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HYPERSTABILITY OF SOME FUNCTIONAL EQUATIONS IN MODULAR SPACES

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Abstract. This paper is devoted to the study of hyperstability phenomena in the context of convex modular spaces. In particular, we investigate the hyperstability of three fundamental functional equations: the quadratic equation

$$\varphi(x+y)+\varphi(x-y)=2\varphi(x)+2\varphi(y) \tag{0.1}$$

the general linear equation

$$\varphi(ax+by)=A\varphi(x)+B\varphi(y)+C \tag{0.2}$$

and the n-dimensional quadratic equation

$$f\left(\sum_{i=1}^{m} x_i\right)+\sum_{1\le i<j\le m} f\big(x_i-x_j\big)=m\sum_{i=1}^{m} f(x_i). \tag{0.3}$$

Using the direct method, we establish sufficient conditions under which every approximate solution of these equations in modular spaces coincides exactly with an exact solution. Our results extend earlier contributions obtained in Banach spaces via fixed point techniques, and provide new insights into the stability of functional equations in the broader context of modular spaces.

1. Introduction and preliminaries

A fundamental question in the theory of functional equations is: under what conditions is a function that approximately satisfies a functional equation necessarily close to an exact solution of that equation? If such conditions are met, the equation is said to be stable. The initial stability problem related to group homomorphisms was proposed by Ulam in 1940 (see [29]). In 1941, the study of stability theory for functional equations began with Hyers'pioneering work [20], where he provided a partial affirmative answer to Ulam's question concerning the additive functional equation, specifically in the setting of Banach spaces. This result was subsequently generalized in 1950 by Aoki [5], Bourgin [8], and Rassias [27], who extended the analysis to additive and linear mappings by addressing the case of unbounded Cauchy differences. Further progress was made in 1994 by Găvruţă [18], who introduced a broader generalization of Rassias'results by replacing the bounded condition with a general control function $\varphi(x,y)$, thereby establishing the existence of a unique linear mapping.
During this period, a special form of stability, known as hyperstability, was identified. Hyperstability of a functional equation requires that any mapping approximately satisfying the equation (in a suitably defined sense) must in fact be an exact solution. Although the term hyperstability was first introduced by Maksa in 2001 [23], its earliest occurrence can be traced back to 1949, as noted by Bourgin [7]. Among the most prominent methods for proving the hyperstability of functional equations is the fixed point approach, which has gained considerable attention over the past two decades, largely due to the contributions of Brzdęk and Ciepliński [11] to fixed point theory, and subsequently developed by various other authors. For additional results and references, see [1, 2, 3, 9, 10, 12, 13].

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Additionally, numerous papers have been published on the hyperstability of functional equations (see, e.g., [6, 14, 17, 19, 21]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.4}$$

is known as the quadratic functional equation, and any solution of this equation is referred to as a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation was first established by Skof [28] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Later, Cholewa [15] observed that Skof's theorem remains valid when the domain X is replaced by an Abelian group. Furthermore, Czerwik [16] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation.
Research on modular and modular spaces as extensions of normed spaces was initiated by Nakano [25]. Since the 1950s, numerous eminent mathematicians [4, 26, 30] have made significant contributions to this field. Examples of applications of modular spaces include Orlicz spaces and interpolation theory (see [22, 24, 26]). We now present the definitions, properties, and standard terminology of the theory of modular spaces.

Definition 1.1. Let Y be a vector space. A functional $\rho : Y \to [0, \infty)$ is called a modular, if for arbitrary $u, v \in Y$, the following conditions hold:

(1) $\rho(u) = 0$ if and only if $u = 0$.
(2) $\rho(\alpha u) = \rho(u)$ for every scalar α with $|\alpha| = 1$.
(3) $\rho(\alpha u + \beta v) \leq \rho(u) + \rho(v)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.
If condition (3) is replaced by:
(4) $\rho(\alpha u + \beta v) \leq \alpha\rho(u) + \beta\rho(v)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, denoted by Y_ρ, which is given by:

$$Y_\rho = \{u \in Y : \rho(\lambda u) \to 0 \text{ as } \lambda \to 0\}.$$

A function modular is said to satisfy the Δ_s-condition if there exists $\tau_s > 0$ such that $\rho(su) \leq \tau_s \rho(u)$ for all $u \in Y_\rho$.

Definition 1.2. Let $\{u_n\}$ be a sequence in Y_ρ and let $u \in Y_\rho$. Then

(1) $\{u_n\}$ is said to be ρ-convergent to u, denoted $u_n \to u$, if $\rho(u_n - u) \to 0$ as $n \to \infty$.
(2) $\{u_n\}$ is called ρ-Cauchy if $\rho(u_n - u_m) \to 0$ as $n, m \to \infty$.
(3) The modular space Y_ρ is said to be ρ-complete if every ρ-Cauchy sequence in Y_ρ is ρ-convergent.

