2}[[q]]$ by

$$\varphi_t\left(\sum_{n=0}^{\infty} q^n f_n,\ \sum_{n=0}^{\infty} q^n g_n\right) = \sum_{n=0}^{\infty} [f_n, g_n]_t\,, \tag{3.10}$$

where $[,]_t$ is the symmetric bilinear form (2.9) on $\mathbb{H}_t$, especially, if $t < 0$, then it is a definite $\mathbb{R}$-inner product, or if $t > 0$, then it is a $\mathbb{R}$-indefinite inner product, or if $t = 0$, then it is a $\mathbb{R}$-indefinite semi-inner product on $\mathbb{H}_t$. Note that, on $\mathbb{H}_t$,

$$|[h,h]_t| = |\tau(hh^{\circledast})| \leq \|\tau\| \, \|hh^{\circledast}\|_t = \|hh^{\circledast}\|_t\,, \quad \forall h \in \mathbb{H}_t, \tag{3.11}$$

where $\|\tau\| \overset{\text{def}}{=} \sup\{|\tau(q)| : \|q\|_t = 1\} = 1$, since $\tau(1) = \text{Re}(1) = 1$, and hence, one has

$$|[h,h]_t| \leq \|h\|_t \, \|h^{\circledast}\|_t = \|h\|_t^2\,,\ \forall h \in \mathbb{H}_t, \tag{3.12}$$

by (3.11). Thus, similar to (3.12), if $f(q) = \sum_{n=0}^{\infty} q^n f_n$, $g(q) = \sum_{n=0}^{\infty} q^n g_n \in \mathscr{H}_t[[U]]$, then we have that

$$|\varphi_t(f(q), g(q))| \leq \sum_{n=0}^{\infty} |[f_n, g_n]_t| \leq \sum_{n=0}^{\infty} \|f_n\|_t \, \|g_n\|_t. \tag{3.13}$$

It shows that the morphism φ_t of (3.10) is bounded from $\mathbf{H}_{t:2}[[q]] \times \mathbf{H}_{t:2}[[q]]$ into $\mathbb{R}$ in the sense that

$$|\varphi_t(f(q), f(q))| \leq \sum_{n=0}^{\infty} \|f_n\|_t^2 < \infty.$$

by (3.12) and (3.13). Moreover, we have

$$\varphi_t(r_1 f(q) + r_2 g(q)\,,\ p(q)) = r_1\varphi_t(f(q), p(q)) + r_2\varphi_t(g(q), p(q))$$

and (3.14)

$$\varphi_t(p(q), r_1 f(q) + r_2 g(p)) = r_1\varphi_t(p(q), f(q)) + r_2\varphi_t(p(q), g(q))\,,$$

for all $r_1, r_2 \in \mathbb{R}$, and $f(q), g(q) \in \mathbf{H}_{t:2}[[q]]$, by the bilinearity of $[,]_t$ on $\mathbb{H}_t$. And, we have that

$$\varphi_t(f(q), g(q)) = \sum_{n=0}^{\infty} [f_n, g_n]_t = \sum_{n=0}^{\infty} [g_n, f_n]_t = \varphi_t(g(q), f(q)). \tag{3.15}$$

i.e., the form φ_t is a symmetric bilinear form on the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$, by (3.14) and (3.15).

This symmetric bilinear form φ_t of (3.10) on the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ also satisfies that: for any fixed $f(q) \in \mathbf{H}_{t:2}[[q]]$, if

$$\varphi_t(f(q), g(q)) = 0, \quad \text{"for all"}\ g(q) \in \mathbf{H}_{t:2}[[q]]\,,$$

then

$$f(q) = \sum_{n=0}^{\infty} q^n 0 = 0, \quad \text{if}\ \ t \neq 0,$$

while, (3.16)

$$f(q) \neq 0,\ \text{in general, if}\ t = 0,$$

by Proposition 2, i.e., by the non-degenerated-ness of $\{[,]_t\}_{t\in\mathbb{R}\setminus\{0\}}$ on $\{\mathbb{H}_t\}_{t\in\mathbb{R}\setminus\{0\}}$, respectively, by the degenerated-ness of $[,]_0$ on $\mathbb{H}_0$. i.e.,

$$t \neq 0 \Longrightarrow \varphi_t \text{ is non-degenerated on } \mathbf{H}_{t:2}[[q]]$$

meanwhile, (3.17)

$$t = 0 \Longrightarrow \varphi_0 \text{ is not non-degenerated on } \mathbf{H}_{0:2}[[q]]$$

by (3.16).

Theorem 3.6. *Let* $\mathbf{H}_{t:2}[[q]]$ *be the* $\mathbb{H}_t$*-Hardy space* (3.6), *and* φ_t, *the form* (3.10). *Then*

$$t < 0 \Longrightarrow (\mathbf{H}_{t:2}[[q]], \varphi_t) \text{ is a definite } \mathbb{R}\text{-inner-product space,}$$

$$t > 0 \Longrightarrow (\mathbf{H}_{t:2}[[q]], \varphi_t) \text{ is a } \mathbb{R}\text{-indefinite-inner-product space,}$$

and (3.18)

$$t = 0 \Longrightarrow (\mathbf{H}_{0:2}[[q]], \varphi_0) \text{ is a } \mathbb{R}\text{-indefinite-semi-inner-product space,}$$

in the sense of [3, 4, 6, 8]. *i.e., the form* φ_0 *is a positive semidefinite and degenerated. Moreover, the form* φ_t *is bounded on* $\mathbf{H}_{t:2}[[q]]$ *in the sense that:*

$$|\varphi_t(f(q), f(q))| < \infty, \quad \forall f(q) \in \mathbf{H}_{t:2}[[q]], \ \forall t \in \mathbb{R}. \tag{3.19}$$

Proof. By (3.14) and (3.15), the form φ_t of (3.10) is a symmetric bilinear form on $\mathbf{H}_{t:2}[[q]]$, for all scales $t \in \mathbb{R}$.

If $t > 0$, then this symmetric bilinear form φ_t is non-degenerated by (3.17), and hence, it forms a $\mathbb{R}$-indefinite inner product on $\mathbf{H}_{t:2}[[q]]$. i.e., the pair $(\mathbf{H}_{t:2}[[q]], \varphi_t)$ forms a $\mathbb{R}$-indefinite-inner-product space. If $t = 0$, then φ_0 is not non-degenerated (or, degenerated) by (3.17). So, the form φ_0 becomes a $\mathbb{R}$-indefinite "semi-inner" product on $\mathbf{H}_{t:2}[[q]]$, saying that the pair $(\mathbf{H}_{0:2}[[q]], \varphi_0)$ is a $\mathbb{R}$-indefinite-semi-inner-product space. If $t < 0$, then this symmetric bilinear form φ_t is not only non-degenerated, but also, satisfying that

$$\varphi_t(f(q), f(q)) = 0 \Longrightarrow f(q) = 0 = \sum_{n=0}^{\infty} q^n 0 \in \mathbf{H}_{t:2}[[q]],$$

since $[,]_t$ is a definite $\mathbb{R}$-inner product on $\mathbb{H}_t$. So, the non-degenerated symmetric bilinear form φ_t becomes a definite $\mathbb{R}$-inner product on $\mathbf{H}_{t:2}[[q]]$. Thus, if $t < 0$, then the pair $(\mathbf{H}_{t:2}[[q]], \varphi_t)$ forms a definite $\mathbb{R}$-inner-product space. Therefore, the structure theorem (3.18) holds.

Also, for any arbitrary scale $t \in \mathbb{R}$, the form φ_t is bounded on $\mathbf{H}_{t:2}[[q]]$ in the sense that

$$|\varphi_t(f(q), f(q))| < \infty, \quad \forall f(q) \in \mathbf{H}_{t:2}[[q]],$$

by (3.12) and (3.13), i.e.,

$$\left| \varphi_t \left(\sum_{n=0}^{\infty} q^n f_n, \sum_{n=0}^{\infty} q^n f_n \right) \right| \underset{(3.13)}{\leq} \sum_{n=0}^{\infty} \|f_n\|_t^2 \underset{(3.6)}{<} \infty.$$

Therefore, the boundedness (3.19) of φ_t on $\mathbf{H}_{t:2}[[q]]$ is shown. □

By (3.18) and (3.19), if we define a map $\|.\|_{\mathbf{H}_{t:2}} : \mathbf{H}_{t:2}[[q]] \to \mathbb{R}$ by

$$\left\| \sum_{n=0}^{\infty} q^n f_n \right\|_{\mathbf{H}_{t:2}} \overset{\text{def}}{=} \sqrt{\sum_{n=0}^{\infty} \|f_n\|_t^2}, \quad \forall \sum_{n=0}^{\infty} q^n f_n \in \mathbf{H}_{t:2}[[q]] \tag{3.20}$$

then it is a well-defined complete $\mathbb{R}$-semi-norm on the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$. More precisely, if $t \neq 0$, then the map $\|.\|_{\mathbf{H}_{t:2}}$ becomes a $\mathbb{R}$-norm on $\mathbf{H}_{t:2}[[q]]$, meanwhile, if $t = 0$, then it is a $\mathbb{R}$-semi-norm on $\mathbf{H}_{0:2}[[q]]$, because $\|.\|_t$ is a $\mathbb{R}$-norm if $t \neq 0$, while, $\|.\|_0$ is a $\mathbb{R}$-semi-norm on $\mathbb{H}_0$ if $t = 0$. The completeness of the $\mathbb{R}$-semi-norm $\|.\|_{\mathbf{H}_{t:2}}$ on $\mathbf{H}_{t:2}[[q]]$ is guaranteed by that of $\|.\|_t$ on $\mathbb{H}_t$, for all $t \in \mathbb{R}$.

Theorem 3.7. *If* $\|.\|_{\mathbf{H}_{t:2}}$ *is the morphism* (3.20) *on the* $\mathbb{H}_t$*-Hardy space* $\mathbf{H}_{t:2}[[q]]$, *then the pair* $\left(\mathbf{H}_{t:2}[[q]], \|.\|_{\mathbf{H}_{t:2}}\right)$ *is a complete* $\mathbb{R}$*-semi-normed space. More precisely,*

$$t \neq 0 \Longrightarrow \left(\mathbf{H}_{t:2}[[q]], \|.\|_{\mathbf{H}_{t:2}}\right) \text{ is a } \mathbb{R}\text{-Banach space,}$$

meanwhile, (3.21)

$$t = 0 \Longrightarrow \left(\mathbf{H}_{0:2}[[q]], \|.\|_{\mathbf{H}_{0:2}}\right) \text{ is a complete } \mathbb{R}\text{-semi-normed space.}$$

Proof. The structure theorem (3.21) of $\mathbf{H}_{t:2}[[q]]$ up to the complete $\mathbb{R}$-semi-norm $\|.\|_{\mathbf{H}_{t:2}}$ is shown by (3.20) and Proposition 3. □

By (3.18), (3.19) and (3.21), one obtains the following corollary.

Corollary 3.8. *If $t \neq 0$, then the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ is a complete $\mathbb{R}$-normed definite, or indefinite $\mathbb{R}$-inner-product space. Meanwhile, if $t = 0$, then $\mathbf{H}_{0:2}[[q]]$ is a complete $\mathbb{R}$-semi-normed $\mathbb{R}$-indefinite-semi-inner-product space.*

Proof. It is shown by (3.18), (3.19) and (3.21). □

The above corollary characterizes the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ as a complete $\mathbb{R}$-semi-normed definite, or indefinite $\mathbb{R}$-semi-inner-product space, for all $t \in \mathbb{R}$.

Define now an action M of $\mathbb{H}_t$ acting on the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ by

$$M : h \in \mathbb{H}_t \longmapsto M(h) \overset{\text{denote}}{=} M_h \in B_{\mathbb{R}}(\mathbf{H}_{t:2}[[q]]),$$

where (3.22)

$$M_h\left(\sum_{n=0}^{\infty} q^n f_n\right) \overset{\text{def}}{=} \sum_{n=0}^{\infty} q^n (hf_n),$$

where $B_{\mathbb{R}}(\mathbf{H}_{t:2}[[q]])$ is the operator $\mathbb{R}$-algebra consisting of all bounded $\mathbb{R}$-linear operators on $\mathbf{H}_{t:2}[[q]]$, equipped with its operator semi-norm $\|.\|$,

$$\|A\| \overset{\text{def}}{=} \sup\left\{\|A(f(q))\|_{\mathbf{H}_{t:2}} : \|f(q)\|_{\mathbf{H}_{t:2}} = 1\right\}, \ \forall A \in B_{\mathbb{R}}(\mathbf{H}_{t:2}[[q]]).$$

Then this function M of (3.22) satisfies that

$$M_{r_1h_1+r_2h_2} = r_1M_{h_1} + r_2M_{h_2}, \quad \forall r_1, r_2 \in \mathbb{R},$$

and (3.23)

$$M_{h_1h_2} = M_{h_1}M_{h_2}, \qquad \forall h_1, h_2 \in \mathbb{H}_t,$$

in $B_{\mathbb{R}}(\mathbf{H}_{t:2}[[q]])$, by (3.22). Moreover, for any $h \in \mathbb{H}_t$ and $f(q) = \sum_{n=0}^{\infty} q^n f_n \in \mathbf{H}_{t:2}[[q]]$ with $\|f(q)\|_{\mathbf{H}_{t:2}} = 1$,

$$\|M_h(f(q))\|^2_{\mathbf{H}_{t:2}} = \left\|\sum_{n=0}^{\infty} q^n (hf_n)\right\|^2_{\mathbf{H}_{t:2}} = \sum_{n=0}^{\infty} \|hf_n\|_t^2$$

$$\leq \sum_{n=0}^{\infty} \|h\|_t^2 \|f_n\|_t^2 = \|h\|_t^2 \|f(q)\|^2_{\mathbf{H}_{t:2}} = \|h\|_t^2,$$

implying that (3.24)

$$\|M_h\| = \|h\|_t, \qquad \forall h \in \mathbb{H}_t.$$

Theorem 3.9. *The function M of (3.22) is an action of the complete $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra $\mathbb{H}_t$ acting on $\mathbf{H}_{t:2}[[q]]$. Equivalently, the subset*

$$\mathfrak{M}_t = \left\{M_h \overset{\text{denote}}{=} M(h) : h \in \mathbb{H}_t\right\} \subseteq B_{\mathbb{R}}(\mathbf{H}_{t:2}[[q]])$$

forms a complete $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra equipped with the $\mathbb{R}$-adjoint ($\circledast$) on $\mathfrak{M}_t$,

$$M_h^{\circledast} \overset{\text{def}}{=} M_{h^{\circledast}} \in B_{\mathbb{R}}(\mathbf{H}_{t:2}[[q]]), \quad \forall h \in \mathbb{H}_t.$$

Proof. It is shown by (3.23), (3.24) and the definition of the $\mathbb{R}$-adjoint $(\circledast)$: $M_h^{\circledast} = M_{h^{\circledast}}$, for all $h \in \mathbb{H}_t$. Indeed, the family $\mathfrak{M}_t$ forms a $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra by an isometric isomorphism,

$$h \in \mathbb{H}_t \longmapsto M_h \in \mathfrak{M}_t,$$

satisfying

$$\|M_h\| = \|h\|_t, \quad \text{in } B_{\mathbb{R}}\left(\mathbf{H}_{t:2}\left[[q]\right]\right), \quad \forall h \in \mathbb{H}_t,$$

and

$$M_h^{\circledast} = M_{h^{\circledast}} \in B_{\mathbb{R}}\left(\mathbf{H}_{t:2}\left[[q]\right]\right), \quad \forall h \in \mathbb{H}_t,$$

because $\mathbb{H}_t$ is a complete $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra. □

The above theorem illustrates that how the t-scaled hypercomplexes $\mathbb{H}_t$ acts as operators of $\mathfrak{M}_t$ inside $B_{\mathbb{R}}\left(\mathbf{H}_{t:2}\left[[q]\right]\right)$. Define the set $\mathbf{l}^2\left(\mathbb{H}_t\right)$ of all square-summable $\mathbb{H}_t$-sequences by

$$\mathbf{l}^2\left(\mathbb{H}_t\right) \overset{\text{def}}{=} \left\{ (h_n)_{n=0}^{\infty} \in \mathbb{H}_t^{\infty} : \sum_{n=0}^{\infty} \|h_n\|_t^2 < \infty \right\}, \tag{3.25}$$

equipped with the addition $(+)$ by

$$(f_n)_{n=0}^{\infty} + (g_n)_{n=0}^{\infty} = (f_n + g_n)_{n=0}^{\infty},$$

and the $\mathbb{R}$-scalar product by

$$r\,(f_n)_{n=0}^{\infty} = (r f_n)_{n=0}^{\infty}, \quad \forall r \in \mathbb{R}.$$

Then it is indeed a well-defined $\mathbb{R}$-vector space, equipped with the $\mathbb{R}$-inner product (if $t < 0$), or the $\mathbb{R}$-indefinite inner product (if $t > 0$), or the $\mathbb{R}$-indefinite semi-inner product (if $t = 0$), also denoted by φ_t,

$$\varphi_t\left((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{\infty} [f_n, g_n]_t\,. \tag{3.26}$$

under the complete $\mathbb{R}$-semi-norm $\|.\|_{\mathbf{l}^{t:2}}$ defined by

$$\|(f_n)_{n=0}^{\infty}\|_{\mathbf{l}^{t:2}} \overset{\text{def}}{=} \sqrt{\sum_{n=0}^{\infty} \|f_n\|_t^2}. \tag{3.27}$$

Theorem 3.10. *Let $\mathbf{l}^2\left(\mathbb{H}_t\right)$ be a $\mathbb{R}$-vector space of* (3.5) *equipped with the bilinear symmetric form φ_t of* (3.26)*, and the $\mathbb{R}$-semi-norm $\|.\|_{\mathbf{l}^{t:2}}$ of* (3.27)*. Then*

$$\left(\mathbf{l}^2\left(\mathbb{H}_t\right), \varphi_t\right) \overset{\text{iso}}{=} \left(\mathbf{H}_{t:2}\left[[q]\right], \varphi_t\right), \ \textit{isometrically}, \tag{3.28}$$

as complete $\mathbb{R}$-semi-normed definite, or indefinite $\mathbb{R}$-semi-inner-product spaces.

Proof. The bijection,

$$\sum_{n=0}^{\infty} q^n f_n \in \mathbf{H}_{t:2}\left[[q]\right] \longmapsto (f_n)_{n=0}^{\infty} \in \mathbf{l}^2\left(\mathbb{H}_2\right),$$

is an isometric $\mathbb{R}$-vector-space isomorphism in the sense that: it is a $\mathbb{R}$-vector-space isomorphism satisfying

$$\varphi_t\left(\sum_{n=0}^{\infty} q^n f_n, \sum_{n=0}^{\infty} q^n g_n\right) = \sum_{n=0}^{\infty} [f_n, g_n]_t = \varphi_t\left((f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty}\right),$$

and

$$\left\|\sum_{n=0}^{\infty} q^n f_n\right\|_{\mathbf{H}_{t:2}} = \sqrt{\sum_{n=0}^{\infty} \|f_n\|_t^2} = \|(f_n)_{n=0}^{\infty}\|_{\mathbf{l}^{t:2}},$$

by (3.16), (3.20), (3.21), (3.25), (3.26) and (3.27). □

The above theorem provides an isometrically isomorphic $\mathbb{R}$-vector space $\mathbf{l}^2(\mathbb{H}_t)$ of the $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2}[[q]]$ by (3.28). By the isomorphism theorem (3.28), we also call the $\mathbb{R}$-vector space $\mathbf{l}^2(\mathbb{H}_t)$ of (3.25), the $\mathbb{H}_t$-Hardy space.

Assumption and Notation 3.1. (in short, **AN 3.1** from below) If there are no confusions, then we denote the $\mathbb{H}_t$-Hardy spaces $\mathbf{H}_{t:2}[[q]]$ and $\mathbf{l}^2(\mathbb{H}_t)$ simply by $\mathbf{H}_{t:2}$, respectively, by $\mathbf{l}^{t:2}$, from now on.