Proposition 1.3. *Let Y_ρ be a modular space. Then:*

(1) *If $u_n \xrightarrow{\rho} u$ and a is a fixed vector in Y_ρ, then $u_n + a \xrightarrow{\rho} u + a$.*
(2) *If $u_n \xrightarrow{\rho} u$ and $v_n \xrightarrow{\rho} v$, then for all $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$, we have $\alpha u_n + \beta v_n \xrightarrow{\rho} \alpha u + \beta v$.*

Remark 1.4. Note that $\rho(u)$ is an increasing function, for all $u \in X$. Suppose $0 < a < b$. Then, by property (4) of Definition 1.1 with $v = 0$, we have $\rho(ax) = \rho\left(\frac{a}{b}bu\right) \leq \rho(bu)$ for all $u \in Y$. Moreover, if ρ is a convex modular on Y and $|\alpha| \leq 1$, then $\rho(\alpha u) \leq \alpha\rho(u)$.
In general, if $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, then

$$\rho(\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n) \leq \lambda_1\rho(u_1) + \lambda_2\rho(u_2) + \cdots + \lambda_n\rho(u_n).$$

If $\{u_n\}$ is ρ-convergent to u, then $\{cu_n\}$ is ρ-convergent to cu, where $|c| \leq 1$. However, the ρ-convergence of a sequence $\{u_n\}$ to u does not imply that $\{\alpha u_n\}$ is ρ-convergent to αu for scalars α with $|\alpha| > 1$.

If ρ is a convex modular satisfying the Δ_s-condition with $0 < \tau_s < s$, then

$$\rho(u) \leq \tau_s \rho(\frac{1}{s}u) \leq \frac{\tau_s}{s}\rho(u) \text{ for all } u.$$

Hence, $\rho = 0$. Consequently, we must have $\tau_s \geq s$ if ρ is convex modular.

2. Hyperstability of Equation (1.4)

In this section, we investigate the hyperstability of the quadratic functional equation (1.4) in the framework of modular spaces. In particular, we establish that every approximate solution of (1.4), under suitable conditions, must in fact be an exact solution. Unless otherwise stated, we assume throughout this section that X is a vector space over a field $\mathbb{K}$ and that Y_ρ is a convex modular space satisfying the Δ_2-condition.

Theorem 2.1. *Suppose that E is a non-empty subset of X that is symmetric with respect to 0 and satisfies the conditions $x+y, x-y \in E$ and $kx \in E$ for all $x, y \in E$ and all $k \in \mathbb{K}$. Let $\alpha : E^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \alpha(x, nx) = 0 \text{ and } \lim_{n\to\infty} \alpha(nx, ny) = 0, \tag{2.5}$$

for all $x, y \in E$. Let $\varphi : E \to Y_\rho$ be a mapping satisfying

$$\rho(\varphi(x+y) + \varphi(x-y) - 2\varphi(x) - 2\varphi(y)\Big) \leq \alpha(x,y),$$

for all $x, y \in E$. Then φ is quadratic on E.

Proof. Letting $y = nx$ in (2.5), we obtain

$$\rho\left(\frac{1}{2}\varphi(x+nx) + \frac{1}{2}\varphi(x-nx) - \varphi(x) - \varphi(nx)\right) \leq \frac{1}{2}\alpha(x,nx),$$

for all $x \in E$. Hence, for all $x, y \in E$, we have

$$\rho\left(\frac{1}{2}\varphi(y+ny) + \frac{1}{2}\varphi(y-ny) - \varphi(y) - \varphi(ny)\right) \leq \frac{1}{2}\alpha(y,ny),$$

$$\rho\left(\frac{1}{2}\varphi\big(x+y+n(x+y)\big) + \frac{1}{2}\varphi\big(x+y-n(x+y)\big) - \varphi(x+y) - \varphi\big(n(x+y)\big)\right) \leq \frac{1}{2}\alpha(x+y, n(x+y)),$$

and

$$\rho\left(\frac{1}{2}\varphi\big(x-y+n(x-y)\big) + \frac{1}{2}\varphi\big(x-y-n(x-y)\big) - \varphi(x-y) - \varphi\big(n(x-y)\big)\right) \leq \frac{1}{2}\alpha(x-y, n(x-y)),$$

for all $x, y \in E$. Letting $n \to \infty$, we get

$$\varphi(x) = \rho - \lim_{n\to\infty}\left(\frac{1}{2}\varphi\big((n+1)x\big) + \frac{1}{2}\varphi\big((1-n)x\big) - \varphi(nx)\right),$$

$$\varphi(y) = \rho - \lim_{n\to\infty}\left(\frac{1}{2}\varphi\big((n+1)y\big) + \frac{1}{2}\varphi\big((1-n)y\big) - \varphi(ny)\right),$$

$$\varphi(x+y) = \rho - \lim_{n\to\infty}\left(\frac{1}{2}\varphi\big((n+1)(x+y)\big) + \frac{1}{2}\varphi\big((1-n)(x+y)\big) - \varphi\big(n(x+y)\big)\right),$$

$$\varphi(x-y) = \rho - \lim_{n\to\infty}\left(\frac{1}{2}\varphi\big((n+1)(x-y)\big) + \frac{1}{2}\varphi\big((1-n)(x-y)\big) - \varphi\big(n(x-y)\big)\right).$$