4. Certain Subspaces of the $\mathbb{H}_t$-Hardy Space

In this section, we construct a certain type of $\mathbb{R}$(-vector)-subspaces of our $\mathbb{H}_t$-Hardy space $\mathbf{H}_{t:2} \overset{\text{iso}}{=} \mathbf{l}^{t:2}$, for a fixed scale $t \in \mathbb{R}$. Throughout this section, fix $N \in \mathbb{N}$, and define a subset $\mathbf{l}^{t:2}_N$ of $\mathbf{l}^{t:2}$ by

$$\mathbf{l}^{t:2}_N \overset{\text{def}}{=} \left\{ (f_n)_{n=0}^{\infty} \in \mathbf{l}^{t:2} : f_k = 0 \in \mathbb{H}_t, \ \forall k \geq N \right\},$$

i.e., (4.1)

$$\mathbf{l}^{t:2}_N = \{(f_0, f_1, ..., f_{N-1}, 0, 0, 0, ...) : f_l \in \mathbb{H}_t, \ \forall l = 0, ..., N-1\}.$$

Then the family $\mathbf{l}^{t:2}_N$ becomes a $\mathbb{R}$-subspace of $\mathbf{l}^{t:2}$, because

$$(f_0, ..., f_{N-1}, 0, 0, ...) + (g_0, ..., g_{N-1}, 0, 0, ...) = (f_0 + g_0, ..., f_{N-1} + g_{N-1}, 0, 0, ...),$$

and (4.2)

$$r(f_0, f_1, ..., f_{N-1}, 0, 0, ...) = (rf_0, rf_1, ..., rf_{N-1}, 0, 0, 0, ...),$$

in $\mathbf{l}^{t:2}_N$, for all $r \in \mathbb{R}$. So, by the isomorphism theorem (3.28), we have the isomorphic $\mathbb{R}$-subspace $\mathbf{H}_{t:2:N}$ of $\mathbf{H}_{t:2}$,

$$\mathbf{H}_{t:2:N} \overset{\text{def}}{=} \left\{ \sum_{n=0}^{N-1} q^n f_n \in \mathbf{H}_{t:2} : f_n \in \mathbb{H}_t, \ \forall n = 0, 1, ..., N-1 \right\}, \tag{4.3}$$

i.e.,

$$\mathbf{H}_{t:2:N} \ni \sum_{n=0}^{N-1} q^n f_n = \left(\sum_{n=0}^{N-1} q^n f_n \right) + \left(\sum_{n=N}^{\infty} q^n 0 \right) \in \mathbf{H}_{t:2}.$$

By the definitions (4.1) and (4.3), these $\mathbb{R}$-vector spaces $\mathbf{l}^{t:2}_N$ and $\mathbf{H}_{t:2:N}$ have their bounded definite, or indefinite $\mathbb{R}$-semi-inner product φ_t,

$$\varphi_{t,N}((f_0, ..., f_{N-1}, 0, 0, ...), (g_0, ..., g_{N-1}, 0, 0, ...)) = \sum_{n=0}^{N-1} [f_n, g_n]_t,$$

and (4.4)

$$\varphi_{t,N}\left(\sum_{n=0}^{N-1} q^n f_n, \sum_{n=0}^{N-1} q^n g_n \right) = \sum_{n=0}^{N-1} [f_n, g_n]_t.$$

Similarly, they have their complete $\mathbb{R}$-semi-norm,

$$\|(f_0, ..., f_{N-1}, 0, 0, ...)\|_{t:N} \overset{\text{denote}}{=} \|(f_0, ..., f_{N-1}, 0, 0, ...)\|_{\mathbf{l}^{t:2}},$$

and (4.5)

$$\left\| \sum_{n=0}^{N-1} q^n f_n \right\|_{t:N} \overset{\text{denote}}{=} \left\| \sum_{n=0}^{N-1} q^n f_n \right\|_{\mathbf{H}_{t:2}},$$

satisfying (4.6)

$$\|(f_0, ..., f_{N-1}, 0, 0, ...)\|_{t:N} = \sqrt{\sum_{n=0}^{N-1} \|f_n\|_t^2} = \left\| \sum_{n=0}^{N-1} q^n f_n \right\|_{t:N},$$

by (4.5). So, up to subspace topology, the $\mathbb{R}$-subspaces $\mathbf{l}_N^{t:2}$ and $\mathbf{H}_{t:2:N}$ form complete $\mathbb{R}$-semi-normed definite, or indefinite $\mathbb{R}$-semi-inner-product spaces inside $\mathbf{l}^{t:2}$, respectively, $\mathbf{H}_{t:2}$, by (4.4) and (4.6).

Corollary 4.1. *For $N \in \mathbb{N}$, the $\mathbb{R}$-subspaces $\mathbf{l}_N^{t:2} \subset \mathbf{l}^{t:2}$ of* (4.1) *and $\mathbf{H}_{t:2:N} \subset \mathbf{H}_{t:2}$ of* (4.3) *are isometrically isomorphic as complete $\mathbb{R}$-semi-normed definite, or indefinite $\mathbb{R}$-semi-inner-product spaces. i.e.,*

$$\mathbf{l}_N^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2:N}, \qquad \forall N \in \mathbb{N}. \tag{4.7}$$

Proof. As we seen in the above paragraph, two $\mathbb{R}$-vector spaces $\mathbf{l}_N^{t:2}$ and $\mathbf{H}_{t:2:N}$ are well-defined complete $\mathbb{R}$-semi-normed definite, or indefinite $\mathbb{R}$-semi-inner-product spaces in the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$. Similar to the proof of (3.25), one can define an isometric isomorphism,

$$\sum_{n=0}^{N-1} q^n f_n \in \mathbf{H}_{t:2:N} \longmapsto (f_0, ..., f_{N-1}, 0, 0, ...) \in \mathbf{l}_N^{t:2},$$

by (4.2), (4.4) and (4.6). Therefore, the structure theorem (4.7) holds. □

The above corollary confirms that the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$ contains its $\mathbb{R}$-subspaces $\left\{\mathbf{l}_N^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2:N}\right\}_{N \in \mathbb{N}}$. Define now the Cartesian product set $\mathbb{H}_t^N$ of N-copies of the t-scaled hypercomplexes $\mathbb{H}_t$, by

$$\mathbb{H}_t^N = \{(f_1, f_2, ..., f_N) : f_l \in \mathbb{H}_t, \ \forall l = 1, 2, ..., N\}. \tag{4.8}$$

Then, this Cartesian product set $\mathbb{H}_t^N$ of (4.8) becomes a $\mathbb{R}$-vector space under the vector-addition,

$$(f_1, ..., f_N) + (g_1, ..., g_N) = (f_1 + g_1, ..., f_N + g_N),$$

and the $\mathbb{R}$-scalar-product, (4.9)

$$r(f_1, ..., f_N) = (rf_1, ..., rf_N), \quad \forall r \in \mathbb{R}.$$

Also, one can define a definite, or indefinite $\mathbb{R}$-semi-inner product $[,]_{t,N}$,

$$[(f_1, ..., f_N), (g_1, ..., g_N)]_{t,N} \overset{\text{def}}{=} \sum_{k=1}^{N} [f_k, g_k]_t. \tag{4.10}$$

In particular, if $t < 0$, then the form $[,]_{t,N}$ of (4.10) becomes a $\mathbb{R}$-inner product on $\mathbb{H}_t^N$, since $[,]_t$ is a $\mathbb{R}$-inner product on $\mathbb{H}_t$; if $t > 0$, then $[,]_{t,N}$ is a $\mathbb{R}$-indefinite inner product on $\mathbb{H}_t^N$, since $[,]_t$ is a $\mathbb{R}$-indefinite inner product on $\mathbb{H}_t$; and if $t = 0$, then it is a $\mathbb{R}$-indefinite semi-inner product on $\mathbb{H}_0^N$, since $[,]_0$ is a $\mathbb{R}$-indefinite semi-inner product on $\mathbb{H}_0$. Clearly, one can define the $\mathbb{R}$-semi-norm on $\mathbb{H}_t^N$ by

$$\|(f_1, ..., f_N)\|_{t,N} \overset{\text{def}}{=} \sqrt{\sum_{k=1}^{N} \|f_k\|_t^2}. \tag{4.11}$$

Especially, if $t \neq 0$, then $\|.\|_{t,N}$ of (4.11) becomes a $\mathbb{R}$-norm on $\mathbb{H}_t^N$, since $\|.\|_t$ is a $\mathbb{R}$-norm on $\mathbb{H}_t$; and if $t = 0$, then it is a $\mathbb{R}$-semi-norm on $\mathbb{H}_0^N$, since $\|.\|_0$ is a $\mathbb{R}$-semi-norm on $\mathbb{H}_0$.

Theorem 4.2. *The Cartesian-product set $\mathbb{H}_t^N$ of* (4.8) *forms a definite, or indefinite $\mathbb{R}$-semi-inner-product complete $\mathbb{R}$-semi-normed space. In particular,*

$$\mathbb{H}_t^N \overset{\text{iso}}{=} \mathbf{l}_N^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2:N}, \qquad \forall N \in \mathbb{N}. \tag{4.12}$$

Proof. Recall that, by (4.7), the $\mathbb{R}$-subspaces $\mathbf{l}_N^{t:2}$ and $\mathbf{H}_{t:2:N}$ are isometrically isomorphic as definite, or indefinite $\mathbb{R}$-semi-inner-product complete $\mathbb{R}$-semi-normed spaces. So, if we show the first relation of (4.12) holds, then one can conclude that the set $\mathbb{H}_t^N$ of (4.8) is a definite, or indefinite $\mathbb{R}$-semi-inner-product complete $\mathbb{R}$-semi-normed space equipped with the form $[,]_{t,N}$ of (4.10), and the morphism $\|.\|_{t,N}$ of (4.11).

Define a bijective morphism $\Psi_{t,N} : \mathbb{H}_t^N \to \mathbf{l}_N^{t:2}$ by

$$\Psi_{t,N}\left((f_1, ..., f_N)\right) \overset{\text{def}}{=} \left(g_0, g_1, ..., g_{N-1}, 0, 0, ...\right),$$

with (4.13)

$$g_n = f_{n+1} \in \mathbb{H}_t, \quad \forall n = 0, 1, ..., N-1.$$

Then this bijection $\Psi_{t,N}$ satisfies that

$$\Psi_{t,N}\left(r_1 W_1 + r_2 W_2\right) = r_1 \Psi_{t,N}\left(W_1\right) + r_2 \Psi_{t,N}\left(W_2\right),$$

for all $r_1, r_2 \in \mathbb{R}$ and $W_1, W_2 \in \mathbb{H}_t^N$, and hence, it is a $\mathbb{R}$-vector-space-isomorphism. Moreover, it is isometric in the sense that

$$\varphi_{t,N}\left(\Psi_{t,N}\left(W_1\right), \Psi_{t,N}\left(W_2\right)\right) = \left[W_1, W_2\right]_{t,N},$$

and

$$\left\|\Psi_{t,N}\left(W_1\right)\right\|_{t:N} = \left\|W_1\right\|_{t,N},$$

where $\varphi_{t,N}$ and $\|.\|_{t:N}$ are in the sense of (4.4) and (4.5), respectively, and where $[,]_{t,N}$ and $\|.\|_{t,N}$ are in the sense of (4.10) and (4.11), respectively. So, the $\mathbb{R}$-vector-space isomorphism $\Psi_{t,N}$ of (4.13) is isometric, too. Therefore, $\mathbb{H}_t^N$ and $\mathbf{l}_N^{t:2}$ are isometrically isomorphic over $\mathbb{R}$, and hence, the isomorphic relation (4.12) holds true. □

By (4.12), one can understand $\mathbb{H}_t^N$, $\mathbf{l}_N^{t:2}$ and $\mathbf{H}_{t:2:N}$ as isomorphic definite, or indefinite $\mathbb{R}$-semi-inner-product complete $\mathbb{R}$-semi-normed spaces embedded in the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$, for all $N \in \mathbb{N}$. In this paper, we focus on the $\mathbb{R}$-vector space $\mathbb{H}_t^N$ of (4.8).

Let $B_{\mathbb{R}}\left(\mathbb{H}_t^N\right)$ be the operator $\mathbb{R}$-algebra consisting of all bounded $\mathbb{R}$-linear operators on $\mathbb{H}_t^N$ equipped with its operator-semi-norm,

$$\|T\| = \sup\left\{\|T(v)\|_{t,N} : \|v\|_{t,N} = 1\right\}, \quad \forall T \in B_{\mathbb{R}}\left(\mathbb{H}_t^N\right).$$

Now, we are interested in a certain type of operators of $B_{\mathbb{R}}\left(\mathbb{H}_t\right)$. Define a subset $M_N\left(\mathbb{H}_t\right)$ by

$$\mathcal{M}_{t,N} \overset{\text{denote}}{=} M_N\left(\mathbb{H}_t\right) = \left\{[h_{i,j}]_{N\times N} : h_{i,j} \in \mathbb{H}_t\right\}, \tag{4.14}$$

where

$$[h_{i,j}]_{N\times N} = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,N} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N,1} & h_{N,2} & \cdots & h_{N,N} \end{pmatrix}, \text{ with } h_{i,j} \in \mathbb{H}_t,$$

acts on

$$\mathbb{H}_t^N = \left\{ \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} : f_l \in \mathbb{H}_t, \ \forall l = 1, ..., N \right\},$$

canonically under the usual block-matrix action, i.e.,

$$[h_{i,j}]_{N\times N}\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{N} h_{1,k}f_k \\ \sum_{k=1}^{N} h_{2,k}f_k \\ \vdots \\ \sum_{k=1}^{N} h_{N,k}f_k \end{pmatrix},$$

having its operator-semi-norm,

$$\left\|[h_{i,j}]_{N\times N}\right\| = \max\left\{\|(h_{k,1},...,h_{k,N})\|_{t,N} : k=1,...,N\right\} < \infty,$$

and hence, $[h_{i,j}]_{N\times N} \in B_{\mathbb{R}}\left(\mathbb{H}_t^N\right)$, implying indeed that

$$\mathcal{M}_{t,N} = M_N\left(\mathbb{H}_t\right) \subseteq B_{\mathbb{R}}\left(\mathbb{H}_t^N\right).$$

Definition 4.3. The family $\mathcal{M}_{t,N} \overset{\text{def}}{=} M_N\left(\mathbb{H}_t\right)$ of (4.14) is called the $\mathbb{H}_t$-matrix algebra (for $N \in \mathbb{N}$).

As we discussed above, the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is a subset of $B_{\mathbb{R}}\left(\mathbb{H}_t^N\right)$. Also, under the usual block-matrix addition,

$$[f_{i,j}]_{N\times N} + [g_{i,j}]_{N\times N} = [f_{i,j}+g_{i,j}]_{N\times N},$$

and the $\mathbb{R}$-scalar product,

$$r\,[f_{i,j}]_{N\times N} = [rf_{i,j}]_{N\times N}, \quad \forall r \in \mathbb{R},$$

and the block-matrix multiplication,

$$\left([f_{i,j}]_{N\times N}\right)\left([g_{i,j}]_{N\times N}\right) = [d_{i,j}]_{N\times N}, \text{ with } d_{i,j} = \sum_{k=1}^{N} f_{i,k}g_{k,j},$$

indeed, our $\mathbb{H}_t$-matrix algebra forms a $\mathbb{R}$-algebra embedded in $B_{\mathbb{R}}\left(\mathbb{H}_t^N\right)$. Moreover, one can get that

$$\left[[f_{i,j}]_{N\times N}\left(W_1\right), W_2\right]_{t,N} = \left[W_1,\ \left[f_{j,i}^{\circledast}\right]_{N\times N}\left(W_2\right)\right]_{t,N}, \tag{4.15}$$

for all $W_1, W_2 \in \mathbb{H}_t^N$.

Theorem 4.4. *The $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ of* (4.14) *is a complete $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-algebra under the $\mathbb{R}$-operator-semi-normed subspace topology for $B_{\mathbb{R}}\left(\mathbb{H}_t^N\right)$. i.e.,*

$$\mathcal{M}_{t,N} \textit{ is a complete } \mathbb{R}\textit{-semi-normed } \mathbb{R}\textit{-}*\textit{-algebra.} \tag{4.16}$$

Proof. As we discussed above, the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is a $\mathbb{R}$-operator-semi-normed $\mathbb{R}$-algebra consisting of all bounded block matrices in $\mathbb{H}_t$ over $\mathbb{R}$. If we define an operation $(*)$ on $\mathcal{M}_{t,N}$ by

$$[h_{i,j}]_{N\times N}^{*} \overset{\text{def}}{=} \left[h_{j,i}^{\circledast}\right]_{N\times N} \in \mathcal{M}_{t,N}, \quad \forall\,[h_{i,j}]_{N\times N} \in \mathcal{M}_{t,N}, \tag{4.17}$$

then it satisfies that

$$T^{**} = T, \ \ (rT)^{*} = rT^{*}, \ \ \forall T \in \mathcal{M}_{t,N}, \ r \in \mathbb{R},$$

and

$$\left(T_1+T_2\right)^{*} = T_1^{*}+T_2^{*}, \ \ \left(T_1T_2\right)^{*} = T_2^{*}T_1^{*},$$

for all $T_1, T_2 \in \mathcal{M}_{t,N}$, by (4.15). i.e., this operation $(*)$ of (4.17) forms a $\mathbb{R}$-adjoint on $\mathcal{M}_{t,N}$. Therefore, the structure theorem (4.16) holds. □

The above theorem shows that the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N} = M_N(\mathbb{H}_t)$ acts on $\mathbb{H}_t^N$ as a complete $\mathbb{R}$-operator-semi-normed $\mathbb{R}$-$*$-algebra of all adjointable bounded operators of $B_{\mathbb{R}}(\mathbb{H}_t^N)$, in the sense of (4.17), by (4.16).

Now, let M be the action (3.19) of the t-scaled hypercomplexes $\mathbb{H}_t$ acting on the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$, i.e.,

$$M_h\left((f_n)_{n=0}^{\infty}\right) = (hf_n)_{n=0}^{\infty} \overset{\text{iso}}{=} \sum_{n=0}^{\infty} q^n (hf_n) = M_h\left(\sum_{n=0}^{\infty} q^n f_n\right).$$

By (4.12), one can restrict the action M of (3.19) as an action of $\mathbb{H}_t$ acting on $\mathbb{H}_t^N$, i.e.,

$$M : h \in \mathbb{H}_t \longmapsto M_h \in B_{\mathbb{R}}(\mathbb{H}_t^N),$$

where (4.18)

$$M_h((h_1, ..., h_N)) = (hh_1, ..., hh_N), \ \forall h \in \mathbb{H}_t.$$

Then the family $\{M_h : h \in \mathbb{H}_t\}$ forms a well-defined $\mathbb{R}$-$*$-algebra on $\mathbb{H}_t^N$, as a realization of $\mathbb{H}_t$ acting on $\mathbb{H}_t^N$.

Theorem 4.5. *Let M be the action* (4.18) *of $\mathbb{H}_t$ acting on $\mathbb{H}_t^N$, as a restriction of the action M of* (3.19)*. Then the realization $M(\mathbb{H}_t) = \{M_h : h \in \mathbb{H}_t\}$ satisfies that*

$$M(\mathbb{H}_t) \overset{\text{set}}{=} \left\{hI \in B_{\mathbb{R}}(\mathbb{H}_t^N) : h \in \mathbb{H}_t\right\} \overset{*\text{-subalgebra}}{\subset} \mathcal{M}_{t,N}, \tag{4.19}$$

where I is the identity operator satisfying $I(W) = W$, for all $W \in \mathbb{H}_t^N$.

Proof. By (4.18), clearly, the realization $M(\mathbb{H}_t)$ is equipotent to

$$\{hI : h \in \mathbb{H}_t\} \subset B_{\mathbb{R}}(\mathbb{H}_t^N).$$

So, the set-equality of (4.19) holds. Note that the identity operator I is identified with the identity matrix $I_N \in \mathcal{M}_{t,N}$,

$$I_N = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{N \times N} \in \mathcal{M}_{t,N},$$

where $1 = 1 + 0i + 0j_t + 0k_t \in \mathbb{H}_t$. It shows that

$$M(\mathbb{H}_t) \overset{\text{iso}}{=} \{hI_N \in \mathcal{M}_{t,N} : h \in \mathbb{H}_t\} \subset \mathcal{M}_{t,N},$$

as the collection of all $\mathbb{H}_t$-constant matrices of $\mathcal{M}_{t,N}$. So, the family $M(\mathbb{H}_t)$ is $*$-homomorphic to $\mathcal{M}_{t,N}$, satisfying

$$(hI_N)^* = I_N h^{\circledast} = h^{\circledast} I_N \in M(\mathbb{H}_t), \text{ in } \mathcal{M}_{t,N}.$$

Therefore, the relation in (4.19) holds, too. □

It is clear that $M(\mathbb{H}_t) \overset{\text{iso}}{=} \mathbb{H}_t$ realized on $\mathbb{H}_t^N$ as $\mathbb{H}_t$-constant matrices by (4.19).

Since our $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is a $\mathbb{R}$-$*$-algebra, one can have the following operator-theoretic properties of $\mathbb{H}_t$-matrices.

Definition 4.6. Let $\mathcal{M}_{t,N}$ be the $\mathbb{H}_t$-matrix algebra.
(1) T is self-adjoint in $\mathcal{M}_{t,N}$, if $T^* = T$ on $\mathbb{H}_t^N$.
(2) T is a projection in $\mathcal{M}_{t,N}$, if $T^* = T = T^2$ on $\mathbb{H}_t^N$.
(3) T is normal in $\mathcal{M}_{t,N}$, if $T^*T = TT^*$ on $\mathbb{H}_t^N$.
(4) T is an isometry in $\mathcal{M}_{t,N}$, if $T^*T = I_N$ on $\mathbb{H}_t^N$.
(5) T is unitary in $\mathcal{M}_{t,N}$, if $T^*T = I_N = TT^*$ on $\mathbb{H}_t^N$.

The following result characterizes the self-adjointness on $\mathcal{M}_{t,N}$.

Theorem 4.7. *An $\mathbb{H}_t$-matrix $[h_{i,j}]_{N\times N}$ is self-adjoint in $\mathcal{M}_{t,N}$, if and only if*

$$h_{j,i} = h_{i,j}^{\circledast} \in \mathbb{H}_t, \quad \forall i,j \in \{1,...,N\},$$

if and only if (4.20)

$$[h_{i,j}]_{N\times N} = \begin{pmatrix} h_{1,1} & h_{2,1}^{\circledast} & h_{3,1}^{\circledast} & \cdots & h_{N,1}^{\circledast} \\ h_{2,1} & h_{2,2} & h_{3,2}^{\circledast} & \cdots & h_{N,2}^{\circledast} \\ h_{3,1} & h_{3,2} & h_{3,3} & \cdots & h_{N,3}^{\circledast} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ h_{N,1} & h_{N,2} & \cdots & h_{N,N-1} & h_{N,N} \end{pmatrix}, \text{ with } h_{k,k}^{\circledast} = h_{k,k}.$$

Proof. By the $\mathbb{R}$-adjoint (4.17) on $\mathcal{M}_{t,N}$, one has $[h_{i,j}]_{N\times N}$ is self-adjoint in $\mathcal{M}_{t,N}$, if and only if

$$[h_{i,j}]_{N\times N}^{*} = \left[h_{j,i}^{\circledast}\right]_{N\times N} = [h_{i,j}]_{N\times N},$$

if and only if

$$h_{j,i}^{\circledast} = h_{i,j} \in \mathbb{H}_t, \quad \forall i,j \in \{1,...,N\}.$$

Therefore, the characterization (4.20) holds. □

The above theorem characterizes the self-adjointness on $\mathcal{M}_{t,N}$ in terms of $\mathbb{H}_t$-entries by (4.20).