Then, we have

$$\begin{aligned}
&\rho\left(\frac{1}{7}\varphi(x+y)+\frac{1}{7}\varphi(x-y)-\frac{2}{7}\varphi(x)-\frac{2}{7}\varphi(y)\right)\\
&\le \frac{1}{7}\rho\left(\varphi(x+y)-\left(\frac{1}{2}\varphi((n+1)(x+y))+\frac{1}{2}\varphi((1-n)(x+y))-\varphi(n(x+y))\right)\right)\\
&+\frac{1}{7}\rho\left(\varphi(x-y)-\left(\frac{1}{2}\varphi((n+1)(x-y))+\frac{1}{2}\varphi((1-n)(x-y))-\varphi(n(x-y))\right)\right)\\
&+\frac{2}{7}\rho\left(\varphi(x)-\left(\frac{1}{2}\varphi((n+1)x)+\frac{1}{2}\varphi((1-n)x)-\varphi(nx)\right)\right)\\
&+\frac{2}{7}\rho\left(\varphi(y)-\left(\frac{1}{2}\varphi((n+1)y)+\frac{1}{2}\varphi((1-n)y)-\varphi(ny)\right)\right)\\
&+\frac{1}{7}\rho\left(\frac{1}{2}\varphi((n+1)(x+y))+\frac{1}{2}\varphi((1-n)(x+y))-\varphi(n(x+y))\right)\\
&+\frac{1}{7}\rho\left(\frac{1}{2}\varphi((n+1)(x-y))+\frac{1}{2}\varphi((1-n)(x-y))-\varphi(n(x-y))\right).\\
&\le \frac{1}{7}\rho\left(\varphi(x+y)-\left(\frac{1}{2}\varphi((n+1)(x+y))+\frac{1}{2}\varphi((1-n)(x+y))-\varphi(n(x+y))\right)\right)\\
&+\frac{1}{7}\rho\left(\varphi(x-y)-\left(\frac{1}{2}\varphi((n+1)(x-y))+\frac{1}{2}\varphi((1-n)(x-y))-\varphi(n(x-y))\right)\right)\\
&+\frac{2}{7}\rho\left(\varphi(x)-\left(\frac{1}{2}\varphi((n+1)x)+\frac{1}{2}\varphi((1-n)x)-\varphi(nx)\right)\right)\\
&+\frac{2}{7}\rho\left(\varphi(y)-\left(\frac{1}{2}\varphi((n+1)y)+\frac{1}{2}\varphi((1-n)y)-\varphi(ny)\right)\right)\\
&+\frac{\tau}{14}\rho\left(\frac{1}{2}\varphi((n+1)(x+y))+\frac{1}{2}\varphi((n+1)(x-y))-\varphi((n+1)x)-\varphi((n+1)y)\right)\\
&+\frac{\tau}{14}\rho\left(\frac{1}{2}\varphi((1-n)(x+y))+\frac{1}{2}\varphi((1-n)(x-y))-\varphi((1-n)x)-\varphi((1-n)y)\right)\\
&\le \frac{1}{7}\rho\left(\varphi(x+y)-\left(\frac{1}{2}\varphi((n+1)(x+y))+\frac{1}{2}\varphi((1-n)(x+y))-\varphi(n(x+y))\right)\right)\\
&+\frac{1}{7}\rho\left(\varphi(x-y)-\left(\frac{1}{2}\varphi((n+1)(x-y))+\frac{1}{2}\varphi((1-n)(x-y))-\varphi(n(x-y))\right)\right)\\
&+\frac{2}{7}\rho\left(\varphi(x)-\left(\frac{1}{2}\varphi((n+1)x)+\frac{1}{2}\varphi((1-n)x)-\varphi(nx)\right)\right)\\
&+\frac{2}{7}\rho\left(\varphi(y)-\left(\frac{1}{2}\varphi((n+1)y)+\frac{1}{2}\varphi((1-n)y)-\varphi(ny)\right)\right)\\
&+\frac{\tau}{28}\alpha((n+1)x,(n+1)y)\\
&+\frac{\tau^2}{28}\rho\left(\frac{1}{2}\varphi((1-n)(x+y))+\frac{1}{2}\varphi((1-n)(x-y))-\varphi((1-n)x)-\varphi((1-n)y)\right)\\
&+\frac{\tau^2}{28}\rho\left(\frac{1}{2}\varphi(n(x+y))+\frac{1}{2}\varphi(n(x-y))-\varphi(nx)-\varphi(ny)\right)
\end{aligned}$$

for all $x, y \in E$. Letting $n \to \infty$, we get

$$\rho\left(\frac{1}{7}\varphi(x+y)+\frac{1}{7}\varphi(x-y)-\frac{2}{7}\varphi(x)-\frac{2}{7}\varphi(y)\right)\to 0$$

which implies that

$$\varphi(x+y)+\varphi(x-y)=2\varphi(x)+2\varphi(y),$$

for $x, y \in E$, which means that φ is quadratic on E. $\square$

Corollary 2.2. *Let $\theta \geq 0$, p and q be two real numbers such that $p+q<0$. Let $\varphi : X \to Y_\rho$ be a mapping satisfying*

$$\rho\Big(\varphi(x+y)+\varphi(x-y)-2\varphi(x)-2\varphi(y)\Big) \leq \theta\|x\|^p\|y\|^q,$$

for all $x, y \in E$. Then φ is quadratic on E.

Proof. Let $\alpha(x,y)=\theta\|x\|^p\|y\|^q$ for all $x, y \in E$. Since $p+q<0$, at least one of p or q must be negative. Assume that $q<0$, we have

$$\lim_{n\to\infty}\alpha(x,ny)=\lim_{n\to\infty}\theta n^q\|x\|^p\|y\|^q=0 \text{ and } \lim_{n\to\infty}\alpha(nx,ny)=\lim_{n\to\infty}\theta n^{p+q}\|x\|^p\|y\|^q=0.$$

Therefore, the conditions in Theorem 2.1 hold which means that φ is quadratic on E. $\square$

3. Hyperstability of a general linear functional equation

In the following section, we investigate the hyperstability of a general linear functional equation. We establish conditions under which every approximate solution necessarily coincides with an exact solution of the functional equation in the framework of modular spaces.

Theorem 3.1. *Suppose that E is a non-empty subset of X, symmetric with respect to 0, and satisfying $x+y, x-y \in E$ and $kx \in E$ for all $x, y \in E$ and all $k \in \mathbb{K}$. Let $a, b \in \mathbb{K}\setminus\{0\}$ and let $\alpha : E^2 \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty}\alpha\big(a^{-1}(n+1)x,-b^{-1}nx\big)=0$$

and

$$\lim_{n\to\infty}\alpha\big(nx,ny\big)=0,$$

for all $x, y \in E\setminus\{0\}$. Let $A, B \in \mathbb{R}$, $C \in Y_\rho$ such that $|A| \leq 1$ and $|B| \leq 1$ and let $\varphi : X \to Y_\rho$ satisfying:

$$\rho\big(\varphi(ax+by)-A\varphi(x)-B\varphi(y)-C\big) \leq \alpha(x,y), \tag{3.6}$$

for all $x, y \in M_\alpha=\{z \in E : \|z\| \geq \alpha\}$ for some $\alpha>0$. Then φ satisfies

$$\varphi(ax+by)=A\varphi(x)+B\varphi(y)+C$$

and

$$(A+B)\varphi(0)=A\varphi(x)+B\varphi\big(-ab^{-1}x\big),$$

for all $x, y \in E$.

Proof. Substituting x by $a^{-1}(n+1)x$ and y by $-b^{-1}nx$ in (3.6), we obtain

$$\rho\big(\varphi(x)-A\varphi(a^{-1}(n+1)x)-B\varphi(-b^{-1}nx)-C\big) \leq \alpha\big(a^{-1}(n+1)x,-b^{-1}nx\big), \tag{3.7}$$

for all $x \in E\setminus\{0\}$ and all positive integers $n \geq m$, where $a^{-1}(m+1)x,\ -b^{-1}mx \in M_\alpha$. Letting $n \to \infty$ in (3.7), we obtain

$$\varphi(x)=\rho-\lim_{n\to\infty}\Big[A\varphi\big(a^{-1}(n+1)x\big)+B\varphi\big(-b^{-1}nx\big)+C\Big], \quad x \in E\setminus\{0\}.$$

Therefore, we have

$$
\begin{aligned}
&\rho\Big(\frac{1}{4}\varphi(ax+by)-\frac{A}{4}\varphi(x)-\frac{B}{4}\varphi(y)-\frac{C}{4}\Big)\\
&\leq \frac{1}{4}\rho\Big(\varphi(ax+by)-\big(A\varphi(a^{-1}(n+1)(ax+by))+B\varphi(-b^{-1}n(ax+by))+C\big)\Big)\\
&+\frac{1}{4}\rho\Big(A\varphi(x)-\big(A^2\varphi(a^{-1}(n+1)x)+AB\varphi(-b^{-1}nx)+AC\big)\Big)\\
&+\frac{1}{4}\rho\Big(B\varphi(y)-\big(AB\varphi(a^{-1}(n+1)y)+B^2\varphi(-b^{-1}nx)+BC\big)\Big)\\
&+\frac{1}{4}\rho\Big(A\varphi(a^{-1}(n+1)(ax+by))+B\varphi(-b^{-1}n(ax+by))+C-A^2\varphi(a^{-1}(n+1)x)\\
&\quad -AB\varphi(-b^{-1}nx)-AC-AB\varphi(a^{-1}(n+1)y)-B^2\varphi(-b^{-1}ny)-BC-C\Big)\\
&\leq \frac{1}{4}\rho\Big(\varphi(ax+by)-\big(A\varphi(a^{-1}(n+1)(ax+by))+B\varphi(-b^{-1}n(ax+by))+C\big)\Big)\\
&+\frac{1}{4}\rho\Big(A\varphi(x)-\big(A^2\varphi(a^{-1}(n+1)x)+AB\varphi(-b^{-1}nx)+AC\big)\Big)\\
&+\frac{1}{4}\rho\Big(B\varphi(y)-\big(AB\varphi(a^{-1}(n+1)y)+B^2\varphi(-b^{-1}nx)+BC\big)\Big)\\
&+\frac{|A|\tau}{8}\rho\Big(\varphi(a^{-1}(n+1)(ax+by))-A\varphi(a^{-1}(n+1)x)-B\varphi(a^{-1}(n+1)y)-C\Big)\\
&+\frac{|A|\tau}{8}\rho\Big(\varphi(-b^{-1}n(ax+by))-A\varphi(-b^{-1}nx)-B\varphi(-b^{-1}ny)-C\Big)\\
&\leq \frac{1}{4}\rho\Big(\varphi(ax+by)-\big(A\varphi(a^{-1}(n+1)(ax+by))+B\varphi(-b^{-1}n(ax+by))+C\big)\Big)\\
&+\frac{1}{4}\rho\Big(A\varphi(x)-\big(A^2\varphi(a^{-1}(n+1)x)+AB\varphi(-b^{-1}nx)+AC\big)\Big)\\
&+\frac{1}{4}\rho\Big(B\varphi(y)-\big(AB\varphi(a^{-1}(n+1)y)+B^2\varphi(-b^{-1}nx)+BC\big)\Big)\\
&+\frac{|A|\tau}{8}\alpha\big(a^{-1}(n+1)x,a^{-1}(n+1)y\big)+\frac{|A|\tau}{8}\alpha\big(-b^{-1}nx,-b^{-1}ny\big)\\
&\to 0 \ \text{ as } \ n\to\infty
\end{aligned}
$$