Corollary 4.8. *An element $M_h \in M(\mathbb{H}_t)$ is self-adjoint in $\mathcal{M}_{t,N}$, if and only if h is $\circledast$-self-adjoint in $\mathbb{H}_t$, i.e.,*

$$M_h \in M(\mathbb{H}_t) \text{ is self-adjoint in } \mathcal{M}_{t,N}, \Longleftrightarrow h^{\circledast} = h \text{ in } \mathbb{H}_t. \tag{4.21}$$

Proof. By (4.19), the realization $M(\mathbb{H}_t)$ is isomorphic to the $*$-subalgebra $\{hI_N : h \in \mathbb{H}_t\}$ of $\mathbb{H}_t$-constant matrices in $\mathcal{M}_{t,N}$. So, by (4.20), $M_h \overset{\text{iso}}{=} hI_N$ is self-adjoint in $\mathcal{M}_{t,N}$, if and only if $h^{\circledast} = h$, in $\mathbb{H}_t$. So, the relation (4.21) holds. □

Let $T = [h_{i,j}]_{N\times N} \in \mathcal{M}_{t,N}$ be an $\mathbb{H}_t$-matrix. Observe that

$$T^2 = [d_{i,j}]_{N\times N}, \text{ with } d_{i,j} = \sum_{k=1}^{N} h_{i,k}h_{k,j}.$$

So, if T is self-adjoint in $\mathcal{M}_{t,N}$, then

$$T^2 = [d_{i,j}]_{N\times N}, \text{ with } d_{i,j} = \sum_{k=1}^{N} h_{i,k}h_{j,k}^{\circledast},$$

where (4.22)

$$h_{k,k}^{\circledast} = h_{k,k} \in \mathbb{H}_t, \quad \forall k = 1,...,N.$$

Theorem 4.9. *An $\mathbb{H}_t$-matrix $[h_{i,j}]_{N\times N}$ is a projection in $\mathcal{M}_{t,N}$, if and only if*

$$h_{i,j} = h_{j,i}^{\circledast} = \sum_{k=1}^{N} h_{i,k}h_{j,k}^{\circledast} \in \mathbb{H}_t, \quad \forall i,j = 1,...,N. \tag{4.23}$$

Proof. Without loss of generality, assume that an $\mathbb{H}_t$-matrix $T = [h_{i,j}]_{N\times N}$ is self-adjoint in $\mathcal{M}_{t,N}$, i.e.,

$$h_{j,i}^{\circledast} = h_{i,j} \in \mathbb{H}_t, \quad \forall i,j = 1,...,N,$$

by (4.20). Then, such a self-adjoint $\mathbb{H}_t$-matrix T is a projection, if and only if

$$T^2 = T, \quad \text{in} \quad \mathcal{M}_{t,N},$$

if and only if

$$h_{i,j} = \sum_{k=1}^{N} h_{i,k} h_{j,k}^{\circledast} \in \mathbb{H}_t, \quad \forall i, j = 1, ..., N,$$

by (4.22). Therefore, the projection-property (4.23) holds on $\mathcal{M}_{t,N}$. □

The above theorem characterizes the projection-property on $\mathcal{M}_{t,N}$ by (4.23).

Corollary 4.10. *An operator $M_h \in M(\mathbb{H}_t)$ with $h \in \mathbb{H}_t$ is a projection on $\mathbb{H}_t^N$, if and only if*

$$\text{either } M_h = M_1 \overset{\text{iso}}{=} I_N, \text{ or } M_h = M_0 \overset{\text{iso}}{=} O_N, \tag{4.24}$$

where O_N is the zero $\mathbb{H}_t$-matrix of $\mathcal{M}_{t,N}$ whose $\mathbb{H}_t$-entries are $0 = 0 + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$.

Proof. By applying (4.23), one has that $M_h \overset{\text{iso}}{=} hI_N$ is a projection, if and only if

$$M_h^* = M_{h^{\circledast}} \overset{\text{iso}}{=} h^{\circledast} I_N = hI_N = h^2 I_N \overset{\text{iso}}{=} M_{h^2} = M_h^2,$$

on $\mathbb{H}_t^N$, if and only if

$$h^{\circledast} = h = h^2 \quad \text{in} \quad \mathbb{H}_t.$$

The first equality $h^{\circledast} = h$ implies that h is a real number in $\mathbb{H}_t$, i.e., $h = h + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$ with $h \in \mathbb{R}$. So, the second equality implies that

$$h^2 = h \Longleftrightarrow h = 1, \text{ or } 0, \text{ in } \mathbb{R}.$$

So, the operator M_h is a projection, if and only if

$$\text{either } M_h = M_1 \overset{\text{iso}}{=} I_N, \text{ or } M_h = M_0 \overset{\text{iso}}{=} O_N, \text{ in } \mathcal{M}_{t,N}.$$

Thus, the relation (4.24) holds. □

As a special case of (4.23), one obtains the projection-property (4.24) on $M(\mathbb{H}_t)$ in $\mathcal{M}_{t,N}$.

Now, let $T = [h_{i,j}]_{N \times N} \in \mathcal{M}_{t,N}$ be an $\mathbb{H}_t$-matrix with its adjoint $T^* = \left[h_{j,i}^{\circledast}\right]_{N \times N} \in \mathcal{M}_{t,N}$. Then

$$T^*T = [d_{i,j}]_{N \times N}, \text{ with } d_{i,j} = \sum_{k=1}^{N} h_{k,i}^{\circledast} h_{k,j},$$

and (4.25)

$$TT^* = [e_{i,j}]_{N \times N}, \text{ with } e_{i,j} = \sum_{k=1}^{N} h_{i,k} h_{j,k}^{\circledast},$$

by the straightforward computations.

Theorem 4.11. *An $\mathbb{H}_t$-matrix $T = [h_{i,j}]_{N \times N}$ is normal in $\mathcal{M}_{t,N}$, if and only if*

$$\sum_{k=1}^{N} \left(h_{k,i}^{\circledast} h_{k,j} - h_{i,k} h_{j,k}^{\circledast} \right) = 0 = 0 + 0i + 0j_t + 0k_t, \tag{4.26}$$

in $\mathbb{H}_t$, for all $i, j = 1, ..., N$.

Proof. By definition, a given $\mathbb{H}_t$-matrix T is normal in $\mathcal{M}_{t,N}$, if and only if $T^*T = TT^*$ in $\mathcal{M}_{t,N}$, if and only if

$$\sum_{k=1}^{N} h_{k,i}^{\circledast} h_{k,j} = \sum_{k=1}^{N} h_{i,k} h_{j,k}^{\circledast}, \text{ in } \mathbb{H}_t, \ \forall i, j = 1, ..., N,$$

by (4.25), if and only if the relation (4.26) holds, for all $i, j = 1, ..., N$. □

The above theorem characterizes the normality on $\mathcal{M}_{t,N}$ in terms of the $\mathbb{H}_t$-entries of $\mathbb{H}_t$-matrices of $\mathcal{M}_{t,N}$, by (4.26).

Corollary 4.12. *Every element $M_h \in M(\mathbb{H}_t)$ for $h \in \mathbb{H}_t$ is normal on $\mathbb{H}_t^N$. i.e.,*

$$\text{All elements of } M(\mathbb{H}_t) \text{ are normal on } \mathbb{H}_t^N. \tag{4.27}$$

Proof. Recall again that if $M_h \in M(\mathbb{H}_t)$, then it is isomorphic to $hI_N \in \mathcal{M}_{t,N}$. So,

$$M_h^* \overset{\text{iso}}{=} (hI_N)^* = h^{\circledast} I_N \overset{\text{iso}}{=} M_{h^{\circledast}}, \text{ on } \mathbb{H}_t^N.$$

Thus, one can get that: M_h is normal on $\mathbb{H}_t^N$, if and only if hI_N is normal in $\mathcal{M}_{t,N}$, if and only if

$$(hI_N)^*(hI_N) = \left(h^{\circledast}h\right) I_N = \left(hh^{\circledast}\right) I_N = (hI_N)(hI_N)^*,$$

in $\mathcal{M}_{t,N}$, if and only if (4.28)

$$h^{\circledast}h = hh^{\circledast}, \quad \text{in} \quad \mathbb{H}_t.$$

However, every t-scaled hypercomplex number $h = a + bj_t \in \mathbb{H}_t$ with $a, b \in \mathbb{C}$ automatically satisfies that

$$h^{\circledast}h = |a|^2 - t|b|^2 = \left(|a|^2 - t|b|^2\right) + 0i + 0j_t + 0k_t = hh^{\circledast},$$

in $\mathbb{H}_t$. It implies that every operator $M_h \in M(\mathbb{H}_t)$, isomorphic to $hI_N \in \mathcal{M}_{t,N}$, satisfies (4.28). Therefore, the normality (4.27) on $M(\mathbb{H}_t)$ holds. □

The above corollary shows that every operator of $M(\mathbb{H}_t)$ is normal on $\mathbb{H}_t^N$ by (4.26) and (4.28).

Also, by (4.25), we obtain the following isometry-property on $\mathcal{M}_{t,N}$.

Theorem 4.13. *An $\mathbb{H}_t$-matrix $T = [h_{i,j}]_{N\times N}$ is an isometry in $\mathcal{M}_{t,N}$, if and only if*

$$\sum_{k=1}^{N} h_{k,i}^{\circledast} h_{k,j} = \begin{cases} 1 & \text{if } i = j \in \{1, ..., N\} \\ 0 & \text{if } i \neq j \in \{1, ..., N\}, \end{cases} \tag{4.29}$$

in $\mathbb{H}_t$, for all $i, j = 1, ..., N$.

Proof. By definition, a given $\mathbb{H}_t$-matrix T is an isometry in $\mathcal{M}_{t,N}$, if and only if $T^*T = I_N$ in $\mathcal{M}_{t,N}$, if and only if all main-diagonal $\mathbb{H}_t$-entries of T^*T are identical to $1 = 1 + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$, and all off-diagonal $\mathbb{H}_t$-entries of T^*T are identical to $0 = 0 + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$, if and only if the relation (4.29) holds by (4.25). □

The above theorem characterizes the isometry-property on $\mathcal{M}_{t,N}$ by (4.29). So, one obtains a following special case.

Corollary 4.14. *An operator $M_h \in M(\mathbb{H}_t)$, with $h = a + bj_t$ for $a, b \in \mathbb{C}$, is an isometry on $\mathbb{H}_t^N$, if and only if*

$$|a|^2 = 1 + t|b|^2, \text{ in } \mathbb{C}. \tag{4.30}$$

Proof. Since $M_h \overset{\text{iso}}{=} hI_N$ in $\mathcal{M}_{t,N}$, it is an isometry on $\mathbb{H}_t^N$, if and only if hI_N is an isometry in $\mathcal{M}_{t,N}$, if and only if

$$(hI_N)^*(hI_N) = \left(h^{\circledast}h\right) I_N = I_N, \text{ in } \mathcal{M}_{t,N},$$

if and only if

$$h^{\circledast}h = 1, \quad \text{in} \quad \mathbb{H}_t,$$

if and only if

$$h^{\circledast}h = |a|^2 - t|b|^2 = 1, \text{ in } \mathbb{H}_t,$$

if and only if the relation (4.30) holds. □

The above corollary characterizes the isometry-property on $M(\mathbb{H}_t)$ on $\mathbb{H}_t^N$ by (4.30). Let's consider an interesting application of (4.30). Suppose $h = x + uj_t$ with $x, u \in \mathbb{R}$ in $\mathbb{H}_t$. i.e., h is a t-hyperbolic number in the sense of [3]. Recall that, in [3], we considered a sub-structure,

$$\mathbb{D}_t = \{x + 0i + uj_t + 0k_t \in \mathbb{H}_t : x, u \in \mathbb{R}\} \subset \mathbb{H}_t,$$

called the t-scaled hyperbolics. Remark that $\mathbb{D}_{-1}$ is isomorphic to the complex field $\mathbb{C}$; and $\mathbb{D}_1$ is isomorphic to the classical hyperbolic numbers $\mathcal{D} = \left\{x + uj : x, u \in \mathbb{R},\ j^2 = 1\right\}$; and $\mathbb{D}_0$ is isomorphic to the dual numbers $\mathsf{D} = \left\{x + uJ : x, u \neq \mathbb{R},\ J^2 = 0\right\}$. If $w = x + uj_t \in \mathbb{D}_t$ in $\mathbb{H}_t$ with $x, u \in \mathbb{R}$, then

$$w^{\circledast} w = x^2 - tu^2 = ww^{\circledast}, \text{ in } \mathbb{D}_t \subset \mathbb{H}_t,$$

as a $\mathbb{R}$-quantity. So, by (4.30), $M_w \overset{\text{iso}}{=} wI_N$ is an isometry on $\mathbb{H}_t^N$, if and only if $x^2 - tu^2 = 1$ in $\mathbb{R}$, if and only if

$$\begin{cases} x^2 + |t|\, u^2 = 1 & \text{if } t = -|t| < 0 \\ x^2 - tu^2 = 1 & \text{if } t > 0 \\ x^2 = 1 & \text{if } t = 0, \end{cases}$$

for $t \in \mathbb{R}$. It shows that: M_w is an isometry on $\mathbb{H}_t^N$, if and only if (i) $(x, u) \in \mathbb{R}^2$ is contained in the boundary of the oval figure $\left\{(x, u) : x^2 + |t|\, u^2 = 1\right\}$ in $\mathbb{R}^2$ if $t < 0$; (ii) $(x, u) \in \mathbb{R}^2$ is contained in the hyperbolic lines $\left\{(x, u) : x^2 = tu^2\right\}$ if $t > 0$; and (iii) $(x, u) \in \mathbb{R}^2$ is contained in the vertical straight lines $\{(\pm 1, u) : u \in \mathbb{R}\}$ in $\mathbb{R}^2$ if $t = 0$.

Theorem 4.15. *An $\mathbb{H}_t$-matrix $T = [h_{i,j}]_{N \times N}$ is unitary in $\mathcal{M}_{t,N}$, if and only if*

$$\sum_{k=1}^{N} h_{k,i}^{\circledast} h_{k,j} = \sum_{k=1}^{N} \left(h_{i,k} h_{j,k}^{\circledast}\right) = \begin{cases} 1 & \text{if } i = j \in \{1, ..., N\} \\ \\ 0 & \text{if } i \neq j \in \{1, ..., N\}, \end{cases} \tag{4.31}$$

in $\mathbb{H}_t$, for all $i, j = 1, ..., N$.

Proof. By definition, a given $\mathbb{H}_t$-matrix T is unitary in $\mathcal{M}_{t,N}$, if and only if it is both a normal operator, and an isometry in $\mathcal{M}_{t,N}$, if and only if

$$\sum_{k=1}^{N} h_{k,i}^{\circledast} h_{k,j} = \sum_{k=1}^{N} \left(h_{i,k} h_{j,k}^{\circledast}\right),$$

and

$$\sum_{k=1}^{N} h_{k,i}^{\circledast} h_{k,j} = \begin{cases} 1 & \text{if } i = j \in \{1, ..., N\} \\ \\ 0 & \text{if } i \neq j \in \{1, ..., N\}, \end{cases}$$

by the normality (4.26), respectively, by the isometry-property (4.29), for all $i, j = 1, ..., N$, if and only if the condition (4.31) holds. $\square$

The unitarity on $\mathcal{M}_{t,N}$ is characterized by (4.31).

Corollary 4.16. *An operator $M_h \in M(\mathbb{H}_t)$, with $h = a + bj_t \in \mathbb{H}_t$ for $a, b \in \mathbb{C}$, is unitary on $\mathbb{H}_t^N$, if and only if it is an isometry in the sense of* (4.30).

Proof. By (4.27), every element of $M(\mathbb{H}_t)$ is automatically normal on $\mathbb{H}_t^N$. So, an operator M_h is unitary on $\mathbb{H}_t^N$, if and only if it is an isometry. And the isometry-property on $M(\mathbb{H}_t)$ is characterized by (4.30). $\square$

5. $\mathbb{H}_t$-Toeplitz Matrices on $\mathbb{H}_t^N$

In this section, we construct, and study a special type of $\mathbb{H}_t$-matrices of $\mathcal{M}_{t,N} = M_N(\mathbb{H}_t)$ acting on the definite, or indefinite $\mathbb{R}$-semi-inner-product complete $\mathbb{R}$-semi-normed space $\mathbb{H}_t^N$ for a fixed $N \in \mathbb{N}$. In particular, we are interested in Toepliz-like matrices. Also, the construction of such $\mathbb{H}_t$-matrices are motivated by those of so-called $\mathbb{H}_t$-Toeplitz operators of [6, 7].

Let's define an $\mathbb{H}_t$-matrix U by

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{N\times N} \in \mathcal{M}_{t,N},$$

having its $\mathbb{R}$-adjoint U^*,

$$U^* = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{N\times N} \in \mathcal{M}_{t,N},$$

where $1 = 1 + 0i + 0j_t + 0k_t,\ 0 = 0 + 0i + 0j_t + 0k_t \in \mathbb{H}_t$. i.e.,

$$U((f_1, f_2, ..., f_{N-1}, f_N)) = (0, f_1, f_2 ..., f_{N-1}),$$

and (5.1)

$$U^*((f_1, f_2, ..., f_{N-1}, f_N)) = (f_2, ..., f_{N-1}, f_N, 0),$$

on $\mathbb{H}_t^N$, for all $(f_1, ..., f_N) \in \mathbb{H}_t^N$.

Definition 5.1. We call the $\mathbb{H}_t$-matrices U and U^* of (5.1), the forward, respectively, the backward shifts on $\mathbb{H}_t^N$.

It is not hard to check that

$$U^N = O_N = (U^*)^N, \quad \text{in} \quad \mathcal{M}_{t,N},$$

more generally, (5.2)

$$U^{N+k} = O_N = (U^*)^{N+k}, \text{ in } \mathcal{M}_{t,N},\ \forall k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Equivalently, the forward, and the backward shifts U and U^* of (5.1) are nilpotent in $\mathcal{M}_{t,N}$ with their nilpotences N, in the sense that: the quantity $N \in \mathbb{N}$ is the smallest quantity making $U^N = O_N = (U^*)^N$ in $\mathcal{M}_{t,N}$.

Proposition 5.2. *Let $U, U^* \in \mathcal{M}_{t,N}$ be the forward, respectively, the backward shifts of* (5.1).

$$U \text{ and } U^* \text{ are nilpotent with their nilpotences } N. \tag{5.3}$$

Proof. By the definition (5.1) of the shifts $U, U^* \in \mathcal{M}_{t,N}$, there exists $N \in \mathbb{N}$, such that

$$U^{N+k} = O_N = (U^*)^{N+k} \in \mathcal{M}_{t,N}, \quad \forall k \in \mathbb{N}_0.$$

Therefore, the nilpotent property (5.3) holds in $\mathcal{M}_{t,N}$. □

From the forward shift U of (5.1) and its $\mathbb{R}$-adjoint U^*, the backward shift of (5.1), satisfying (5.3), we define a certain type of $\mathbb{H}_t$-matrices.

Definition 5.3. Let U and U^* be the forward, and the backward shifts (5.1) in $\mathcal{M}_{t,N}$. An $\mathbb{H}_t$-matrix,

$$T = \sum_{k=1}^{N-1} (U^*)^k (h_{-k} I_N) + \sum_{k=0}^{N-1} U^k (h_k I_N) \in \mathcal{M}_{t,N}$$

with axiomatization: (5.4)

$$U^0 = I_N = (U^*)^0 \in \mathcal{M}_{t,N},$$

is called a $\mathbb{H}_t$-Toeplitz matrix, where $h_j \in \mathbb{H}_t$, for all $j \in \{0, \pm 1, ..., \pm (N-1)\}$. i.e., an $\mathbb{H}_t$-matrix,

$$T = \begin{pmatrix} h_0 & h_{-1} & h_{-2} & \cdots & h_{-(N-1)} \\ h_1 & h_0 & h_{-1} & \ddots & \vdots \\ h_2 & h_1 & h_0 & \ddots & h_{-2} \\ \vdots & \vdots & \ddots & \ddots & h_{-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{pmatrix} \in \mathcal{M}_{t,N},$$

is called an $\mathbb{H}_t$-Toeplitz matrix of $\mathcal{M}_{t,N}$.

By (5.4), every $\mathbb{H}_t$-Toeplitz matrix $T = [h_{i-j}]_{N \times N} \in \mathcal{M}_{t,N}$ is isomorphic to

$$T = \sum_{k=1}^{N-1} (U^*)^k M_{h_{-k}} + \sum_{k=0}^{N-1} U^k M_h \in B_{\mathbb{R}} \left(\mathbb{H}_t^N \right),$$

where $M_h \in B_{\mathbb{R}} \left(\mathbb{H}_t^N \right)$ are in the sense of (3.22), isomorphic to $hI_N \in \mathcal{M}_{t,N}$, for all $h \in \mathbb{H}_t$.