Hence, we obtain

$$\varphi(ax+by)=A\varphi(x)+B\varphi(y)+C.$$

Moreover, we have

$$
\begin{aligned}
&\rho\left(\frac{1}{3}(A+B)\varphi(0)-\frac{1}{3}A\varphi(x)-\frac{1}{3}B\varphi\left(-ab^{-1}x\right)\right)\\
&\leq \frac{1}{3}\rho\left(A\varphi(x)-\left(A^2\varphi\left(a^{-1}(n+1)x\right)+AB\varphi\left(-b^{-1}nx\right)+AC\right)\right)\\
&\quad+\frac{1}{3}\rho\left(B\varphi\left(-ab^{-1}nx\right)-\left(AB\varphi\left(-b^{-1}(n+1)x\right)+B^2\varphi\left(ab^{-2}nx\right)+BC\right)\right)\\
&\quad+\frac{1}{3}\rho\left((A+B)\varphi(0)-A^2\varphi\left(a^{-1}(n+1)x\right)+AB\varphi\left(-b^{-1}nx\right)-AC\right.\\
&\quad\left.-AB\varphi\left(-b^{-1}(n+1)x\right)-B^2\varphi\left(ab^{-2}nx\right)-BC\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{1}{3}\rho\left(A\varphi(x) - \left(A^2\varphi\left(a^{-1}(n+1)x\right) + AB\varphi\left(-b^{-1}nx\right) + AC\right)\right)\\
&\quad + \frac{1}{3}\rho\left(B\varphi\left(-ab^{-1}nx\right) - \left(AB\varphi\left(-b^{-1}(n+1)x\right) + B^2\varphi\left(ab^{-2}nx\right) + BC\right)\right)\\
&\quad + \frac{|A|\tau}{6}\rho\left(\varphi(0) - A\varphi\left(a^{-1}(n+1)x\right) - B\varphi\left(-b^{-1}(n+1)x\right) - C\right)\\
&\quad + \frac{|B|\tau}{6}\rho\left(\varphi(0) - A\varphi\left(-b^{-1}nx\right) - B\varphi\left(ab^{-2}nx\right) - C\right)\\
&\leq \frac{1}{3}\rho\left(A\varphi(x) - \left(A^2\varphi\left(a^{-1}(n+1)x\right) + AB\varphi\left(-b^{-1}nx\right) + AC\right)\right)\\
&\quad + \frac{1}{3}\rho\left(B\varphi\left(-ab^{-1}nx\right) - \left(AB\varphi\left(-b^{-1}(n+1)x\right) + B^2\varphi\left(ab^{-2}nx\right) + BC\right)\right)\\
&\quad + \frac{|A|\tau}{6}\alpha\left(a^{-1}(n+1)x, -b^{-1}(n+1)x\right) + \frac{|B|\tau}{6}\alpha\left(-b^{-1}nx, ab^{-2}nx\right)\\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence,

$$(A+B)\varphi(0) = A\varphi(x) + B\varphi(-ab^{-1}x). \tag{3.8}$$

Now, if we replace x by bnx and y by $-anx$ in the inequality (3.6), we obtain:

$$\rho\big(\varphi(0) - A\varphi(bnx) - B\varphi(-anx) - C\big) \leq \alpha(bnx, -anx) \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Hence

$$\varphi(0) = \rho - \lim_{n\to\infty}\big[A\varphi(bnx) + B\varphi(-anx) + C\big].$$

On the other hand, if we replace x by bnx in (3.8), we get:

$$(A+B)\varphi(0) = A\varphi(bnx) + B\varphi(-anx).$$

Then

$$\rho\big((1-A-B)\varphi(0) - C\big) = \rho\big(\varphi(0) - A\varphi(bnx) - B\varphi(-anx) - C\big)$$

and therefore

$$C = (1-A-B)\varphi(0).$$

□

Corollary 3.2. *Let $a, b \in \mathbb{K} \setminus \{0\}$ and let $\varphi : E \to Y_\rho$ be a function. Take $\theta, \theta' \geq 0$, and p, q, r be real numbers. Suppose that one of the following conditions holds:*
(i) $p + q + r \leq 0$ and

$$\rho\big(\varphi(ax+by) - A\varphi(x) - B\varphi(y) - C\big) \leq \|x\|^p\|y\|^q\big(\theta\|x+y\|^r + \theta\|x-y\|^r\big)$$

(ii) $p + q \leq 0$ and

$$\rho\big(\varphi(ax+by) - A\varphi(x) - B\varphi(y) - C\big) \leq \theta\|x\|^p\|y\|^q$$

(iii) $p, q < 0$ and

$$\rho\big(\varphi(ax+by) - A\varphi(x) - B\varphi(y) - C\big) \leq \theta\|x\|^p + \theta'\|y\|^q.$$

Then φ satisfies

$$\varphi(ax+by) = A\varphi(x) + B\varphi(y) + C$$

and

$$(A+B)\varphi(0) = A\varphi(x) + B\varphi\big(-ab^{-1}x\big),$$

for all $x, y \in M_\alpha = \{z \in E : \|z\| \geq \alpha\}$ for some $\alpha > 0$.