If the readers check the forward shift $\mathbf{U}$, and the backward shift $\mathbf{U}^*$ acting on the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$ in [6, 7], i.e.,

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \text{ and } \mathbf{U}^* = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

on (5.5)

$$\mathbf{l}^{t:2} = \left\{ \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{pmatrix} : \sum_{n=0}^{\infty} \|f_n\|_t^2 < \infty \right\},$$

then they are "not" nilpotent in the sense that: there does not exist any natural quantity $n \in \mathbb{N}$, such that $\mathbf{U}^n = O = (\mathbf{U}^*)^n$ on $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$, where O is the zero operator on $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$. Also, the readers can check, in [7], that the $\mathbb{H}_t$-Toeplitz operators $\mathbf{T}$ are defined by

$$\mathbf{T} = \sum_{n=1}^{\infty} (\mathbf{U}^*) (h_{-n} I) + \sum_{n=0}^{\infty} \mathbf{U}^n (h_n I),$$

with (5.6)

$$(h_{-n})_{n=1}^{\infty}, \quad (h_n)_{n=0}^{\infty} \in \mathbf{l}^{\infty},$$

satisfying

$$\sup\left\{\left\|\sum_{n=1}^{\infty} q^n h_{-n}\right\|_t : q \in \mathbb{U}_t\right\} < \infty,$$

and

$$\sup\left\{\left\|\sum_{n=0}^{\infty} q^n h_n\right\|_t : q \in \mathbb{U}_t\right\} < \infty,$$

on the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$, where $\mathbf{U}$ and $\mathbf{U}^*$ are in the sense of (5.5) and $\mathbb{U}_t$ is the unit open ball of $\mathbb{H}_t$, and where I is the identity operator on $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$. So, if we compress the $\mathbb{H}_t$-Toeplitz operators (5.6) acting on the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2}$ to those on $\mathbf{l}_N^{t:2} \overset{\text{iso}}{=} \mathbf{H}_{t:2:N} \overset{\text{iso}}{=} \mathbb{H}_t^N$, then the compressions of $\mathbb{H}_t$-Toeplitz operators becomes our $\mathbb{H}_t$-Toeplitz matrices of $\mathcal{M}_{t,N}$.

Theorem 5.4. *Let* $\mathbf{T} \in B_{\mathbb{R}}\left(\mathbf{l}^{t:2}\right)$ *be an* $\mathbb{H}_t$*-Toeplitz operator* (5.6)*, introduced in* [7]*. For* $N \in \mathbb{N}$*, if*

$$P_{[N]} \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 & \cdots & 0 & & \cdots & \cdots \\ 0 & 1 & \ddots & \vdots & & \cdots & \\ 0 & 0 & \ddots & 0 & \ddots & & \\ & \ddots & \ddots & \underbrace{1}_{(N,N)\text{-}th} & 0 & \ddots & \\ \vdots & & \ddots & 0 & 0 & 0 & \\ & \vdots & & \ddots & 0 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix},$$

in $B_{\mathbb{R}}\left(\mathbf{l}^{t:2}\right)$*, then*

$$P_{[N]}\mathbf{T}P_{[N]} \in B_{\mathbb{R}}\left(\mathbf{l}_N^{t:2}\right),$$

and (5.7)

$$P_{[N]}\mathbf{T}P_{[N]} \overset{\text{iso}}{=} T, \text{ the } \mathbb{H}_t\text{-Toeplitz matrix (5.4) in } \mathcal{M}_{t,N}.$$

Proof. By (5.5) and (5.6), one has

$$\mathbf{T} = \begin{pmatrix} h_0 & h_{-1} & h_{-2} & h_{-3} & \cdots \\ h_1 & h_0 & h_{-1} & h_{-2} & \ddots \\ h_2 & h_1 & h_0 & h_{-1} & \ddots \\ h_3 & h_2 & h_1 & h_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \in B_{\mathbb{R}}\left(\mathbf{l}^{t:2}\right),$$

and hence, if $P_{[N]} \in B_{\mathbb{R}}\left(\mathbf{l}^{t:2}\right)$ is the above projection, satisfying $P_{[N]}^{*} = P_{[N]} = P_{[N]}^{2}$ on $\mathbf{l}^{t:2}$ (e.g., see [7]), then

$$P_{[N]}\mathbf{T}P_{[N]} = \begin{pmatrix} h_0 & h_{-1} & \cdots & h_{-(N-1)} & 0 & \cdots \\ h_1 & h_0 & \ddots & \vdots & \vdots & \cdots \\ \vdots & \ddots & \ddots & h_{-1} & \vdots & \cdots \\ h_{N-1} & \cdots & h_1 & h_0 & 0 & \cdots \\ 0 & \cdots & \cdots & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

identified with

$$P_{[N]}\mathbf{T}P_{[N]} = \begin{pmatrix} T & O \\ O & O \end{pmatrix}, \text{ as a operator-block matrix,}$$

where T is the $\mathbb{H}_t$-Toeplitz matrix (5.4) in $\mathcal{M}_{t,N}$. Thus, this compression $P_{[N]}\mathbf{T}P_{[N]}$ is a well-defined on the $\mathbb{R}$-subspace $\mathbf{l}_N^{t:2}$ of the $\mathbb{H}_t$-Hardy space $\mathbf{l}^{t:2}$, and hence,

$$P_{[N]}\mathbf{T}P_{[N]} \overset{\text{iso}}{=} T, \quad \text{on} \quad \mathbb{H}_t^N \overset{\text{iso}}{=} \mathbf{l}_N^{t:2}.$$

Therefore, the compressions $P_{[N]}\mathbf{T}P_{[N]}$ of $\mathbb{H}_t$-Toeplitz operators $\mathbf{T}$ of (5.6) by the projection $P_{[N]}$ are (isomorphic to) our $\mathbb{H}_t$-Toeplitz matrices T of (5.4). □

The above theorem shows the relation between $\mathbb{H}_t$-Toeplitz operators of [7] and our $\mathbb{H}_t$-Toeplitz matrices by (5.7). The $\mathbb{H}_t$-Toeplitz operators (5.4) of $\mathcal{M}_{t,N}$ are (isomorphic to) the $P_{[N]}$-compression of $\mathbb{H}_t$-Toeplitz operators (5.6) of $B_{\mathbb{R}}\left(\mathbf{l}^{t:2}\right)$.

Now, consider the following projections P_k and Q_k of $\mathcal{M}_{t,N}$,

$$P_k = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & \underbrace{1}_{(k,k)\text{-th}} & & & \\ & & & & 0 & & \\ 0 & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}_{N \times N} \in \mathcal{M}_{t,N},$$

and (5.8)

$$Q_k = \begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \underbrace{1}_{(N-k,N-k)\text{-th}} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}_{N \times N} \in \mathcal{M}_{t,N},$$

for all $k \in \{1, ..., N\}$.

Theorem 5.5. *If P_k and Q_k are the projections* (5.8) *in $\mathcal{M}_{t,N}$, for $k = 1, ..., N$, then the forward, and the backward shifts U and U^* of* (5.1) *satisfy that:*

$$(U^*)^{n_1} U^{n_2} = \begin{cases} (U^*)^{n_1 - n_2} P_{N-n_2} & \text{if } n_1 + n_2 < 2N-1, \ n_1 \geq n_2 \\ P_{N-n_1} U^{n_2 - n_1} & \text{if } n_1 + n_2 < 2N-1, \ n_1 \leq n_2 \\ O_N & \text{otherwise,} \end{cases}$$

and (5.9)

$$U^{n_1} (U^*)^{n_2} = \begin{cases} U^{n_1 - n_2} Q_{N-n_2} & \text{if } n_1 + n_2 < 2N-1, \ n_1 \geq n_2 \\ Q_{N-n_1} (U^*)^{n_2 - n_1} & \text{if } n_1 + n_2 < 2N-1, \ n_1 \leq n_2 \\ O_N & \text{otherwise,} \end{cases}$$

for all $n_1, n_2 \in \mathbb{N}$.

Proof. By (5.1), one obtains that

$$U^*U = P_{N-1}, \text{ and } UU^* = Q_{N-1}, \text{ in } \mathcal{M}_{t,N}.$$

Inductively, one can get that

$$(U^*)^n U^n = P_{N-n}, \ U^n (U^*)^n = Q_{N-n}, \ \forall n = 1, ..., N, \tag{5.10}$$

and, by the nilpotent-property (5.3), if either $n_1 \geq N$, or $n_2 \geq N$, then

$$(U^*)^{n_1} U^{n_2} = O_N = U^{n_2} (U^*)^{n_1}.$$

Equivalently, if $n_1 + n_2 \geq 2N - 1$, then

$$(U^*)^{n_1} U^{n_2} = O_N = (U^*)^{n_2} U^{n_1}, \text{ in } \mathcal{M}_{t,N}.$$

Now, suppose $n_1 + n_2 < 2N - 1$. If $n_1 \geq n_2$, then

$$(U^*)^{n_1} U^{n_2} = (U^*)^{n_1-n_2} ((U^*)^{n_2} U^{n_2}) = (U^*)^{n_1-n_2} P_{N-n_2},$$

and

$$U^{n_1} (U^*)^{n_2} = U^{n_1-n_2} (U^{n_2} (U^*)^{n_2}) = U^{n_1-n_2} Q_{N-n_2},$$

by (5.10). Meanwhile, if $n_1 \leq n_2$, then

$$(U^*)^{n_1} U^{n_2} = ((U^*)^{n_1} U^{n_1}) U^{n_2-n_1} = P_{N-n_1} U^{n_2-n_1},$$

and

$$U^{n_1} (U^*)^{n_2} = (U^{n_1} (U^*)^{n_1}) (U^*)^{n_2-n_1} = Q_{N-n_1} (U^*)^{n_2-n_1},$$

in $\mathcal{M}_{t,N}$, by (5.10). Therefore, the formulas in (5.9) hold true. □

The formulas of (5.9) illustrate the following properties of U and U^* on $\mathcal{M}_{t,N}$.

Corollary 5.6. (1) *The forward, and the backward shifts U and U^* are not self-adjoint in $\mathcal{M}_{t,N}$, and hence, they are not projections in $\mathcal{M}_{t,N}$, either.*
(2) *U and U^* are not normal in $M_{t,N}$.*
(3) *U and U^* are not isometries in $\mathcal{M}_{t,N}$, and hence, they are not unitary in $\mathcal{M}_{t,N}$, either.*

Proof. Clearly, the forward shift U is not self-adjoint, since its $\mathbb{R}$-adjoint is the backward shift U^* in $\mathcal{M}_{t,N}$. By the non-self-adjointness, these $\mathbb{H}_t$-matrices cannot be projections in $\mathcal{M}_{t,N}$.

By (5.8) and (5.9), one has that

$$U^*U = P_{N-1} \neq Q_{N-1} = UU^*, \text{ in } \mathcal{M}_{t,N},$$

implying the non-normality of both U and U^* in $\mathcal{M}_{t,N}$. It implies also that neither U nor U^* is an isometry in $\mathcal{M}_{t,N}$, and hence, they cannot be unitary in $\mathcal{M}_{t,N}$. □

The above corollary shows that the generating operators $\{U, U^*\}$ of all $\mathbb{H}_t$-Toeplitz matrices (5.4) of $\mathcal{M}_{t,N}$ disobey the fundamental operator-theoretic properties, self-adjointness, projection-property, normality, isometry-property, and unitarity. However, such a nilpotent $\mathbb{H}_t$-matrices satisfy the following additional property.

Definition 5.7. An $\mathbb{H}_t$-matrix $T \in \mathcal{M}_{t,N}$ is said to be a partial isometry, if T^*T is a projection in $\mathcal{M}_t$.

As in the usual operator theory, by definition, it is not difficult to check that T is a partial isometry, if and only if T^*T is a projection, if and only if TT^* is a projection, if and only if T^* is a partial isometry, in $\mathcal{M}_{t,N}$, if and only if $T = TT^*T$, if and only if $T^* = T^*TT^*$, in $\mathcal{M}_{t,N}$.

Theorem 5.8. *The forward shift U is a partial isometry in $\mathcal{M}_{t,N}$, equivalently, the backward shift U^* is a partial isometry in $\mathcal{M}_{t,N}$. i.e.,*

$$U \text{ and } U^* \text{ are partial isometries in } \mathcal{M}_{t,N}. \tag{5.11}$$

Proof. The operator-theoretic property (5.11) is immediately proven by (5.9), especially, by the special case (5.10). Indeed, the operators U^*U and UU^* are identified with the projections P_{N-1}, respectively, Q_{N-1} of (5.8), in $\mathcal{M}_{t,N}$. Therefore, the relation (5.11) holds true.

Independently, one can check that

$$U(f_1, f_2, ..., f_N) = (0, f_1, ..., f_{N-1}),$$

and

$$\begin{aligned} UU^*U(f_1, ..., f_N) &= UP_{N-1}(f_1, ..., f_N) \\ &= U(f_1, ..., f_{N-1}, 0) \\ &= (0, f_1, ..., f_{N-1}), \end{aligned}$$

for all $(f_1, ..., f_N) \in \mathbb{H}_t^N$, implying that

$$U = UU^*U, \quad \text{in} \quad \mathcal{M}_{t,N}.$$

Thus, the forward shift U is a partial isometry, and hence, its $\mathbb{R}$-adjoint U^*, the backward shift, is a partial isometry, too. Therefore, the relation (5.11) is re-proven. □

The above corollary and theorem show that even though the $\mathbb{H}_t$-matrices U and U^* do not satisfy fundamental operator-theoretic properties introduced in Section 4, they are characterized to be partial isometries by (5.11). The following corollary summarize the operator-theoretic properties of U and U^* in $\mathcal{M}_{t,N}$.

Corollary 5.9. *The forward, and the backward shifts U and U^* are nilpotent partial isometries in $\mathcal{M}_{t,N}$ with their nilpotences N.*

Proof. It is shown by (5.3) and (5.11). □

From the partial isometries U and U^*, if

$$n_1, n_2 \in \mathbb{N} \text{ satisfy } n_1 + n_2 < 2N - 1,$$

and (5.12)

$$S_{n_1,n_2} \overset{\text{denote}}{=} (U^*)^{n_1} U^{n_1} + U^{n_2} (U^*)^{n_2} = P_{N-n_1} + Q_{N-n_2},$$

in $\mathcal{M}_{t,N} \setminus \{O_N\}$, then

$$S_{1,1} = P_{N-1} + Q_{N-1} = \begin{pmatrix} 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ 0 & & & & 1 \end{pmatrix},$$

more generally, if $|n_1 - n_2| > \frac{N}{2}$ under the condition of (5.12), then

$$S_{n_1,n_2} = \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 2 & & & & & \\ & & & |n_1-n_2|\text{-times} & \ddots & & & & \\ & & & & & 2 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 1 \end{pmatrix},$$

meanwhile (5.13)

$$S_{n_1,n_2} = \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & |n_1-n_2|\text{-times} & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 1 \end{pmatrix},$$

if $|n_1 - n_2| < \frac{N}{2}$ under the condition of (5.12).

Proposition 5.10. *Under the condition* (5.12)*, an* $\mathbb{H}_t$*-matrix* $S_{n_1,n_2} \in \mathcal{M}_{t,N}$ *satisfies that*

$$S_{n_1,n_2} = P_{N-n_1} + Q_{N-n_2} = [h_{i,j}]_{N \times N},$$

with (5.14)

$$h_{k,k} = \begin{cases} 1 & \text{if } k = 1, ..., N-n_2, N-n_1, ..., N \\ 2 & \text{if } |n_1-n_2| > \frac{N}{2},\ k = N-n_2+1, ..., N-|n_1-n_2|, \\ 0 & \text{if } |n_1-n_2| < \frac{N}{2},\ k = N-n_2+1, ..., N-|n_1-n_2| \\ 1 & \text{if } |n_1-n_2| = 0, \quad \forall k = 1, ..., N, \end{cases}$$

and

$$h_{k_1,k_2} = 0 = 0 + 0i + 0j_t + 0k_t \in \mathbb{H}_t, \text{ if } k_1 \neq k_2.$$

Proof. Under the condition (5.12), an $\mathbb{H}_t$-matrix S_{n_1,n_2} is a non-zero operator of $\mathcal{M}_{t,N}$ by (5.9). Moreover, by (5.13), one can get the resulted $\mathbb{H}_t$-matrix (5.14). In particular, the last result of (5.14) for the case where $|n_1 - n_2| = 0$ is verified again by (5.13). Remark that this case can happen only when $n_1 = n_2$ in $\mathbb{N}$, and $N = n_1 + n_2$ is even in $\mathbb{N}$. $\square$

By (5.14), one can obtain the following corollary immediately.

Corollary 5.11. *Under the condition* (5.12), *if* $S_{n_1,n_2} \in \mathcal{M}_{t,N}$ *is in the sense of* (5.12), *then*

$$S_{n_1,n_2} \text{ is a projection in } \mathcal{M}_{t,N}, \Longleftrightarrow |n_1 - n_2| = 0, \text{ or } |n_1 - n_2| < \tfrac{N}{2}. \tag{5.15}$$

Proof. By (5.12), the self-adjointness of S_{n_1,n_2} is guaranteed because

$$S_{n_1,n_2} = P_{N-n_1} + Q_{N-n_2} \in \mathcal{M}_{t,N}$$

is the sum of two projections, and hence,

$$S^*_{n_1,n_2} = (P_{N-n_1} + Q_{N-n_2})^* = P_{N-n_1} + Q_{N-n_2} = S_{n_1,n_2},$$

in $\mathcal{M}_{t,N}$. So, to check the projection-property of S_{n_1,n_2}, it is sufficient to check its idempotence; $S^2_{n_1,n_2} = S_{n_1,n_2}$ in $\mathcal{M}_{t,N}$. However, by (5.14), we have that

$$S_{n_1,n_2} = \begin{pmatrix} 1 & & & & & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 2 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ 0 & & & & & & & & 1 \end{pmatrix},$$

or

$$S_{n_1,n_2} = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix},$$

or

$$S_{n_1,n_2} = I_N, \Longleftrightarrow n_1 = n_2, \text{ and } N = n_1 + n_2 \text{ is even in } \mathbb{N}.$$

It is easy to check that the first case where $|n_1 - n_2| > \frac{N}{2}$ does not provide S_{n_1,n_2} as a projection, since $S^2_{n_1,n_2} \neq S_{n_1,n_2}$ in $\mathcal{M}_{t,N}$. However, the other two cases give us a projection S_{n_1,n_2}, satisfying

$$S^2_{n_1,n_2} = S_{n_1,n_2}, \quad \text{in} \quad \mathcal{M}_{t,N}.$$

So, the projection-property (5.15) holds for S_{n_1,n_2}, where $n_1, n_2 \in \mathbb{N}$ satisfy the condition of (5.12). $\square$

6. Some Statistical-Analytic Data on $\mathcal{M}_{t,N}$

In this section, we establish two different types of statistical-analytic structures on our $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$, for a fixed scale $t \in \mathbb{R}$, and a fixed quantity $N \in \mathbb{N}$, acting on the definite, or indefinite $\mathbb{R}$-semi-inner-product complete $\mathbb{R}$-semi-normed space $\mathbb{H}_t^N$. In particular, we are considering some statistical data up to the two non-equivalent $\mathbb{R}$-linear functionals on $\mathcal{M}_{t,N}$. This study is motivated by the well-known free probability theory (e.g., see [22, 25]). But the free probability theory is established over the complex field $\mathbb{C}$ on noncommutative algebras "over $\mathbb{C}$." As we have seen above, our structures are "over the real field $\mathbb{R}$." So, we cannot use, or apply the concepts, methods, and languages from free probability, however, we mimic the free-probabilistic techniques and tools on our structure $\mathcal{M}_{t,N}$ over $\mathbb{R}$.

6.1. **The Noncommutative Statistical Space $(\mathcal{M}_{t,N}, \varphi_1)$ over $\mathbb{R}$.** On the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$, let's define a $\mathbb{R}$-linear functional $\varphi_1 : \mathcal{M}_{t,N} \to \mathbb{R}$ by

$$\varphi_1(T) \overset{\text{def}}{=} [T(\mathbf{v_1}), \mathbf{v_1}]_{t,N}, \quad \forall T \in \mathcal{M}_{t,N},$$

where (6.1.1)

$$\mathbf{v}_1 = (1, 0, 0, ..., 0) \in \mathbb{H}_t^N,$$

where $[,]_{t,N}$ is the definite, or indefinite semi-inner product (4.10) on $\mathbb{H}_t^N$. Then, by the bilinearity of $[,]_{t,N}$ on $\mathbb{H}_t^N$, the morphism φ_1 of (6.1.1) is indeed a well-defined $\mathbb{R}$-linear functional on $\mathcal{M}_{t,N}$. Moreover, it is bounded since

$$|\varphi_1(T)| = \left|[T(\mathbf{v}_1), \mathbf{v_1}]_{t,N}\right| \leq \|T\| \|\mathbf{v}_1\|_{t,N}^2 = \|T\|,$$

for all $T \in \mathcal{M}_{t,N}$, where $\|\mathbf{v}_1\|_{t,N} = \sqrt{\|1\|_t^2 + \|0\|_t^2 + ... + \|0\|_t^2} = 1$ by (4.11).

By (6.1.1), if $T = [h_{i,j}]_{N \times N} \in \mathcal{M}_{t,N}$, then we have that

$$\varphi_1(T) = \left[[h_{i,j}]_{N\times N} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]_{t,N} = \left[\begin{pmatrix} h_{1,1} \\ h_{2,1} \\ h_{3,1} \\ \vdots \\ h_{N,1} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right]_{t,N}$$

$$= [h_{1,1}, 1]_t + [h_{2,1}, 0]_t + [h_{3,1}, 0]_t + ... + [h_{N,1}, 0]_t$$

$$= \tau(h_{1,1}1^{\circledast}) + \tau(h_{2,1}0^{\circledast}) + \tau(h_{3,1}0^{\circledast}) + ... + \tau(h_{N,1}0^{\circledast})$$

$$= \tau(h_{1,1}) = \text{Re}(h_{1,1}),$$

implying that (6.1.2)

$$\varphi_1\left([h_{i,j}]_{N\times N}\right) = \tau(h_{1,1}) = \text{Re}(h_{1,1}).$$

Thus, we obtain that

$$\varphi_1(I_N) = \tau(1) = \text{Re}(1) = 1, \tag{6.1.3}$$

Furthermore, one can get that

$$\varphi_1\left([h_{i,j}]_{N\times N}^*\right) = \text{Re}\left(h_{1,1}^{\circledast}\right) = \text{Re}(h_{1,1}) = \varphi\left([h_{i,j}]_{N\times N}\right), \tag{6.1.4}$$

demonstrating that, indeed, the linear functional φ_1 of (6.1.1) is $\mathbb{R}$-valued up to the $\mathbb{R}$-adjoint $(*)$ on $\mathcal{M}_{t,N}$.

Definition 6.1. The pair (A, ψ) of a (commutative, or noncommutative) $\mathbb{R}$-algebra A and a $\mathbb{R}$-linear functional ψ on A is called a (commutative, respectively, noncommutative) statistical space over $\mathbb{R}$ (in short, $\mathbb{R}$-statistical space). In particular, if the $\mathbb{R}$-algebra A contains its unity 1_A, and $\psi(1_A) = 1$, then the $\mathbb{R}$-statistical space (A, ψ) is said to be unital. Also, if A is a topological $\mathbb{R}$-algebra, and if ψ is bounded, then (A, ψ) is called a topological $\mathbb{R}$-statistical space. Similarly, if A is a $\mathbb{R}$-$*$-algebra, then (A, ψ) is also called a $*$-statistical space over $\mathbb{R}$ (in short, $\mathbb{R}$-$*$-statistical space).

By definition, one can get the following result.

Proposition 6.2. *The pair $(\mathcal{M}_{t,N}, \varphi_1)$ is a complete semi-normed unital noncommutative $\mathbb{R}$-$*$statistical space, satisfying*

$$\varphi_1\left([h_{i,j}]^*_{N\times N}\right) = \operatorname{Re}\left(h^{\circledast}_{1,1}\right) = \operatorname{Re}(h_{1,1}) = \varphi_1\left([h_{i,j}]_{N\times N}\right).$$

Proof. By definition, the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is a well-defined complete (operator-)semi-normed(-topological) noncommutative $\mathbb{R}$-$*$-algebra. Also, by (6.1.1), the linear functional φ_1 is bounded, and unital by (6.1.3). Thus, the pair $(\mathcal{M}_{t,N}, \varphi_1)$ forms a complete semi-normed unital $\mathbb{R}$-$*$-statistical space. The formula is shown by (6.1.2) and (6.1.4). □

The above proposition characterizes the structure $(\mathcal{M}_{t,N}, \varphi_1)$ as a noncommutative topological unital $\mathbb{R}$-$*$-statistical space. On it, let's consider some statistical data on $\mathcal{M}_{t,N}$ up to φ_1.

Theorem 6.3. *Let $T_l = \left[h^{(l)}_{i,j}\right]_{N\times N} \in (\mathcal{M}_{t,N}, \varphi_1)$, for $l = 1, ..., n$, for $n \in \mathbb{N}$. Then*

$$\varphi_1\left(\prod_{l=1}^{n} T_l\right) = \operatorname{Re}\left(\sum_{(k_1,...,k_{n-1})\in\{1,...,N\}^{n-1}} h^{(1)}_{1,k_1} h^{(2)}_{k_1,k_2} h^{(3)}_{k_2,k_3} ... h^{(n)}_{k_{n-1},1}\right). \tag{6.1.5}$$

Proof. Under hypothesis, one has that

$$\prod_{l=1}^{n} T_l = \prod_{l=1}^{n} \left[h^{(l)}_{i,j}\right]_{N\times N} = \left(\left[h^{(1)}_{i,j}\right]_{N\times N} \left[h^{(2)}_{i,j}\right]_{N\times N}\right)(T_3...T_n)$$

$$= \left(\left[\sum_{k_1=1}^{N} h^{(1)}_{i,k_1} h^{(2)}_{k_1,j}\right]_{N\times N} \left[h^{(3)}_{i,j}\right]_{N\times N}\right)(T_4...T_n)$$

$$= \left[\sum_{k_2=1}^{N}\left(\sum_{k_1=1}^{N} h^{(1)}_{i,k_1} h^{(2)}_{k_1,k_2}\right) h^{(3)}_{k_2,j}\right]_{N\times N} (T_4...T_n)$$

$$= \left[\sum_{(k_1,k_2)\in\{1,...,N\}^2} h^{(1)}_{i,k_1} h^{(2)}_{k_1,k_2} h^{(3)}_{k_2,j}\right]_{N\times N} (T_4...T_n) = ...$$

$$... = \left[\sum_{(k_1,...,k_{n-1})\in\{1,...,N\}^{n-1}} h^{(1)}_{i,k_1} h^{(2)}_{k_1,k_2} ... h^{(n)}_{k_{n-1},j}\right]_{N\times N},$$

in $\mathcal{M}_{t,N}$, having its $(1,1)$-entry, (6.1.6)

$$\sum_{(k_1,...,k_{n-1})\in\{1,...,N\}^{n-1}} h^{(1)}_{1,k_1} h^{(2)}_{k_1,k_2} h^{(3)}_{k_2,k_3} ... h^{(n)}_{k_{n-1},1} \in \mathbb{H}_t.$$

Thus,

$$\varphi_1\left(\prod_{l=1}^{n} T_l\right) = \operatorname{Re}\left(\sum_{(k_1,...,k_{n-1})\in\{1,...,N\}^{n-1}} h^{(1)}_{1,k_1} h^{(2)}_{k_1,k_2} ... h^{(n)}_{k_{n-1},1}\right),$$

by (6.1.2) and (6.1.6). Therefore, the analytic data (6.1.5) holds. □

The analytic data (6.1.5) provides a general tool to compute the statistical information on $(\mathcal{M}_{t,N}, \varphi_1)$.

Theorem 6.4. *Let $M_{h_l} \in M(\mathbb{H}_t)$ with $h_l \in \mathbb{H}_t$, for $l = 1, ..., s$, for $s \in \mathbb{N}$, isomorphic to $T_l = h_l I_N \in (\mathcal{M}_{t,N}, \varphi_1)$, for all $l = 1, ..., s$. Then, for any*

$$(l_1, ..., l_n) \in \{1, ..., s\}^n, \quad \forall n \in \mathbb{N},$$

we have (6.1.7)

$$\varphi_1\left(\prod_{k=1}^{n} T_{l_k}\right) = \text{Re}\left(\prod_{k=1}^{n} h_{l_k}\right) = \tau\left(\prod_{k=1}^{n} h_{l_k}\right).$$

Proof. Recall that every multiplication operator $M_h \in M(\mathbb{H}_t)$ acting on $\mathbb{H}_t^N$ is isomorphic to the $\mathbb{H}_t$-matrix $hI_N \in \mathcal{M}_{t,N}$, for all $h \in \mathbb{H}_t$. So, under hypothesis, we have that

$$\prod_{k=1}^{n} T_{l_k} = \prod_{k=1}^{n} (h_{l_k} I_N) = \left(\prod_{k=1}^{n} h_{l_k}\right) I_N \in \mathcal{M}_{t,N},$$

by (6.1.6), having its $(1,1)$-entry,

$$\prod_{k=1}^{n} h_{l_k} \in \mathbb{H}_t, \ \forall (l_1, ..., l_n) \in \{1, ..., s\}^n, \ \forall n \in \mathbb{N}.$$

Thus, by (6.1.2), one has that

$$\varphi_1\left(\prod_{k=1}^{n} T_{l_k}\right) = \text{Re}\left(\prod_{k=1}^{n} h_{l_k}\right),$$

for all $(l_1, ..., l_n) \in \{1, ..., s\}^n$, for all $n \in \mathbb{N}$. Therefore the analytic data (6.1.7) holds. □

If we understand the pair $(\mathbb{H}_t, \tau)$ as a complete semi-normed unital $\mathbb{R}$-$*$-statistical space in the sense of Definition 42, then one can conclude from (6.1.7) as follow.

Corollary 6.5. *If $\mathcal{M}(\mathbb{H}_t) = \{hI_N : h \in \mathbb{H}_t\}$ is a $\mathbb{R}$-$*$-subalgebra of $\mathcal{M}_{t,N}$, consisting of all $\mathbb{H}_t$-constant matrices, then*

$$\left(\mathcal{M}(\mathbb{H}_t), \varphi_1 = \varphi_1 \mid_{\mathcal{M}(\mathbb{H}_t)}\right) \overset{\text{equi}}{=} (\mathbb{H}_t, \tau),$$

in the sense that: (6.1.8)

$$\exists \textit{isometric isomorphism } \Psi : \mathcal{M}(\mathbb{H}_t) \to \mathbb{H}_t,$$

such that

$$\tau(\Psi(T)) = \varphi_1(T), \quad \forall T \in \mathcal{M}(\mathbb{H}_t).$$

Proof. By (4.19), the family $\mathcal{M}(\mathbb{H}_t) = \{hI_N : h \in \mathbb{H}_t\}$ is isometrically isomorphic to $M(\mathbb{H}_t)$. Note and recall that $M(\mathbb{H}_t)$ is isometrically isomorphic to $\mathbb{H}_t$. So, there exists an isometric isomorphism,

$$\Psi : hI_N \in \mathcal{M}(\mathbb{H}_t) \longmapsto h \in \mathbb{H}_t.$$

Moreover, by (6.1.7), one has that

$$\varphi_1(hI_N) = \text{Re}(h) = \tau(h) = \tau(\Psi(hI_N)) \in \mathbb{R}, \ \forall h \in \mathbb{H}_t.$$

Therefore, the equivalence (6.1.8) of $(\mathcal{M}(\mathbb{H}_t), \varphi_1)$ and $(\mathbb{H}_t, \tau)$ holds. □

The equivalence (6.1.8) seems trivial, but it means that the statistical data on $(\mathbb{H}_t, \tau)$ are applicable into those on $(\mathcal{M}_{t,N}, \varphi_1)$, via the isomorphic relation,

$$\mathbb{H}_t \overset{\text{iso}}{=} \mathcal{M}(\mathbb{H}_t) \overset{\text{iso}}{=} M(\mathbb{H}_t), \quad \text{on} \quad \mathbb{H}_t^N.$$

Proposition 6.6. *Let U and U^* be the forward, and the backward shifts of $(\mathcal{M}_{t,N}, \varphi_1)$, and let $n_1, n_2 \in \mathbb{N}$. Then*

$$\varphi_1(U^n) = 0 = \varphi((U^*)^n), \quad \forall n \in \mathbb{N}; \tag{6.1.9}$$

Also, we have that

$$n_1 = n_2 \overset{\text{say}}{=} n < N \Longrightarrow \varphi_1((U^*)^n U^n) = 1,\ \varphi_1(U^n (U^*)^n) = 0$$

meanwhile, (6.1.10)

$$\varphi_1(U^{n_1}(U^*)^{n_2}) = 0 = \varphi_1((U^*)^{n_1} U^{n_2}), \text{ otherwise.}$$

Proof. By definition, the $\mathbb{H}_t$-matrices $\{U^n, (U^*)^n\}_{n \in \mathbb{N}}$ have their $(1,1)$-entries $0 = 0 + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$. So, the analytic data (6.1.9) holds by (6.1.2). Recall that

$$(U^*)^{n_1} U^{n_2} = \begin{cases} (U^*)^{n_1 - n_2} P_{N-n_2} & \text{if } n_1 + n_2 < 2N-1,\ n_1 \geq n_2 \\ \\ P_{N-n_1} U^{n_2 - n_1} & \text{if } n_1 + n_2 < 2N-1,\ n_1 \leq n_2 \\ \\ O_N & \text{otherwise,} \end{cases}$$

and

$$(U^*)^{n_1} U^{n_2} = \begin{cases} Q_{N-n_1} U^{n_1 - n_2} & \text{if } n_1 + n_2 < 2N-1,\ n_1 \geq n_2 \\ \\ P_{N-n_1} (U^*)^{n_2 - n_1} & \text{if } n_1 + n_2 < 2N-1,\ n_1 \leq n_2 \\ \\ O_N & \text{otherwise,} \end{cases}$$

by (5.9). So, if $n_1 \neq n_2$, then the forward, or the backward shift is involved in computing $(U^*)^{n_1} U^{n_2}$, and $U^{n_1}(U^*)^{n_2}$, making their $(1,1)$-entries be $0 \in \mathbb{H}_t$, because they "shift" the main-diagonals of the $\mathbb{H}_t$-diagonal matrices P_k's, or Q_k's, for $k = 1, ..., N$. So, if $n_1 \neq n_2$ in $\mathbb{N}$, then

$$\varphi_1((U^*)^{n_1} U^{n_2}) = 0 = \varphi_1(U^{n_1}(U^*)^{n_2}),$$

by (6.1.9). Meanwhile, if $n_1 = n = n_2 < N$ in $\mathbb{N}$, then

$$\varphi_1((U^*)^n U^n) = \varphi_1(P_{N-n}) = \mathrm{Re}(1) = 1,$$

but

$$\varphi_1(U^n (U^*)^n) = \varphi_1(Q_{N-n}) = \mathrm{Re}(0) = 0,$$

since the projections P_k have their $(1,1)$-entries $1 \in \mathbb{H}_t$, while, the projections Q_k have their $(1,1)$-entries $0 \in \mathbb{H}_t$, whenever $k = 1, ..., N-1$. Of course, if $n_1 = n_2 \geq N$, then, by the nilpotent-property of both U and U^*,

$$\varphi_1((U^*)^n U^n) = 0 = \varphi_1(U^n (U^*)^n).$$

Therefore, the analytic data (6.1.10) holds, too. □

The above proposition allows us to verify that "most of" the analytic data of $\{U, U^*\}$,

$$\varphi_1\left(\prod_{l=1}^{n} U^{e_l}\right) = \varphi_1(U^{e_1} U^{e_2} ... U^{e_n}),$$

for all $(e_1, ..., e_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, become 0, by (6.1.9) and (6.1.10). In particular, if n is odd in $\mathbb{N}$, the above quantities would be 0. The possible non-zero data would be only

$$\varphi_1\left(\prod_{l=1}^{k} (U^*)^{n_l} U^{n_l}\right) = \varphi_1\left(\prod_{l=1}^{k} P_{N-n_l}\right) = \mathrm{Re}(1) = 1,$$

with (6.1.11)

$$n_1, ..., n_k < N, \text{ in } \mathbb{N}, \ \forall k \in \mathbb{N}.$$

Theorem 6.7. *Let U and U^* be the forward, resp., the backward shifts of $(\mathcal{M}_{t,N}, \varphi_1)$. Then the "only" "non-zero" analytic (or, distributional) data of $\{U, U^*\}$ (up to φ_1) are*

$$\varphi_1 \left(\prod_{l=1}^{k} (U^*)^{n_l} U^{n_l} \right) = 1,$$

whenever (6.1.12)

$$n_1, ..., n_k < N \quad \text{in} \quad \mathbb{N}, \quad \forall k \in \mathbb{N}.$$

Proof. As we discussed in the very above paragraph, by (6.1.2) and (6.1.9), if n is odd in $\mathbb{N}$, then

$$\varphi_1 \left(\prod_{l=1}^{n} U^{e_l} \right) = 0, \ \forall (e_1, ..., e_n) \in \{1, *\}^n, \ \forall n \in \mathbb{N},$$

because the $\mathbb{H}_t$-matrices $\prod_{l=1}^{n} U^{e_l}$ have their $(1,1)$-entries $0 \in \mathbb{H}_t$, whenever $n$is odd in $\mathbb{N}$. So, let's focus on the cases where n is even in $\mathbb{N}$. However, as one can check in (5.9) and (6.1.10), the only possible non-zero analytic data would be

$$\varphi_1 \left(\prod_{l=1}^{k} (U^*)^{n_l} U^{n_l} \right), \text{ with } n_1, ..., n_k < N.$$

Indeed, in such a case,

$$\varphi_1 \left(\prod_{l=1}^{k} (U^*)^{n_l} U^{n_l} \right) = \varphi_1 \left(\prod_{l=1}^{k} P_{N-n_l} \right) = \mathrm{Re}\,(1) = 1,$$

because the $(1,1)$-entry of $\prod_{l=1}^{k} P_{N-n_l}$ is $1 \in \mathbb{H}_t$. Therefore, the analytic data (6.1.12) holds on $(\mathcal{M}_t, \varphi_1)$. □

The above theorem characterizes the distributional data of U (equivalently, that of U^*) in $(\mathcal{M}_{t,N}, \varphi_1)$.

6.2. **The Noncommutative $\mathbb{R}$-$*$-Statistical Space** $(\mathcal{M}_{t,N}, \varphi)$**.** In this section, we define a new bounded $\mathbb{R}$-linear functional φ on the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$, and construct a new noncommutative topological $\mathbb{R}$-$*$-statistical space $(\mathcal{M}_{t,N}, \varphi)$. And then some analytic data on $\mathcal{M}_{t,N}$ are studied up to φ. Define a $\mathbb{R}$-linear functional φ on $\mathcal{M}_{t,N}$ by

$$\varphi \left([h_{i,j}]_{N \times N} \right) \overset{\text{def}}{=} \tfrac{1}{N} \sum_{k=1}^{N} \tau(h_{k,k}), \quad \forall [h_{i,j}]_{N \times N} \in \mathcal{M}_{t,N}. \tag{6.2.1}$$

Since τ is a bounded $\mathbb{R}$-linear functional on $\mathbb{H}_t$, the morphism φ of (6.2.1) is indeed a bounded $\mathbb{R}$-linear functional on $\mathcal{M}_{t,N}$. Also, it satisfies the unital property,

$$\varphi(I_N) = \tfrac{1}{N} \sum_{n=1}^{N} \tau(1) = \tfrac{N}{N} = 1. \tag{6.2.2}$$

Proposition 6.8. *The pair $(\mathcal{M}_{t,N}, \varphi)$ of the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ and the bounded $\mathbb{R}$-linear functional φ of* (6.2.1) *forms a unital $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-statistical space.*

Proof. The proof is done by the very definition (6.2.1) and the unital property (6.2.2). □

One can realize that if we restrict the $\mathbb{R}$-linear functional φ to the $\mathbb{R}$-$*$-subalgebra,

$$\mathcal{M}(\mathbb{H}_t) = \{hI_N : h \in \mathbb{H}_t\} \overset{\text{iso}}{=} M(\mathbb{H}_t),$$

of $\mathcal{M}_{t,N}$, then the sub-structure $\left(\mathcal{M}(\mathbb{H}_t), \varphi = \varphi \mid_{\mathcal{M}(\mathbb{H}_t)} \right)$ is equivalent to the $\mathbb{R}$-$*$-statistical space $(\mathbb{H}_t, \tau)$.

Theorem 6.9. *The $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-statistical spaces $(\mathcal{M}(\mathbb{H}_t), \varphi)$ and $(\mathbb{H}_t, \tau)$ are equivalent in the sense that there exists an isometric isomorphism,*

$$\Psi : h \in \mathbb{H}_t \longmapsto hI_N \in \mathcal{M}(\mathbb{H}_t),$$

such that (6.2.3)

$$\varphi(\Psi(h)) = \tau(h), \quad \forall h \in \mathbb{H}_t.$$

Proof. Observe first that, the morphism Ψ in (6.2.3) is an isometric $\mathbb{R}$-$*$-algebra-isomorphism satisfying the bijectivity, and the $\mathbb{R}$-linearity,

$$\Psi(r_1h_1 + r_1h_2) = (r_1h_1 + r_2h_2) I_N = r_1\Psi(h_1) + r_2\Psi(h_2),$$

for all $r_1, r_2 \in \mathbb{R}$ and $h_1, h_2 \in \mathbb{H}_t$, and the multiplication-preserving property,

$$\Psi(h_1h_2) = h_1h_2I_N = (h_1I_N)(h_2I_N) = \Psi(h_1)\Psi(h_2),$$

for all $h_1, h_2 \in \mathbb{H}_t$, and the adjoint-perserving property,

$$\Psi\left(h^{\circledast}\right) = h^{\circledast}I_N = h^{\circledast}I_N^* = (hI_N)^* = \Psi(h)^*, \ \forall h \in \mathbb{H}_t,$$

in $\mathcal{M}(\mathbb{H}_t) \subset \mathcal{M}_{t,N}$, and the isometric property,

$$\|\Psi(h)\| = \|hI_N\| = \|h\|_t, \quad \forall h \in \mathbb{H}_t.$$

Moreover, for any $h_l \in \mathbb{H}_t$ assigning to $\Psi(h_l) = h_lI_N \in \mathcal{M}(\mathbb{H}_t)$, for all $l = 1, ..., s$, for any $s \in \mathbb{N}$, one has that

$$\varphi\left(\Psi\left(\prod_{k=1}^{n} h_{l_k}\right)\right) = \varphi\left(\left(\prod_{k=1}^{n} h_{l_k}\right) I_N\right) = \frac{1}{N}\sum_{n=1}^{N}\tau\left(\prod_{k=1}^{n} h_{l_k}\right),$$

i.e.,

$$\varphi\left(\Psi\left(\prod_{k=1}^{n} h_{l_k}\right)\right) = \tau\left(\prod_{k=1}^{n} h_{l_k}\right),$$

for all $(l_1, ..., l_n) \in \{1, ..., s\}^n$, for all $n \in \mathbb{N}$. Therefore, the equivalence (6.2.3) holds. □

The above theorem shows that if we define a bounded $\mathbb{R}$-linear functional $\varphi_{t,N} : M(\mathbb{H}_t) \to \mathbb{R}$ on $M(\mathbb{H}_t)$ by

$$\varphi_{t,N}(M_h) \overset{\text{def}}{=} \varphi(hI_N), \quad \forall M_h \in M(\mathbb{H}_t),$$

then the pairs $(M(\mathbb{H}_t), \varphi_{t,N})$, $(\mathcal{M}(\mathbb{H}_t), \varphi)$, and $(\mathbb{H}_t, \tau)$ are equivalent $\mathbb{R}$-semi-normed $\mathbb{R}$-$*$-statistical spaces.