Remark 3.3. (i) If $a = b = A = B = 1$ and $C = 0$, we obtain the hyperstability result for the additive functional equation $\varphi(x+y) = \varphi(x) + \varphi(y)$ in modular space.

(ii) If $a = b = A = B = \dfrac{1}{2}$ and $C = 0$, we obtain the hyperstability result for the Jensen functional equation $\varphi\big(\frac{x+y}{2}\big) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$ in modular space.

4. Hyperstability of the n-dimensional quadratic functional equation

In this section, we investigate the hyperstability of the n-dimensional quadratic functional equation.

Theorem 4.1. *Suppose that E is a non-empty subset of X that is symmetric with respect to 0 and satisfies $x+y, x-y \in E$ and $kx \in E$ for all $x, y \in E$ and all $k \in \mathbb{K}$. Let $f : E \to Y$ and $\varphi : E^m \to [0,\infty)$ be two functions that satisfy the following conditions*

$$\lim_{n\to\infty} \varphi\big(x, nx, \ldots, nx\big) = 0, \tag{4.9}$$

$$\lim_{n\to\infty} \varphi\big(nx_1, nx_2, \ldots, nx_m\big) = 0$$

and

$$\rho\left(f\left(\sum_{i=1}^{m} x_i\right) + \sum_{1\le i<j\le m} f(x_i - x_j) - m\sum_{i=1}^{m} f(x_i)\right) \le \varphi(x_1, x_2, \ldots, x_m), \tag{4.10}$$

for all $x_1, x_2, \ldots, x_m \in E$. Then f satisfies equation (0.3) on E.

Proof. Letting $x_i = nx$ with $i \ge 2$ and $n \in \mathbb{N}$ in (4.10), we obtain

$$\rho\left(f\big((1+mn-n)x\big) + (m-1)f((1-n)x) - m(m-1)f(nx) - mf(x)\right) \le \varphi\big(x, nx, \ldots, nx\big),$$

Hence

$$\rho\left(\frac{1}{m}f\big((1+mn-n)x\big) + \frac{m-1}{m}f((1-n)x) - (m-1)f(nx) - f(x)\right) \le \frac{1}{m}\varphi\big(x, nx, \ldots, nx\big),$$

for all $x \in X$ and for all $n \in \mathbb{N}$. In view of (4.9), we deduce that

$$f(x) = \rho - \lim_{n\to\infty}\left[\frac{1}{m}f\big((1+mn-n)x\big) + \frac{(m-1)}{m}f((1-n)x) - (m-1)f(nx)\right],$$

for all $x \in X$. Therefore,

$$f\left(\sum_{i=1}^{m} x_i\right) = \rho - \lim_{n\to\infty}\left\{\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m} x_i\right) - \frac{(m-1)}{m}f\left((1-n)\sum_{i=1}^{m} x_i\right) - (m-1)f\left(n\sum_{i=1}^{m} x_i\right)\right\},$$

$$f(x_i) = \rho - \lim_{n\to\infty}\left\{f\big((1+mn-n)x_i\big) - (m-1)f((1-n)x_i) - m(m-1)f\big(nx_i\big)\right\},$$

and

$$f\big(x_i - x_j\big) = \rho - \lim_{n\to\infty}\left\{\frac{1}{m}f\big((1+mn-n)(x_i - x_j)\big) + \frac{(m-1)}{m}f\big((1-n)(x_i - x_j)\big) - (m-1)f\big(n(x_i - x_j)\big)\right\},$$

for all $x_1, x_2, \ldots, x_m \in E$. Now, we have:

$$
\begin{aligned}
&\rho\left[\frac{2}{3m^2-m+4}\left(f\left(\sum_{i=1}^{m} x_i\right)+\sum_{1\le i<j\le m} f\left(x_i-x_j\right)-m\sum_{i=1}^{m} f\left(x_i\right)\right)\right.\\
&\le \frac{2}{3m^2-m+4}\rho\left[f\left(\sum_{i=1}^{m} x_i\right)-\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m} x_i\right)\right.\\
&\left.-\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m} xi\right)+(m-1)f\left(n\sum_{i=1}^{m} xi\right)\right]\\
&+\frac{2}{3m^2-m+4}\sum_{1\le i<j\le m}\rho\left[f\left(x_i-x_j\right)-\frac{1}{m}f\left(\left(1+mn-n\right)\left(x_i-x_j\right)\right)\right.\\
&\left.-\frac{m-1}{m}f\left(\left(1-n\right)\left(x_i-x_j\right)\right)+(m-1)f\left(n\left(x_i-x_j\right)\right)\right]\\
&+\frac{2m}{3m^2-m+4}\sum_{i=1}^{m}\rho\left[f\left(x_i\right)-\frac{1}{m}f\left((1+mn-n)x_i\right)\ -\frac{m-1}{m}f\left((1-n)x_i\right)+(m-1)f\left(nx_i\right)\right]\\
&+\frac{2}{3m^2-m+4}\rho\left[\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m} x_i\right)+\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m} x_i\right)\right]\\
&-(m-1)f\left(n\sum_{i=1}^{m} x_i\right)+\frac{1}{m}\sum_{1\le i<j\le m} f\left(\left(1+mn-n\right)\left(x_i-x_j\right)\right)\\
&+\frac{m-1}{m}\sum_{1\le i<j\le m} f\left(\left(1-n\right)\left(x_i-x_j\right)\right)-(m-1)\sum_{1\le i<j\le m} f\left(n\left(x_i-x_j\right)\right)\\
&\left.-\sum_{i=1}^{m} f\left((1+mn-n)x_i\right)-(m-1)\sum_{i=1}^{m} f\left((1-n)x_i\right)\ +m(m-1)\sum_{i=1}^{m} f\left(nx_i\right)\right]\\
&\le \frac{2}{3m^2-m+4}\rho\left[f\left(\sum_{i=1}^{m} x_i\right)-\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m} x_i\right)\right.\\
&\left.-\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m} xi\right)+(m-1)f\left(n\sum_{i=1}^{m} xi\right)\right]\\
&+\frac{2}{3m^2-m+4}\sum_{1\le i<j\le m}\rho\left[f\left(x_i-x_j\right)-\frac{1}{m}f((1+mn-n)(x_i-x_j))\right.\\
&\left.-\frac{m-1}{m}f\left(\left(1-n\right)\left(x_i-x_j\right)\right)+(m-1)f\left(n\left(x_i-x_j\right)\right)\right]\\
&+\frac{2m}{3m^2-m+4}\sum_{i=1}^{m}\rho\left[f\left(x_i\right)-\frac{1}{m}f\left((1+mn-n)x_i\right)\ -\frac{m-1}{m}f\left((1-n)x_i\right)+(m-1)f\left(nx_i\right)\right]\\
&+\frac{k_2}{3m^2-m+4}\rho\left\{\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m} x_i\right)\right.\\
&\left.+\frac{1}{m}\sum_{1\le i<j\le m} f\left(\left(1+mn-n\right)\left(x_i-x_j\right)\right)-\sum_{i=1}^{m} f\left((1+mn-n)x_i\right)\right\}\\
&+\frac{k_2}{3m^2-m+4}\rho\left\{\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m} x_i\right)\right.\\
&+\frac{m-1}{m}\sum_{1\le i<j\le m} f\left(\left(1-n\right)\left(x_i-x_j\right)\right)-(m-1)\sum_{i=1}^{m} f\left((1-n)x_i\right)\\
&\left.-(m-1)f\left(n\sum_{i=1}^{m} x_i\right)-(m-1)\sum_{1\le i<j\le m} f\left(n\left(x_i-x_j\right)\right)+m(m-1)\sum_{i=1}^{m} f\left(nx_i\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{2}{3m^2-m+4}\rho\left[f\left(\sum_{i=1}^{m}x_i\right)-\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m}x_i\right)\right.\\
&\left.-\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m}x_i\right)+(m-1)f\left(n\sum_{i=1}^{m}x_i\right)\right]\\
&+\frac{2}{3m^2-m+4}\sum_{1\leq i<j\leq m}\rho\left[f\left(x_i-x_j\right)-\frac{1}{m}f((1+mn-n)(x_i-x_j))\right.\\
&\left.-\frac{m-1}{m}f\left((1-n)\left(x_i-x_j\right)\right)+(m-1)f\left(n\left(x_i-x_j\right)\right)\right]+\frac{2m}{3m^2-m+4}\\
&\times\sum_{i=1}^{m}\rho\left[f\left(x_i\right)-\frac{1}{m}f\left((1+mn-n)x_i\right)\ -\frac{m-1}{m}f\left((1-n)x_i\right)+(m-1)f\left(nx_i\right)\right]\\
&+\frac{k_2}{3m^3-m^2+4m}\rho\left\{f\left((1+mn-n)\sum_{i=1}^{m}x_i\right)\right.\\
&\left.+\sum_{1\leq i<j\leq m}f\left((1+mn-n)\left(x_i-x_j\right)\right)-m\sum_{i=1}^{m}f\left((1+mn-n)x_i\right)\right\}\\
&+\frac{k_2^2}{6m^2-2m+8}\rho\left\{\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m}x_i\right)\right.\\
&\left.+\frac{m-1}{m}\sum_{1\leq i<j\leq m}f\left((1-n)\left(x_i-x_j\right)\right)-(m-1)\sum_{i=1}^{m}f\left((1-n)x_i\right)\right\}+\frac{k_2^2}{6m^2-2m+8}\\
&\times\rho\left\{(m-1)f\left(n\sum_{i=1}^{m}xi\right)+(m-1)\sum_{1\leq i<j\leq m}f\left(n\left(x_i-x_j\right)\right)-m(m-1)\sum_{i=1}^{m}f\left(nx_i\right)\right\}\\
&\leq \frac{2}{3m^2-m+4}\rho\left[f\left(\sum_{i=1}^{m}x_i\right)-\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^{m}x_i\right)\right.\\
&\left.-\frac{m-1}{m}f\left((1-n)\sum_{i=1}^{m}x_i\right)+(m-1)f\left(n\sum_{i=1}^{m}x_i\right)\right]\\
&+\frac{2}{3m^2-m+4}\sum_{1\leq i<j\leq m}\rho\left[f\left(x_i-x_j\right)-\frac{1}{m}f((1+mn-n)(x_i-x_j))\right.\\
&\left.-\frac{m-1}{m}f\left((1-n)\left(x_i-x_j\right)\right)+(m-1)f\left(n\left(x_i-x_j\right)\right)\right]\\
&+\frac{2m}{3m^2-m+4}\sum_{i=1}^{m}\rho\left[f\left(x_i\right)-\frac{1}{m}f\left((1+mn-n)x_i\right)\ -\frac{m-1}{m}f\left((1-n)x_i\right)+(m-1)f\left(nx_i\right)\right]\\
&+\frac{k_2}{3m^3-m^2+4m}\varphi\left((1+mn-n)x_1,\ldots,(1+mn-n)x_n\right)\\
&+\frac{k_2^2(m-1)}{6m^3-2m^2+8m}\varphi\left((1-n)x_1,\ldots,(1-n)x_n\right)+\frac{k_2^2k_{m-1}}{6m^2-2m+8}\varphi\left(nx_1,\ldots,nx_n\right)\to 0
\end{aligned}
$$