Now, let U and U^* be the forward, and the backward shifts on $\mathbb{H}_t^N$. Then

$$\varphi(U^n) = 0 = \varphi((U^*)^n), \quad \forall n \in \mathbb{N}, \tag{6.2.4}$$

because (i) if $n \geq N$, then $U^n = O = (U^*)^n$ in $\mathcal{M}_{t,N}$, whose main diagonal $\mathbb{H}_t$-entries are $0 = 0 + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$, by the nilpotent property of U and U^*, and (ii) if $n < N$ in $\mathbb{N}$, then the $\mathbb{H}_t$-matrices U^n and $(U^*)^n$ have their main diagonal $\mathbb{H}_t$-entries $0 \in \mathbb{H}_t$.

Recall that

$$(U^*)^{n_1} U^{n_2} = \begin{cases} (U^*)^{n_1-n_2} P_{N-n_2} & \text{if } n_1 + n_2 < 2N-1, \ n_1 \geq n_2 \\ P_{N-n_1}U^{n_2-n_1} & \text{if } n_1 + n_2 < 2N-1, \ n_1 \leq n_2 \\ O_N & \text{otherwise,} \end{cases}$$

and (6.2.5)

$$U^{n_1}(U^*)^{n_2} = \begin{cases} U^{n_1-n_2}Q_{N-n_2} & \text{if } n_1+n_2<2N-1,\ n_1\geq n_2 \\ Q_{N-n_1}(U^*)^{n_2-n_1} & \text{if } n_1+n_2<2N-1,\ n_1\leq n_2 \\ O_N & \text{otherwise,} \end{cases}$$

by (5.9). By (6.2.5), one obtains the general results of (6.2.4).

Theorem 6.10. *If U and U^* are the forward, resp., the backward shifts of $(\mathcal{M}_{t,N},\varphi)$, then*

$$\varphi(U^n) = 0 = \varphi((U^*)^n), \quad \forall n \in \mathbb{N},$$

$$\varphi((U^*)^{n_1}U^{n_2}) = \begin{cases} \frac{N-n_1}{N} & \text{if } n_1=n_2<N \\ 0 & \text{otherwise,} \end{cases}$$

and (6.2.6)

$$\varphi(U^{n_1}(U^*)^{n_2}) = \begin{cases} \frac{N-n_2}{N} & \text{if } n_1=n_2<N \\ 0 & \text{otherwise,} \end{cases}$$

for all $n_1, n_2 \in \mathbb{N}$. So, the "possible non-zero" analytic data of $\{U, U^*\}$ in $(\mathcal{M}_{t,N},\varphi)$ are

$$\varphi\left(\prod_{k=1}^{n}((U^*)^{n_k}U^{n_k})\right) = \frac{N - \max\limits_{k=1,\ldots,n}\{n_k\}}{N},$$

and (6.2.7)

$$\varphi\left(\prod_{k=1}^{n}(U^{n_k}(U^*)^{n_k})\right) = \frac{N - \max\limits_{k=1,\ldots,n}\{n_k\}}{N},$$

for all $n_1, n_2 \in \{1, ..., N-1\}$, for all $n \in \mathbb{N}$; and if

$$S_1(k) = (U^*)^n U^n, \ \ S_2(n) = U^n(U^*)^n, \ \forall k \in \{1, ..., N-1\},$$

then (6.2.8)

$$\varphi\left(\prod_{l=1}^{n} S_{k_l}(n_l)\right) \in \left\{0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N}, 1\right\},$$

for all $(k_1, ..., k_n) \in \{1,2\}^n$, for all $n \in \mathbb{N}$.

Proof. The first-lined analytic data of (6.2.6) holds by the analytic data (6.2.4) on $(\mathcal{M}_{t,N},\varphi)$. By the formulas of (6.2.5), if $n_1 \neq n_2$ in $\mathbb{N}$, and $(U^*)^{n_1}U^{n_2} \neq O_N$ in $\mathcal{M}_{t,N}$, then the $\mathbb{H}_t$-matrix $(U^*)^{n_1}U^{n_2}$ is either $(U^*)^{k_1}P_{k_1}$, or $P_{k_1}U^{k_2}$, for suitable $k_1, k_2 \in \mathbb{N}$. Note that such non-zero $\mathbb{H}_t$-matrices have their main-diagonal $\mathbb{H}_t$-entries $0 \in \mathbb{H}_t$. Thus, if $n_1 \neq n_2$, then the $\mathbb{H}_t$-matrices $\{(U^*)^{n_1}U^{n_2}\}_{n_1\neq n_2}$ have their main-diagonal $\mathbb{H}_t$-entries $0 \in \mathbb{H}_t$. Similarly, if $n_1 \neq n_2$ in $\mathbb{N}$, and if $U^{n_1}(U^*)^{n_2} \neq O_N$ in $\mathcal{M}_{t,N}$, then the $\mathbb{H}_t$-matrix $U^{n_1}(U^*)^{n_2}$ is either $U^{k_1}Q_{k_2}$, or $Q_{k_1}(U^*)^{k_2}$, for suitable $k_1, k_2 \in \mathbb{N}$, by (6.2.5), and these $\mathbb{H}_t$-matrices have their mian-diagonal $\mathbb{H}_t$-entries $0 \in \mathbb{H}_t$. It implies also that if $n_1 \neq n_2$ in $\mathbb{N}$, then the $\mathbb{H}_t$-matrices $\{U^{n_1}(U^*)^{n_2}\}_{n_1\neq n_2}$ have their main-diagonal $\mathbb{H}_t$-entries $0 \in \mathbb{H}_t$. Therefore,

$$n_1 \neq n_2 \in \mathbb{N} \Longrightarrow \varphi((U^*)^{n_1}U^{n_2}) = 0 = \varphi(U^{n_1}(U^*)^{n_2}).$$

Suppose now that $n_1 = n_2 \overset{\text{say}}{=} n < N$ in $\mathbb{N}$. Then

$$(U^*)^n U^n = P_{N-n}, \ \text{ and } \ U^n(U^*)^n = Q_{N-n},$$

having $(N-n)$-many non-zero main-diagonal $\mathbb{H}_t$-entries $1 = 1 + 0i + 0j_t + 0k_t \in \mathbb{H}_t$. It implies that

$$\text{if} \quad n_1 = n_2 \overset{\text{say}}{=} n < N \text{ in } \mathbb{N},$$

then

$$\varphi\left((U^*)^n U^n\right) = \varphi\left(P_{N-n}\right) = \frac{1}{N}\sum_{l=1}^{N-n}\tau(1) = \frac{N-n}{N},$$

and

$$\varphi\left(U^n (U^*)^n\right) = \varphi\left(Q_{N-n}\right) = \frac{N-n}{N}.$$

Of course,

$$\text{if } n_1 = n_2 \overset{\text{say}}{=} n \geq N \text{ in } \mathbb{N},$$

then

$$\varphi\left((U^*)^n U^n\right) = \varphi\left(O_N\right) = 0 = \varphi\left(U^n (U^*)^n\right).$$

Therefore, the analytic data (6.2.6) on $(\mathcal{M}_{t,N}, \varphi)$ hold true.

By the analytic data (6.2.6), if we consider the analytic data of $\{U, U^*\}$ in $\mathcal{M}_{t,N}$ up to φ, determined by

$$\varphi\left(\prod_{l=1}^{n} U^{e_l}\right), \quad \forall (e_1, ..., e_n) \in \{1, *\}^n, \quad \forall n \in \mathbb{N},$$

have the only "non-zero" data from either

$$\varphi\left(\prod_{k=1}^{n}\left((U^*)^{n_k} U^{n_k}\right)\right) = \varphi\left(\prod_{k=1}^{n} P_{N-n}\right) = \varphi\left(P_{N-\max\limits_{k=1,\ldots,N-1} n_k}\right),$$

and (6.2.9)

$$\varphi\left(\prod_{k=1}^{n}\left(U^{n_1} (U^*)^{n_2}\right)\right) = \varphi\left(\prod_{k=1}^{n} Q_{N-n}\right) = \varphi\left(Q_{N-\max\limits_{k=1,\ldots,N-1} n_k}\right),$$

by (5.8). Remark that, by (5.8), if $n_1 + n_2 < 2N - 1$ in $\mathbb{N}$, then

$$P_{N-n_1} P_{N-n_2} = P_{N-\max\{n_1, n_2\}} \in \mathcal{M}_{t,N},$$

and

$$Q_{N-n_1} Q_{N-n_2} = Q_{N-\max\{n_1, n_2\}} \in \mathcal{M}_{t,N}.$$

So, the formulas of (6.2.9) hold inductively. Since

$$\varphi\left(P_{N-k}\right) = \frac{N-k}{N} = \varphi\left(Q_{N-k}\right), \quad \forall k \in \{1, ..., N-1\},$$

by (5.8), the formulas of (6.2.9) go to

$$\varphi\left(\prod_{k=1}^{n}\left((U^*)^{n_k} U^{n_k}\right)\right) = \varphi\left(P_{N-\max\limits_{k=1,\ldots,N-1} n_k}\right) = \frac{N - \max\limits_{k=1,\ldots,N-1} n_k}{N},$$

and

$$\varphi\left(\prod_{k=1}^{n}\left(U^{n_1} (U^*)^{n_2}\right)\right) = \varphi\left(Q_{N-\max\limits_{k=1,\ldots,N-1} n_k}\right) = \frac{N - \max\limits_{k=1,\ldots,N-1} n_k}{N}.$$

It shows that the "non-zero" analytic data (6.2.7) hold.

By (6.2.6) and (6.2.7), one can verify that the other "possible" "non-zero" analytic data of $\{U, U^*\}$ in $(\mathcal{M}_{t,N}, \varphi)$ would be

$$\varphi\left(\prod_{l=1}^{n} S_{k_l}(n_l)\right) = \varphi\left(S_{k_1}(n_1) S_{k_2}(n_2) ... S_{k_n}(n_n)\right),$$

for all $(k_1, ..., k_n) \in \{1, 2\}^n$, and $(n_1, ..., n_n) \in \{1, ..., N-1\}^n$, for all $n \in \mathbb{N}$, where

$$S_1(n) = (U^*)^n U^n = P_{N-n}, \quad \forall n = 1, ..., N-1,$$

and

$$S_2(n) = U^n (U^*)^n = Q_{N-n}, \quad \forall n = 1, ..., N-1,$$

including the cases of (6.2.7). Clearly, it contains the case where

$$S_1(n_1) S_2(n_2) = O_N = S_2(n_2) S_1(n_1), \text{ if } n_1 > \frac{N}{2}, \ \& \ n_2 > \frac{N}{2}.$$

(For example, if $N = 3$, then

$$S_1(2) = P_{3-2} = P_1, \text{ and } S_2(2) = Q_{3-2} = Q_1,$$

where

$$P_1 Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = O_N = Q_1 P_1,$$

in $\mathcal{M}_{t,3}$.) So, it is possible that $\varphi\left(\prod_{l=1}^{n} S_{k_l}(n_l)\right) = 0$. It not,

$$\prod_{l=1}^{n} S_{k_l}(n_l) \in \{P_k, \ Q_k\}_{k=1}^{N-1} \cup \left\{ P_{n_1} Q_{n_2} = Q_{n_2} P_{n_1} \;\middle|\; \begin{array}{c} n_1, n_2 \in \{1, ..., N-1\} \\ \frac{N}{2} < n_1 + n_2 < 2N-1 \end{array} \right\},$$

in $\mathcal{M}_{t,N}$, satisfying

$$\varphi\left(\prod_{l=1}^{n} S_{k_l}(n_l)\right) \in \left\{ \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N} \right\},$$

because if $\frac{N}{2} < n_1 + n_2 < 2N - 1$, then

$$P_{n_1} Q_{n_2} = Q_{n_2} P_{n_1} = \begin{pmatrix} 0 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & \underbrace{\ddots}_{|n_1 - n_2|} & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ 0 & & & & & & & 0 \end{pmatrix},$$

in $\mathcal{M}_{t,N}$, satisfying

$$\varphi(P_{n_1} Q_{n_2}) = \frac{|n_1 - n_2|}{N} \in \left\{ \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N} \right\}.$$

In summary,

$$\varphi\left(\prod_{l=1}^{n} S_{k_l}(n_l)\right) \in \left\{ 0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N} \right\},$$

for all $(k_1, ..., k_n) \in \{1, 2\}^n$, $(n_1, ..., n_n) \in \{1, ..., N-1\}^n$, for all $n \in \mathbb{N}$. Therefore, the "only" possible non-zero analytic data (6.2.8) is obtained in $(\mathcal{M}_{t,N}, \varphi)$. $\square$

If we compare the only possible non-zero analytic data (6.2.8) (including (6.2.7)) of $\{U, U^*\}$ in the $\mathbb{R}$-$*$-statistical space $(\mathcal{M}_{t,N}, \varphi)$ and the only non-zero analytic data (6.1.12) of $\{U, U^*\}$ in the $\mathbb{R}$-$*$-statistical space $(\mathcal{M}_{t,N}, \varphi_1)$ of Section 6.1, then it is clear that two $\mathbb{R}$-$*$-statistical spaces $(\mathcal{M}_{t,N}, \varphi)$ and $(\mathcal{M}_{t,N}, \varphi_1)$ are "not" equivalent.

7. A Certain Representation of $\mathbb{H}_t$-matrices of $\mathcal{M}_{t,N}$

In this section, we fix $N \in \mathbb{N}$ and $t \in \mathbb{R}$, and the corresponding $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N} = M_N(\mathbb{H}_t)$, and study how our structure $\mathcal{M}_{t,N}$ acts in the usual operator-theoretic, or matrix-theoretic settings "over the complex field $\mathbb{C}$." i.e., we consider a realization of $\mathcal{M}_{t,N}$ over $\mathbb{C}$. To consider such a usual setting, we recall the canonical representation $\left(\mathbb{C}^2, \pi_t\right)$ of $\mathbb{H}_t$, introduced in [1, 2, 3, 4]. If

$$h = x + yi + uj_t + vk_t = (x + yi) + (u + vi)\, j_t \in \mathbb{H}_t,$$

re-expressed to be

$$h = a + bj_t \in \mathbb{H}_t, \text{ with } a = x + yi,\ b = u + vi \in \mathbb{C},$$

one can define an action $\pi_t : \mathbb{H}_t \to M_2(\mathbb{C})$ of $\mathbb{H}_t$ acting on $\mathbb{C}^2$ by

$$\pi_t(a + bj_t) \overset{\text{def}}{=} \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \pi_t(\mathbb{H}_t),\ \text{in } M_2(\mathbb{C}),\ \forall a, b \in \mathbb{C}. \tag{7.1}$$

Then, as one can check from [1, 2], this morphism π_t satisfies

$$\pi_t(r_1 h_1 + r_2 h_2) = r_1 \pi_t(h_1) + r_2 \pi_t(h_2),$$

and (7.2)

$$\pi_t(h_1 h_2) = \pi_t(h_1)\, \pi_t(h_1),\ \ \forall r_1, r_2 \in \mathbb{R},\ h_1, h_2 \in \mathbb{H}_t,$$

where the right-hand sides of (7.2) mean the matrix-addition, respectively, the matrix-multiplication on $M_2(\mathbb{C})$. So, indeed, the morphism π_t of (7.1) is a $\mathbb{R}$-algebra-action of the $\mathbb{R}$-algebra $\mathbb{H}_t$ acting on $\mathbb{C}^2$. By (7.1), we also have that

$$\pi_t\left((a + bj_t)^{\circledast}\right) = \pi_t(\overline{a} - bj_t) = \begin{pmatrix} \overline{a} & t(-b) \\ -\overline{b} & a \end{pmatrix},\ \ \forall a, b \in \mathbb{C},$$

satisfying (7.3)

$$\pi_t(h)^{\circledast\circledast} = \pi_t\left(h^{\circledast\circledast}\right) = \pi_t(h), \quad \forall h \in \mathbb{H}_t,$$

$$\pi_t(rh)^{\circledast} = r\pi_t(h)^{\circledast}, \quad \forall r \in \mathbb{R},\ h \in \mathbb{H}_t,$$

$$\pi_t(h_1 + h_2)^{\circledast} = \pi_t(h_1)^{\circledast} + \pi_t(h_2)^{\circledast}, \quad \forall h_1, h_2 \in \mathbb{H}_t,$$

and

$$\pi_t(h_1 h_2)^{\circledast} = \pi_t(h_2)^{\circledast}\, \pi_t(h_1)^{\circledast},\ \ \forall h_1, h_2 \in \mathbb{H}_t,$$

by (7.2) and (7.3). Thus, this $\mathbb{R}$-algebra-action π_t becomes a $\mathbb{R}$-$*$-algebra action of $\mathbb{H}_t$ acting on $\mathbb{C}^2$.

By applying this canonical action π_t of (7.1), we define an action Π_t of $\mathcal{M}_{t,N}$ acting on $\mathbb{C}^{2N}$,

$$\Pi_t : \mathcal{M}_{t,N} \to M_{2N}(\mathbb{C}),$$

by (7.4)

$$\Pi_t\left([h_{i,j}]_{N\times N}\right) \overset{\text{def}}{=} [\pi_t(h_{i,j})]_{2N\times 2N} \in M_{2N}(\mathbb{C}),\ \forall\, [h_{i,j}]_{N\times N} \in \mathcal{M}_{t,N}.$$

Then, since π_t of (7.1) is a well-defined $\mathbb{R}$-$*$-algebra-action of $\mathbb{H}_t$ by (7.2) and (7.3), the morphism Π_t of (7.4) is a well-defined $\mathbb{R}$-algebra-action of $\mathcal{M}_{t,N}$ acting on $\mathbb{C}^{2N}$.

Proposition 7.1. *The pair* $\left(\mathbb{C}^{2N},\Pi_t\right)$ *is a well-determined* $\mathbb{C}$*-vector-space representation of the* $\mathbb{H}_t$*-matrix* $\mathbb{R}$*-algebra* $\mathcal{M}_{t,N}$*, i.e., the morphism* Π_t *of* (7.4) *forms a* $\mathbb{R}$*-algebra-action of* $\mathcal{M}_{t,N}$ *acting on* $\mathbb{C}^2$*.*

Proof. Observe that

$$\Pi_t\left([h_{i,j}]_{N\times N}+[f_{i,j}]_{N\times N}\right)=\Pi_t\left([h_{i,j}+f_{i,j}]_{N\times N}\right)$$
$$=[\pi_t(h_{i,j}+f_{i,j})]_{2N\times 2N}=[\pi_t(h_{i,j})+\pi_t(f_{i,j})]_{2N\times 2N}$$

since π_t is an action of $\mathbb{H}_t$ acting on $\mathbb{C}^2$

$$=[\pi_t(h_{i,j})]_{2N\times 2N}+[\pi_t(f_{i,j})]_{2N\times 2N}=\Pi_t\left([h_{i,j}]_{N\times N}\right)+\Pi_t\left([f_{i,j}]_{N\times N}\right),$$

where $(+)$ is the matrix addition on $M_{2N}(\mathbb{C})$; and

$$\Pi_t\left(r\,[h_{i,j}]_{N\times N}\right)=[\pi_t(rh_{i,j})]_{2N\times 2N}=r\,[\pi_t(h_{i,j})]_{2N\times 2N}=r\Pi_t\left([h_{i,j}]_{N\times N}\right),$$

for all $r\in\mathbb{R}$; and

$$\Pi_t\left([h_{i,j}]_{N\times N}[f_{i,j}]_{N\times N}\right)=\Pi_t\left(\left[\sum_{k=1}^{N}h_{i,k}f_{k,j}\right]_{N\times N}\right)$$
$$=\left[\pi_t\left(\sum_{k=1}^{N}h_{i,k}f_{k,j}\right)\right]_{2N\times 2N}=\left[\sum_{k=1}^{n}\pi_t(h_{i,k}f_{k,j})\right]_{2N\times 2N}$$
$$=\left[\sum_{k=1}^{n}\pi_t(h_{i,j})\,\pi_t(f_{i,j})\right]_{2N\times 2N}$$

since π_t is an action of $\mathbb{H}_t$ acting on $\mathbb{C}^2$

$$=[\pi_t(h_{i,j})]_{2N\times 2N}[\pi_t(f_{i,j})]_{2N\times 2N}=\Pi_t\left([h_{i,j}]_{N\times N}\right)\Pi_t\left([f_{i,j}]_{N\times N}\right),$$

where the multiplication $(\cdot)$ means the matrix multiplication on $M_{2N}(\mathbb{C})$. Therefore, our $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ acts on $\mathbb{C}^{2N}$ via the action Π_t of (7.4), equivalently, the pair $\left(\mathbb{C}^{2N},\Pi_t\right)$ forms a $\mathbb{C}$-vector-space representation of $\mathcal{M}_{t,N}$. ☐

The above proposition shows that every element $[h_{i,j}]_{N\times N}\in\mathcal{M}_{t,N}$ is regarded as the $(2N\times 2N)$-$\mathbb{C}$-matrix $[\pi_t(h_{i,j})]_{2N\times 2N}\in M_{2N}(\mathbb{C})$ via the action Π_t of (7.4). Remark that, since the canonical action π_t of $\mathbb{H}_t$ is injective from $\mathbb{H}_t$ into $M_2(\mathbb{C})$ (which is not surjective, e.g., see [1, 2]), the action Π_t of $\mathcal{M}_{t,N}$ is injective from $\mathcal{M}_{t,N}$ into $M_{2N}(\mathbb{C})$. Remark also that the matrix algebra $M_{2N}(\mathbb{C})$, which is defined "over $\mathbb{C}$," is understood to be a "$\mathbb{R}$-algebra" in the sense that:

$$T_1,T_2\in M_{2N}(\mathbb{C})\Longrightarrow T_1+T_2,\ T_1T_2\in M_{2N}(\mathbb{C}),$$

and (7.5)

$$r\in\mathbb{R},\ T\in M_{2N}(\mathbb{C})\Longrightarrow rT=(rI_{2N})\,T\in M_{2N}(\mathbb{C}).$$

So, by regarding $M_{2N}(\mathbb{C})$ as a $\mathbb{R}$-algebra satisfying (7.5), one can define a $\mathbb{R}$-subalgebra $\mathscr{M}_{t,N}$ by

$$\mathscr{M}_{t,N}\overset{\text{def}}{=}\Pi_t(\mathcal{M}_{t,N})=\{\Pi_t(T)\in M_{2N}(\mathbb{C}):T\in\mathcal{M}_{t,N}\}. \tag{7.6}$$

Meanwhile, if $[h_{i,j}]_{N\times N}\in\mathcal{M}_{t,N}$ has its $\mathbb{R}$-adjoint $[h_{i,j}]^*_{N\times N}=\left[h^{\circledast}_{j,i}\right]_{N\times N}$ in $\mathcal{M}_{t,N}$, then

$$\Pi_t\left([h_{i,j}]^*_{N\times N}\right)=\Pi_t\left(\left[h^{\circledast}_{j,i}\right]_{N\times N}\right)=\left[\pi_t\left(h^{\circledast}_{j,i}\right)\right]_{2N\times 2N}\in M_{2N}(\mathbb{C}), \tag{7.7}$$

where $\pi_t\left(h^{\circledast}_{j,i}\right)$ are in the sense of (7.3). It shows that the $\mathbb{R}$-adjoint $(*)$ on the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is closed on its realization $\mathscr{M}_{t,N}$ of (7.6) by (7.7).

Proposition 7.2. *The $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is isometrically isomorphic to the $\mathbb{R}$-subalgebra $\mathscr{M}_{t,N}$ of $M_{2N}(\mathbb{C})$ as $\mathbb{R}$-algebras. i.e.,*

$$\mathcal{M}_{t,N} \overset{\text{iso}}{=} \mathscr{M}_{t,N}, \quad \text{as } \mathbb{R}\text{-algebras}. \tag{7.8}$$

Remark that

$$\Pi_t(T^*) \neq \Pi_t(T)^*, \text{ in } M_{2N}(\mathbb{C}), \text{ in general,}$$

where $(*)$ on the right-hand side means the usual $\mathbb{C}$-adjoint of $\mathbb{C}$-matrices, i.e., the conjugate-transpose on $M_{2N}(\mathbb{C})$.

Proof. By the above proposition, the morphism Π_t of (7.4) is a well-defined $\mathbb{R}$-algebra-action of the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ acting on $M_{2N}(\mathbb{C})$. Moreover, by the injectivity of Π_t, the $\mathbb{R}$-subalgebra $\mathscr{M}_{t,N} = \Pi_t(\mathcal{M}_{t,N})$ of (7.6), satisfying (7.5), is a isomorphic to $\mathcal{M}_{t,N}$ in $M_{2N}(\mathbb{C})$. So, the structure theorem (7.8) holds true.

It is immediately checked that

$$\Pi_t(T^*) \neq \Pi_t(T)^*, \quad \text{in general,}$$

in $M_{2N}(\mathbb{C})$, for all $T \in \mathcal{M}_{t,N}$, where $(*)$ in the left-hand side is the $\mathbb{R}$-adjoint (4.17) on $\mathcal{M}_{t,N}$, and $(*)$ in the right-hand side is the usual $\mathbb{C}$-matrix-adjoint, the conjugate-transpose on $M_{2N}(\mathbb{C})$. So, even though the isomorphic relation (7.8) is satisfied, two $\mathbb{R}$-algebras $\mathcal{M}_{t,N}$ and its injective realization $\mathscr{M}_{t,N}$ are not $*$-isomorphic over $\mathbb{R}$. □

The above proposition shows that inside the matrix algebra $M_{2N}(\mathbb{C})$, there exists a well-established $\mathbb{R}$-subalgebra $\mathscr{M}_{t,N}$, isomorphic to our $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ by (7.8). So, motivated by the above theorem, we define an operation, denoted by $< * >$ on the realization $\mathscr{M}_{t,N}$ of $\mathcal{M}_{t,N}$ by

$$\left(\Pi_t\left([h_{i,j}]_{N\times N}\right)\right)^{<*>} \overset{\text{def}}{=} \Pi_t\left([h_{i,j}]^*_{N\times N}\right) = \left[\pi_t\left(h^{\circledast}_{j,i}\right)\right]_{N\times N} \in \mathscr{M}_{t,N}, \tag{7.9}$$

for all $[h_{i,j}]_{N\times N} \in \mathcal{M}_{t,N}$, where $\pi_t\left(h^{\circledast}_{j,i}\right)$ are in the sense of (7.3), for all $i, j = 1, ..., N$. Then this operation $(< * >)$ of (7.9) is a well-defined $\mathbb{R}$-adjoint on the $\mathbb{R}$-algebra $\mathscr{M}_{t,N}$, by the injectivity of π_t and Π_t, because $(\circledast)$ is a $\mathbb{R}$-adjoint on $\mathbb{H}_t$, and hence, that on $\pi_t(\mathbb{H}_t)$ in the sense of (7.3).

Theorem 7.3. *The $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ and the $\mathbb{R}$-algebra $\mathscr{M}_{t,N}$ of (7.6), equipped with the $\mathbb{R}$-adjoint $(< * >)$ of (7.9) are $*$-isomorphic over $\mathbb{R}$. i.e.,*

$$\mathcal{M}_{t,N} \overset{*\text{-iso}}{=} \mathscr{M}_{t,N}, \quad \text{as } \mathbb{R}\text{-}*\text{-algebras}. \tag{7.10}$$

Proof. By (7.8), the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ and its injective realization $\mathscr{M}_{t,N}$ are isomorphic as $\mathbb{R}$-algebras. By defining the $\mathbb{R}$-adjoint $(< * >)$ of (7.9) on $\mathscr{M}_{t,N}$, the $\mathbb{R}$-algebra $\mathscr{M}_{t,N}$ becomes a well-defined $\mathbb{R}$-$*$-algebra. Indeed, the operation (7.9) satisfies

$$\left(\Pi_t(T_1)^{<*>}\right)^{<*>} = \Pi_t(T_1^*)^{<*>} = \Pi_t(T_1^{**}) = \Pi_t(T_1);$$

$$(r\Pi_t(T_2))^{<*>} = \Pi_t(rT_2)^{<*>} = \Pi_t(rT_2^*) = r\Pi_t(T_2^*) = r\Pi_t(T_2)^{<*>};$$

$$(\Pi_t(T_1) + \Pi_t(T_2))^{<*>} = \Pi_t(T_1^* + T_2^*) = \Pi_t(T_1)^{<*>} + \Pi_t(T_2)^{<*>};$$

and

$$(\Pi_t(T_1)\Pi_t(T_2))^{<*>} = \Pi_t(T_2^*T_1^*) = \Pi_t(T_2)^{<*>}\Pi_t(T_1)^{<*>},$$

on $\mathscr{M}_{t,N}$, for all $T_1, T_2 \in \mathcal{M}_{t,N}$, and $r \in \mathbb{R}$. Since the isomorphic $\mathbb{R}$-algebra action Π_t satisfies

$$\Pi_t(T^*) = \Pi_t(T)^{<*>}, \text{ by definition (7.9),}$$

two $\mathbb{R}$-$*$-algebras $\mathcal{M}_{t,N}$ and $\mathscr{M}_{t,N}$ are $*$-isomorphic, too. Therefore, the structure theorem (7.10) holds with help of (7.9). □

By (7.10), we understand the $\mathbb{R}$-subalgebra $\mathscr{M}_{t,N} = \Pi_t(\mathcal{M}_{t,N})$ of $M_{2N}(\mathbb{C})$ as a $\mathbb{R}$-$*$-algebra equipped with its $\mathbb{R}$-adjoint $(<*>)$ of (7.9).

Since $\mathscr{M}_{t,N}$ is a $\mathbb{R}$-$*$-algebra itself, one can obtain the following result immediately. Since $\mathscr{M}_{t,N}$ is a $\mathbb{R}$-$*$-algebra under (7.9), one can define the following operator-theoretic properties on $\mathscr{M}_{t,N}$;

(i) S is $<*>$-self-adjoint in $\mathscr{M}_{t,N}$, if $S^{<*>} = S$ in $\mathscr{M}_{t,N}$,
(ii) S is a $<*>$-projection in $\mathscr{M}_{t,N}$, if $S^{<*>} = S = S^2$ in $\mathscr{M}_{t,N}$,
(iii) S is $<*>$-normal in $\mathscr{M}_{t,N}$, if $S^{<*>}S = SS^{<*>}$ in $\mathscr{M}_{t,N}$,
(iv) S is a $<*>$-isometry in $\mathscr{M}_{t,N}$, if $S^{<*>}S = I_{2N}$ in $\mathscr{M}_{t,N}$,
(v) S is $<*>$-unitary in $\mathscr{M}_{t,N}$, if $S^{<*>}S = I_{2N} = SS^{<*>}$ in $\mathscr{M}_{t,N}$,

where I_{2N} is the identity $\mathbb{C}$-matrix of $M_{2N}(\mathbb{C})$, which becomes the unity of the $\mathbb{R}$-$*$-algebra $\mathscr{M}_{t,N}$.

By the structure theorem (7.10), one can realize that the operator-theoretic properties on $\mathcal{M}_{t,N}$ of Section 5 up to the $\mathbb{R}$-adjoint $(*)$ of (4.17) have their equivalent properties on $\mathscr{M}_{t,N}$ up to the $\mathbb{R}$-adjoint $(<*>)$ of (7.9).

Corollary 7.4. *Let $\mathscr{M}_{1,N} = \Pi_t(\mathcal{M}_t)$ be the $*$-isomorphic realization of the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ in $M_{2N}(\mathbb{C})$.*
(1) $\Pi_t\left([h_{i,j}]_{N\times N}\right)$ *is $<*>$-self-adjoint in $\mathscr{M}_{t,N}$, if and only if* (4.20) *holds.*
(2) $\Pi_t\left([h_{i,j}]_{N\times N}\right)$ *is a $<*>$-projection in $\mathscr{M}_{t,N}$, if and only if* (4.23) *holds.*
(3) $\Pi_t\left([h_{i,j}]_{N\times N}\right)$ *is $<*>$-normal in $\mathscr{M}_{t,N}$, if and only if* (4.26) *holds.*
(4) $\Pi_t\left([h_{i,j}]_{N\times N}\right)$ *is a $<*>$-isometry in $\mathscr{M}_{t,N}$, if and only if* (4.29) *holds.*
(5) $\Pi_t\left([h_{i,j}]_{N\times N}\right)$ *is $<*>$-unitary in $\mathscr{M}_{t,N}$, if and only if* (4.31) *holds*

Proof. By (7.9) and (7.10), an element $\Pi_t(T)$ satisfies an operator-theoretic property in $\mathscr{M}_{t,N}$ up to the $\mathbb{R}$-adjoint $(<*>)$, if and only if T satisfies the same operator-theoretic property in $\mathcal{M}_{t,N}$ up to the $\mathbb{R}$-adjoint $(*)$ of (4.17). □

Note that the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ is acting on the complete $\mathbb{R}$-semi-normed definite, or indefinite $\mathbb{R}$-semi-inner-product space,

$$\mathbb{H}_t^N = \left\{(h_k)_{k=1}^N : h_k \in \mathbb{H}_t\right\}.$$

So, it is natural to consider where the $*$-isomorphic realization $\mathscr{M}_{t,N}$ of $\mathcal{M}_{t,N}$ is acting. Remark now that, by the very construction of $\mathscr{M}_{t,N}$, it acts on the $(4N)$-dimensional $\mathbb{R}$-vector space $\mathbb{C}^{2N} \overset{\text{iso}}{=} \mathbb{R}^{4N}$ over $\mathbb{R}$, as a sub-structure of $M_{2N}(\mathbb{C})$. However, such a vector space $\mathbb{C}^{2N}$ is not directly related to $\mathbb{H}_t^N$ where $\mathcal{M}_{t,N}$ is acting structurally, because $\mathbb{C}^{2N}$ is over $\mathbb{C}$, and $\mathbb{H}_t^N$ is over $\mathbb{R}$. Thus, we need to consider the isomorphic $\mathbb{R}$-vector space of $\mathbb{H}_t^N$ where the $\mathbb{R}$-$*$-algebra $\mathscr{M}_{t,N}$ is acting.

From the canonical action π_t of $\mathbb{H}_t$ acting on $\mathbb{C}^2$, define a $\mathbb{R}$-vector-space action π_t^N of $\mathbb{H}_t^N$ by

$$\pi_t^N \overset{\text{def}}{=} \pi_t^{\times N} = \underbrace{\pi_t \times \pi_t \times \pi_t \times ... \times \pi_t}_{N\text{-times}},$$

i.e., (7.11)

$$\pi_t^N\left((h_k)_{k=1}^N\right) = \pi_t^N\left(\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix}\right) = \begin{pmatrix} \pi_t(h_1) \\ \pi_t(h_2) \\ \vdots \\ \pi_t(h_N) \end{pmatrix} = (\pi_t(h_k))_{k=1}^N,$$

in $\pi_t^N\left(\mathbb{H}_t^N\right)$, for all $(h_k)_{k=1}^N \in \mathbb{H}_t^N$. Note that, by the injectivity of the canonical action π_t, this morphism π_t^N of (7.11) is also injective (and hence, bijective) from $\mathbb{H}_t^N$ onto $\pi_t\left(\mathbb{H}_t^N\right)$. Then, by (7.3) and (7.11), the image $\pi_t^N\left(\mathbb{H}_t^N\right)$ is actually a subset of the $(2N \times 2)$-$\mathbb{C}$-matrix set,

$$M_{2N\times 2}\left(\mathbb{C}\right) = \left\{[z_{i,j}]_{2N\times 2} : z_{i,j} \in \mathbb{C}\right\}.$$

Note that this $\mathbb{C}$-matrix set $M_{2N\times 2}(\mathbb{C})$ is not a $\mathbb{C}$-algebra because the matrix-multiplication is undefined on it, however, it is a well-defined "$\mathbb{C}$-vector" space satisfying

$$z_1, z_2 \in \mathbb{C},\ A_1, A_2 \in M_{2N\times 2}\left(\mathbb{C}\right) \Longrightarrow z_1A_1 + z_2A_2 \in M_{2N\times 2}\left(\mathbb{C}\right).$$

Therefore, the subset $\pi_t^N\left(\mathbb{H}_t^N\right)$ of the $\mathbb{C}$-vector space $M_{2N\times 2}(\mathbb{C})$ forms a well-determined "$\mathbb{R}$-vector" space, i.e.,

$$r_1, r_2 \in \mathbb{R},\ V_1, V_2 \in \pi_t^N\left(\mathbb{H}_t^N\right) \Longrightarrow r_1V_1 + r_2V_2 \in \pi_t\left(\mathbb{H}_t^N\right). \tag{7.12}$$

Indeed, the $\mathbb{R}$-vector-space property (7.12) holds by (7.11), i.e.,

$$\pi_t^N\left((h_k)_{k=1}^N\right) + \pi_t^N\left((f_k)_{k=1}^N\right) = \pi_t^N\left((h_k+f_k)_{k=1}^N\right) = \left(\pi_t\left(h_k\right) + \pi_t\left(f_k\right)\right)_{k=1}^N,$$

and (7.13)

$$r\pi_t^N\left((h_k)_{k=1}^N\right) = r\left(\pi_t\left(h_k\right)\right)_{k=1}^N = \left(r\pi_t\left(h_k\right)\right)_{k=1}^N = \left(\pi_t\left(rh_k\right)\right)_{k=1}^N,$$

are well-defined vectors of $\pi_t^N\left(\mathbb{H}_t^N\right)$, too, for all $(h_k)_{k=1}^N, (f_k)_{k=1}^N \in \mathbb{H}_t^N$, and $r \in \mathbb{R}$.

Definition 7.5. The $\mathbb{R}$-vector space $\pi_t^N\left(\mathbb{H}_t^N\right)$, satisfying (7.12) or (7.13), is denoted simply by $\mathfrak{H}_t^N$ from below, where π_t^N is the $\mathbb{R}$-vector-space action (7.11) of $\mathbb{H}_t^N$ in $M_{2N\times 2}(\mathbb{C})$. i.e.,

$$\mathfrak{H}_t^N \overset{\text{denote}}{=} \pi_t^N\left(\mathbb{H}_t^N\right) \overset{\text{subset}}{\subset} M_{2N\times 2}\left(\mathbb{C}\right). \tag{7.14}$$

And we call $\mathfrak{H}_t^N$ of (7.14), the $\mathbb{H}_t^N$-realization (by π_t^N).

By (7.11) and (7.14), one has the following result.

Proposition 7.6. *The $\mathbb{R}$-vector spaces $\mathbb{H}_t^N$ and its $\mathbb{H}_t^N$-realization $\mathfrak{H}_t^N$ of (7.14) are isomorphic. i.e.,*

$$\mathbb{H}_t^N \overset{\text{iso}}{=} \mathfrak{H}_t^N, \quad \textit{as } \mathbb{R}\textit{-vector spaces.} \tag{7.15}$$

Proof. Since $\mathbb{H}_t^N$ and $\mathfrak{H}_t^N$ are well-defined $\mathbb{R}$-vector spaces, the isomorphic relation (7.15) holds by (7.14) and the injectivity of π_t^N into $M_{2N\times 2}(\mathbb{C})$ (and hence, the bijectivity of it onto $\pi_t^N\left(\mathbb{H}_t^N\right) = \mathfrak{H}_t^N$). □

By (7.10) and (7.15), we have the following result showing how the realization $\mathscr{M}_{t,N} = \Pi_t\left(\mathcal{M}_{t,N}\right)$ naturally acts on the $\mathbb{H}_t$-realization $\mathfrak{H}_{t,N}$ of (7.14).

Theorem 7.7. *The realization $\mathscr{M}_{t,N} = \Pi_t\left(\mathcal{M}_{t,N}\right)$ of the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ acting on $\mathbb{H}_t^N$ is acting on the $\mathbb{H}_t^N$-realization $\mathfrak{H}_t^N$ of (7.14). And such an action is identical to the action of $(2N \times 2N)$-matrices on $(2N \times 2)$-matrices up to the usual matrix multiplication.*

Proof. Since our $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ acts on $\mathbb{H}_t^N$ under the block-matrix action, the realization $\mathscr{M}_{t,N} = \Pi_t\left(\mathcal{M}_{t,N}\right)$ acts on $\mathfrak{H}_t^N = \pi_t^N\left(\mathbb{H}_t^N\right)$, by (7.10) and (7.15). □

As one can see, all main results of this section are summarized by the above theorem, i.e., the main results of this section illustrate that the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$ acting on $\mathbb{H}_t^N$ is realized to be $\mathscr{M}_{t,N} = \Pi_t\left(\mathcal{M}_{t,N}\right)$ acting on $\mathfrak{H}_t^N = \pi_t^N\left(\mathbb{H}_t^N\right)$.

We finish this section with an example. Let

$$T = [h_{i,j}]_{2\times 2} \in \mathcal{M}_{t,2},\ \text{for } h_{i,j} = a_{i,j} + b_{i,j}j_t \in \mathbb{H}_t,$$

where $a_{i,j}, b_{i,j} \in \mathbb{C}$, for all $i, j = 1, 2$. Then

$$\pi_t(h_{i,j}) = \begin{pmatrix} a_{i,j} & tb_{i,j} \\ \overline{b_{i,j}} & \overline{a_{i,j}} \end{pmatrix}, \quad \forall i, j = 1, 2,$$

and hence,

$$\Pi_t(T) = \begin{pmatrix} a_{1,1} & tb_{1,1} & a_{1,2} & tb_{1,2} \\ \overline{b_{1,1}} & \overline{a_{1,1}} & \overline{b_{1,2}} & \overline{a_{1,2}} \\ a_{2,1} & tb_{2,1} & a_{2,2} & tb_{2,2} \\ \overline{b_{2,1}} & \overline{a_{2,1}} & \overline{b_{2,2}} & \overline{a_{2,2}} \end{pmatrix} \in \mathscr{M}_{t,2} = \Pi_t(\mathcal{M}_{t,2}).$$

And let

$$v = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} c_1 + d_1 j_t \\ c_2 + d_2 j_t \end{pmatrix} \in \mathbb{H}_t^2, \text{ with } c_1, c_2, d_1, d_2 \in \mathbb{C},$$

where $q_1 = c_1 + d_1 j_t$, $q_2 = c_2 + d_2 j_t \in \mathbb{H}_t$. Then

$$\pi_t^2(v) = \begin{pmatrix} c_1 & td_1 \\ \overline{d_1} & \overline{c_1} \\ c_2 & td_2 \\ \overline{d_2} & \overline{c_2} \end{pmatrix} \in \mathfrak{H}_t^2 = \pi_t^2\left(\mathbb{H}_t^2\right).$$

Observe that

$$T(v) = \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} h_{1,1}q_1 + h_{1,2}q_2 \\ h_{2,1}q_1 + h_{2,2}q_2 \end{pmatrix},$$

with

$$\pi_t(h_{i,j}q_l) = \pi_t(h_{i,j})\pi_t(q_l) = \begin{pmatrix} a_{i,j}c_l + tb_{i,j}\overline{d_l} & t\left(a_{i,j}d_l + b_{i,j}\overline{c_l}\right) \\ \overline{a_{i,j}d_l + b_{i,j}\overline{c_l}} & \overline{a_{i,j}c_l + tb_{i,j}\overline{d_l}} \end{pmatrix},$$

for all $i, j, l = 1, 2$. Thus, by (7.10), (7.15), and the above theorem, we have that

$$\Pi_t(T)\left(\pi_t^N(v)\right)$$

$$= \begin{pmatrix} a_{1,1}c_1 + a_{1,2}c_1 + t\left(b_{1,1}\overline{d_1} + b_{1,2}\overline{d_1}\right) & t\left(a_{1,1}d_1 + a_{1,2}d_1 + b_{1,1}\overline{c_1} + b_{1,2}\overline{c_1}\right) \\ \overline{a_{1,1}d_1 + a_{1,2}d_1 + b_{1,1}\overline{c_1} + b_{1,2}\overline{c_1}} & \overline{a_{1,1}c_1 + a_{1,2}c_2 + t\left(b_{1,1}\overline{d_1} + b_{1,2}\overline{d_1}\right)} \\ a_{2,1}c_2 + a_{2,2}c_2 + t\left(b_{2,1}\overline{d_2} + b_{2,2}\overline{d_2}\right) & t\left(a_{2,1}d_2 + a_{2,2}d_2 + b_{2,1}\overline{c_2} + b_{2,2}\overline{c_2}\right) \\ \overline{a_{2,1}d_2 + a_{2,2}d_2 + b_{2,1}\overline{c_2} + b_{2,2}\overline{c_2}} & \overline{a_{2,1}c_2 + a_{2,2}c_2 + t\left(b_{2,1}\overline{d_2} + b_{2,2}\overline{d_2}\right)} \end{pmatrix},$$

in $\mathfrak{H}_t^2$.

8. Certain Invariant $\mathbb{R}$-Subspaces of $\mathbb{H}_t^N$ Induced by $\mathbb{H}_t$-Matrices

In this section, as a continuation of Section 7, we apply the usual spectral theory on $M_{2N}(\mathbb{C})$, and then we consider certain invariant $\mathbb{R}$-subspaces of $\mathbb{H}_t^N$ induced by $\mathbb{H}_t$-matrices of the $\mathbb{H}_t$-matrix algebra $\mathcal{M}_{t,N}$. By (7.10), (7.15) and Theorem 58, every $\mathbb{H}_t$-matrix $T = [h_{i,j}]_{N\times N} \in \mathcal{M}_{t,N}$ acting on the $\mathbb{H}_t$-vectors $v = (q_k)_{k=1}^N \in \mathbb{H}_t^N$ is equivalent (or, isomorphic) to the matrix,

$$\Pi_t(T) = [\pi_t(h_{i,j})]_{2N\times 2N} \in \mathscr{M}_{t,N}, \text{ in } M_{2N}(\mathbb{C}),$$

acting on

$$\pi_t^N(v) = (\pi_t(q_k))_{k=1}^N \in \mathfrak{H}_t^N, \text{ in } M_{2N\times 2}(\mathbb{C}).$$

Note that, by the usual spectral theory, every $\mathbb{C}$-matrix A of $M_{2N}(\mathbb{C})$ has its non-empty spectrum $\text{spec}(A) \subset \mathbb{C}$, inducing its eigenspace $\mathcal{E}_\lambda \subset \mathbb{C}^{2N}$, satisfying

$$A(v) = \lambda v, \text{ for } v \in \mathcal{E}_\lambda, \text{ whenever } \lambda \in \text{spec}(A).$$

Then such an eigenspace $\mathcal{E}_{\lambda}$ for $\lambda \in \text{spec}(A)$ forms an invariant subspace of $\mathbb{C}^{2N}$ (over $\mathbb{C}$), satisfying

$$A(\mathcal{E}_{\lambda}) \subseteq \mathcal{E}_{\lambda}, \quad \forall \lambda \in \text{spec}(A).$$

It means that the realization $\Pi_t(T) \in \mathscr{M}_{t,N}$ of an $\mathbb{H}_t$-matrix $T \in \mathcal{M}_{t,N}$ has its spectrum $\text{spec}(\Pi_t(T))$ as an element of $M_{2N}(\mathbb{C})$. Motivated by this observation, we consider certain invariant "$\mathbb{R}$-subspaces of $\mathbb{H}_t^N$" induced by $\mathbb{H}_t$-matrices of $\mathcal{M}_{t,N}$.

Theorem 8.1. *For an $\mathbb{H}_t$-matrix $T \in \mathcal{M}_{t,N}$, there exist $v \in \mathbb{H}_t^N$ and $q \in \mathbb{H}_t$, such that $T(v) = vq$. i.e.,*

$$\forall T \in \mathcal{M}_{t,N}, \ \exists v \in \mathbb{H}_t^N, \text{ and } q \in \mathbb{H}_t, \text{ s.t., } T(v) = vq,$$

where (8.1)

$$vq = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} q = \begin{pmatrix} q_1 q \\ q_2 q \\ \vdots \\ q_N q \end{pmatrix}, \quad \text{whenever} \quad v = (q_k)_{k=1}^N.$$

Proof. Let $T = [h_{i,j}]_{N \times N} \in \mathcal{M}_{t,N}$ be an arbitrary $\mathbb{H}_t$-matrix with

$$h_{i,j} = a_{i,j} + b_{i,j} j_t \in \mathbb{H}_t, \text{ for } a_{i,j}, b_{i,j} \in \mathbb{C}, \ \forall i, j = 1, ..., N.$$

Consider the realization $\Pi_t(T) \in \mathscr{M}_{t,N}$ of T, as an element of $M_{2N}(\mathbb{C})$. Then, by the usual spectral theory on $M_{2N}(\mathbb{C})$, this $\mathbb{C}$-matrix $\Pi_t(T)$ has its non-empty spectrum $\text{spec}(\Pi_t(T))$ as a subset of $\mathbb{C}$, and if $\lambda \in \text{spec}(\Pi_t(T))$, then there exists the corresponding eigenspace $\mathcal{E}_{\lambda}$, satisfying

$$\Pi_t(T)(\mathcal{E}_{\lambda}) \subseteq \mathcal{E}_{\lambda}, \quad \text{in} \quad \mathbb{C}^{2N}.$$

i.e., for $\Pi_t(T) \in \mathscr{M}_{t,N} \subset M_{2N}(\mathbb{C})$, there exist $V \in \mathbb{C}^{2N}$ and $\lambda \in \mathbb{C}$, such that

$$\Pi_t(T)(V) = \lambda V = V\lambda, \quad \text{in} \quad \mathbb{C}^{2N}. \tag{8.2}$$

Now, for convenience, we write the vector $V \in \mathbb{C}^{2N}$ by

$$V = \left(a_1, \overline{b_1}, a_2, \overline{b_2}, ..., a_N, \overline{b_N}\right) = \begin{pmatrix} a_1 \\ \overline{b_1} \\ a_2 \\ \overline{b_2} \\ \vdots \\ a_N \\ \overline{b_N} \end{pmatrix},$$

and define a new vector $W \in \mathbb{C}^{2N}$ by

$$W = \left(tb_1, \overline{a_1}, tb_2, \overline{a_2}, ..., tb_N, \overline{a_N}\right) = \begin{pmatrix} tb_1 \\ \overline{a_1} \\ tb_2 \\ \overline{a_2} \\ \vdots \\ tb_N \\ \overline{a_N} \end{pmatrix} \in \mathbb{C}^{2N}.$$

Remark that the new vector W in terms of the eigenvector V is constructed to establish

$$\begin{pmatrix} V & W \end{pmatrix} \overset{\text{denote}}{=} \begin{pmatrix} \frac{a_1}{b_1} & \frac{tb_1}{a_1} \\ \frac{a_2}{b_2} & \frac{tb_2}{a_2} \\ \vdots & \vdots \\ \frac{a_N}{b_N} & \frac{tb_N}{a_N} \end{pmatrix} \in \mathfrak{H}_t^N = \pi_t^N \left(\mathbb{H}_t^N\right),$$

having its pre-image,

$$\left(\pi_t^N\right)^{-1} \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} a_1 + b_1 j_t \\ a_2 + b_2 j_t \\ \vdots \\ a_N + b_N j_t \end{pmatrix} \in \mathbb{H}_t^N.$$

Remark that, since π_t^N is bijective from $\mathbb{H}_t^N$ onto $\mathfrak{H}_t^N$, actually, the above pre-image is uniquely determined in $\mathbb{H}_t^N$.

By the straightforward computation, one can re-write the above relation (8.2) by its equivalent relation,

$$\sum_{k=1}^{N} \begin{pmatrix} \frac{a_{i,k}}{b_{i,k}} & \frac{tb_{i,k}}{a_{i,k}} \end{pmatrix} \begin{pmatrix} \frac{a_k}{b_k} \end{pmatrix} = \lambda \begin{pmatrix} \frac{a_i}{b_i} \end{pmatrix}, \ \forall i = 1, ..., N. \tag{8.3}$$

This relation (8.3) is equivalent to

$$\sum_{k=1}^{N} \left(a_{i,k} a_k + t b_{i,k} \overline{b_k}\right) = \lambda a_i, \quad \forall i = 1, ..., N$$

and (8.4)

$$\sum_{k=1}^{N} \left(\overline{b_{i,k}} a_k + \overline{a_{i,k}} \overline{b_k}\right) = \lambda \overline{b_i}, \quad \forall i, = 1, ..., N.$$

By the formulas of (8.4), we have that

$$\sum_{k=1}^{N} \left(a_{i,k} b_k + b_{i,k} \overline{b_k}\right) = \overline{\lambda} b_i, \quad \forall i = 1, ..., N,$$

and (8.5)

$$\sum_{k=1}^{N} \left(\overline{b_{i,k}} b_k + \overline{a_{i,k} a_k}\right) = \overline{\lambda} \overline{a_i}, \quad \forall i = 1, ..., N,$$

implying that (8.6)

$$\sum_{k=1}^{N} \begin{pmatrix} \frac{a_{i,k}}{b_{i,k}} & \frac{tb_{i,k}}{a_{i,k}} \end{pmatrix} \begin{pmatrix} \frac{tb_k}{a_k} \end{pmatrix} = \overline{\lambda} \begin{pmatrix} \frac{b_i}{a_i} \end{pmatrix}, \quad \forall i = 1, ..., N,$$

by (8.5). However, by (8.3) and (8.6), we have that

$$\Pi_t (T) \begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} V & W \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix},$$

where (8.7)

$$\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = \begin{pmatrix} \lambda & t(0) \\ \overline{0} & \overline{\lambda} \end{pmatrix} \in \pi_t \left(\mathbb{H}_t\right).$$

Therefore, by (8.7), one can conclude that, for any realization $\Pi_t(T) \in \mathscr{M}_{t,N}$ of an $\mathbb{H}_t$-matrix $T \in \mathcal{M}_{t,N}$, there exists $\pi_t^N(v) \in \mathfrak{H}_t^N$ with $v \in \mathbb{H}_t^N$, and $\lambda \in \mathbb{C}$ regarded as

$$\lambda + (0 + 0i)\, j_t \in \mathbb{H}_t,$$

such that

$$\Pi_t(T)\left(\pi_t^N(v)\right) = \pi_t^N(v)\,\lambda \in \mathfrak{H}_t^N \Longleftrightarrow T(v) = v\lambda \in \mathbb{H}_t^N.$$

Therefore, the relation (8.1) holds true. □

The above theorem shows that, for every $\mathbb{H}_t$-matrix $T \in \mathcal{M}_{t,N}$, there exist $v \in \mathbb{H}_t^N$ and $\lambda \in \mathbb{C} \subset \mathbb{H}_t$, such that $T(v) = v\lambda$ in $\mathbb{H}_t^N$, by (8.1).

Theorem 8.2. *Suppose $T \in \mathcal{M}_{t,N}$ satisfies $T(v) = v\lambda \in \mathbb{H}_t^N$, for $v \in \mathbb{H}_t^N$ and $\lambda \in \mathbb{C} \subset \mathbb{H}_t$ as in* (8.1)*. Define a $\mathbb{R}$-subspace $\mathcal{E}(T, v, \lambda)$ of $\mathbb{H}_t^N$ by*

$$\mathcal{E}(T, v, \lambda) \overset{\text{def}}{=} \text{span}_{\mathbb{R}}\left(\left\{v\lambda^n \in \mathbb{H}_t^N : n \in \mathbb{N}_0\right\}\right), \tag{8.8}$$

where $\text{span}_{\mathbb{R}}X$ *is the $\mathbb{R}$-vector space spanned by a subset X of $\mathbb{H}_t^N$. Then*

$$T(\mathcal{E}(T, v, \lambda)) \subseteq \mathcal{E}(T, v, \lambda), \quad \text{in} \quad \mathbb{H}_t^N,$$

i.e., (8.9)

$$\mathcal{E}(T, v, \lambda) \;\; \textit{is } T\textit{-invariant in } \mathbb{H}_t^N.$$

Proof. By (8.1), for any $T \in \mathcal{M}_{t,N}$, there are $v \in \mathbb{H}_t^N$ and $\lambda \in \mathbb{C}$ satisfying $\lambda + (0 + 0i)\, j_t \in \mathbb{H}_t$, such that $T(v) = v\lambda$ in $\mathbb{H}_t^N$. Now, note that

$$\Pi_t(T) \text{ is a } (2N \times 2N)\text{-matrix over } \mathbb{C},$$

$$\pi_t^N(v) \text{ is a } (2N \times 2)\text{-matrix over } \mathbb{C},$$

and

$$\pi_t(\lambda) \text{ is a } (2 \times 2)\text{-matrix over } \mathbb{C},$$

satisfying the matrix multiplication,

$$(\Pi_t(T))\left(\pi_t^N(v)\right) = \left(\pi_t^N(v)\right)(\pi_t(\lambda)),$$

in the sense of (8.7) by (8.1). So, one can get that

$$(\Pi_t(T))^2\left(\pi_t^N(v)\right) = (\Pi_t(T))\left(\pi_t^N(v)\right)(\pi_t(\lambda)) = \left(\pi_t^N(v)\right)(\pi_t(\lambda))^2,$$

as in (8.7), and

$$(\Pi_t(T))^3\left(\pi_t^N(v)\right) = (\Pi_t(T))^2\left(\pi_t^N(v)\right)(\pi_t(\lambda)) = \left(\pi_t^N(v)\right)(\pi_t(\lambda))^3,$$

up to the "associative" matrix multiplication. So, inductively, we have that

$$(\Pi_t(T))^n\left(\pi_t^N(v)\right) = \left(\pi_t^N(v)\right)(\pi_t(\lambda))^n, \;\; \forall n \in \mathbb{N},$$

up to the matrix multiplication. Equivalently,

$$T^n(v) = v\lambda^n \in \mathbb{H}_t^N, \qquad \forall n \in \mathbb{N},$$

by the injectivity of Π_t, π_t^N and π_t. Thus, if we define a $\mathbb{R}$-vector space,

$$\mathcal{E}(T, v, \lambda) = \text{span}_{\mathbb{R}}\{v\lambda^n : n \in \mathbb{N}_0\} \subset \mathbb{H}_t^N,$$

as in (8.8), then it is not only a well-defined $\mathbb{R}$-subspace of $\mathbb{H}_t^N$, but also a T-invariant subspace in the sense that:

$$T(V) \in \mathcal{E}(T, v, \lambda), \quad \forall V \in \mathcal{E}(T, v, \lambda).$$

Therefore, the relation (8.9) holds true. □

The above theorem shows that our $\mathbb{H}_t$-matrices of $\mathcal{M}_{t,N}$ have their invariant subspaces of $\mathbb{H}_t^N$ by (8.9).

Corollary 8.3. *Every* $\mathbb{H}_t$*-matrix* $T \in \mathcal{M}_{t,N}$ *has its* T*-invariant* $\mathbb{R}$*-subspace in* $\mathbb{H}_t^N$*.*

Proof. The proof is done by (8.9). Indeed, one can take a T-invariant $\mathbb{R}$-subspace $\mathcal{E}(T, v, \lambda)$ of (8.8) by (8.1). □

Declaration.

Ethical Statement. This submitted paper is the very original, not published, or submitted to elsewhere in any form or language, partially, or fully.

Competeing Interests. None.

Author′s Contribution. The co-author, Ilwoo Cho and Daniel Alpay, contributed to this manuscript equally, in theoretic and structural points.

Funding. No fund for authors.

Data Availability Statement. The co-authors confirms that no data known is used in the manuscript.

References

[1] D. Alpay, and I. Cho, Operators Induced by Certain Hypercomplex Systems, Opuscula Math., 43, issue 3, (2023) 275 - 333.

[2] D. Alpay, and I. Cho, Dynamical Systems of Operators Induced by Scaled Hypercomplex Rings, Adv. Appl. Clifford Alg., 33, (2023) article number 33.

[3] D. Alpay, and I.Cho, On Sclaed-Hyperbolic Numbers Induced by Sclaed Hypercomplex Rings, Pure Appl. Funct. Anal., (2024) To Appear.

[4] D. Alpay, and I. Cho, Regular Functions on the Scaled Hypercomplex Numbers, Integral Equat. Oper. Theo., 96, DOI:10.1007/s00020-024-02765-9, (2024) Article Number 9.

[5] D. Alpay, and I. Cho, Schur Analysis over the Open Unit Ball, Linea Alg. Appl., (2024) To Appear.

[6] D. Alpay, and I. Cho, Toeplitz-Like Operators Induced by Non-Zero Scaled Hypercomplex Numbers over the Real Field $\mathbb{R}$, (2024) Submitted to Pure Appl. Funct. Anal..

[7] D. Alpay, and I. Cho, Block-Toeplitz Operators Induced by Scaled Hypercomplex Numbers over the Real Field $\mathbb{R}$, J. Oper. Theo., (2025) To Appear.

[8] D. Alpay, I. Cho, and M. Vajiak, Scaled Global Operators and Fueter Variables on Non-Zero Scaled Hypercomplex Numbers, Adv. Appl. Clifford Alg., (2024) To Appear.

[9] D. Alpay, M. E. Luna-Elizarraras, M. Shapiro, and D. C. Struppa, Gleason's Problem, Rational Functions and Spaces of Left-regular Functions: The Split-Quaternioin Settings, Israel J. Math., 226, (2018) 319 - 349.

[10] D. Alpay, I. L. Paiva, and D. C. Struppa, A General Setting for Functions of Fueter Variables: Differentiability, Rational Functions, Fock Module and Related Topics, Israel J. Math., 236, (2020) 207 - 246.

[11] D. Alpay, and M. Shapiro, Probléme de Gleason et Interpolation pour let Fonctions hyper-analytiques, C. R. Math. Acad. Sci. Paris, 335, (2022)

[12] M. L. Curtis, Matrix Groups, ISBN: 0-387-90462-X, (1979) Published by Springer-Verlag, NY.

[13] R. Delanghe, On Regular-Analytic Functions with Values in a Clifford Algebra, Math. Ann., 425, (1970) 91 - 111.

[14] I. Frenkel, and M. Libine, Split Quaternionic Analysis and Separation of the Series for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C}) / SL(2, \mathbb{R})$, Adv. Math., 228, (2011) 678 - 763.

[15] P. R. Halmos, Linear Algebra Problem Book, ISBN: 978-0-88385-322-1, (1995) Published by Math. Assoc. Amer.

[16] P. R. Halmos, Hilbert Space Problem Book, ISBN: 978-038-790685-0, (1982) Published by Springer-Verlag (NY).

[17] V. Kravchenko, Applied Quaternionic Analysis, ISBN: 3-88538-228-8, (2003) Published by Heldemann Verlag.

[18] M. Kobayashi, Hyperbolic Hopfield Neural Networks with Directional Multistate Activation Function, Neurocomputing, 275, (2018) 2217 - 2226.

[19] M. Libine, An Invitation to Split Quaternionic Analysis, in Hypercomplex Analysis and Applications, Trends in Math., Birkhauser / Springer, Basel AG, Basel, (2011) 161 - 180.

[20] L. Rodman, Topics in Quaternion Linear Algebra, ISBN:978-0-691-16185-3, (2014) Published by Prinston Univ. Press, NJ.

[21] B. A. Rozenfeld, The History of non-Eucledean Geometry: Evolution of the Concept of a Geometric Spaces, ISBN: 978-038-796458-4, (1988) Published by Springer.

[22] R. Speicher, Combinatorial Theory of the Free Product with Amalgamation and Operator-Valued Free Probability Theory, Amer. Math. Soc., Memoire, ISBN:978-0-8218-0693-7, (1998) Published by Amer. Math. Soc..

[23] A. Sudbery, Quaternionic Analysis, DOI: 10.1017/s0305004100053638, (1998) Published by Math. Proc. Cambridge Philosop. Soc..

[24] J. A. Vince, Geometric Algebra for Computer Graphics, ISBN: 978-1-84628-996-5, (2008) Published by Springer.

[25] D.V. Voiculescu, K. J. Dykema, and A. Nica, Free Random Variables, ISBN:978-0-8218-1140-5, (1992) Published by Amer. Math. Soc..

[26] J. Voight, Quaternion Algebra, Available on http://math.dartmouth.edu/~jvoight/quat-book.pdf, (2019) Dept. of Math., Dartmouth Univ..

Daniel Alpay: alpay@chapman.edu
Chapman University, Dept. of Math., 381 Keck Center, 1 University Dr. Orange, California, 92866, U. S. A.

Ilwoo Cho: choilwoo@sau.edu
St. Ambrose Univ., Dept. of Math. and Stat., 421 Ambrose Hall, 518 W. Locust St., Davenport, Iowa, 52803, U. S. A.

Received 24/01/2025; Revised 13/09/2025