as $n\to\infty$ for all $x_1,\ldots,x_m\in E$. It means that equation (0.3) is hyperstable on E. □

Corollary 4.2. *Let θ and p be two real numbers such that $\theta\geq 0$ and $p<0$. Let $f:E\to Y_\rho$ be a mapping satisfying*

$$\rho\left(f\left(\sum_{i=1}^{m}x_i\right)+\sum_{1\leq i<j\leq m}f(x_i-x_j)-m\sum_{i=1}^{m}f(x_i)\right)\leq\theta\prod_{i=1}^{m}\|x_i\|^p,$$

for all $x_1,\ldots,x_m\in E\setminus\{0\}$. Then f satisfies equation (0.3) on E

Proof. In Theorem 4.1, we suppose that

$$\varphi(x_1, \ldots, x_m) := \theta \prod_{i=1}^{m} \|x_i\|^p$$

for all $x_1, \ldots, x_m \in E$. We notice that

$$\lim_{n\to\infty} \varphi\big(x, nx, \ldots, nx\big) = \lim_{n\to\infty} \theta n^{(m-1)p} \|x\|^p = 0$$

and

$$\lim_{n\to\infty} \varphi\big(nx, nx, \ldots, nx\big) = \lim_{n\to\infty} \theta n^{mp} \|x\|^p = 0,$$

for all $x \in E$. This implies that f satisfies equation (0.3) on E. □

5. Hyperstability of the n-dimensional quadratic equation in Banach space

This last section investigates the hyperstability of the n-dimensional quadratic equation within the context of Banach spaces.

Theorem 5.1. *Suppose that E is a non-empty subset of X, symmetric with respect to 0, and satisfying $x + y, x - y \in E$ and $kx \in E$ for all $x, y \in E$ and all $k \in \mathbb{K}$. Let $\varphi : E^m \to [0, \infty)$ be a function such that:*

$$\lim_{n\to\infty} \varphi\big(x, nx, \ldots, nx\big) = 0 \tag{5.11}$$

and

$$\lim_{n\to\infty} \varphi\big(nx_1, nx_2, \ldots, nx_m\big) = 0.$$

Let Y be a Banach space, and $f : E \longrightarrow Y$ be a mapping satisfying:

$$\left\| f\left(\sum_{i=1}^{m} x_i\right) + \sum_{1\le i<j\le m} f(x_i - x_j) - m \sum_{i=1}^{m} f(x_i) \right\| \le \varphi(x_1, x_2, \ldots, x_m), \tag{5.12}$$

for all $x_1, x_2, \ldots, x_m \in E$. Then f satisfies equation (0.3) on E .

Proof. Let $x_i = nx$ with $i \ge 2$ and $n \in \mathbb{N}$ in (5.12), we get

$$\Big\| f\big((1 + mn - n)x\big) + (m - 1)f((1 - n)x) - m(m - 1)f(nx) - mf(x) \Big\| \le \varphi\big(x, nx, \ldots, nx\big),$$

for all $x \in E$, and all $n \in \mathbb{N}$. In view of (5.11), we deduce that

$$f(x) = \lim_{n\to\infty} \frac{1}{m} f\big((1 + mn - n)x\big) + \frac{(m-1)}{m} f((1 - n)x) - (m - 1)f(nx),$$

for all $x \in X$. On the other hand, we have

$$f\left(\sum_{i=1}^{m} x_i\right) = \lim_{n\to\infty} \left\{ \frac{1}{m} f\left((1 + mn - n)\sum_{i=1}^{m} x_i\right) - \frac{(m-1)}{m} f\left((1 - n)\sum_{i=1}^{m} x_i\right) - (m - 1)f\left(n\sum_{i=1}^{m} x_i\right) \right\},$$

$$m\sum_{i=1}^{m} f(x_i) = \lim_{n\to\infty} \left\{ \sum_{i=1}^{m} f\big((1 + mn - n)x_i\big) - (m - 1)\sum_{i=1}^{m} f((1 - n)x_i) - m(m - 1)\sum_{i=1}^{m} f\big(nx_i\big) \right\},$$

and

$$\sum_{1\le i<j\le m} f\big(x_i-x_j\big)=\lim_{n\to\infty}\Bigg\{\frac{1}{m}\sum_{1\le i<j\le m} f\big((1+mn-n)(x_i-x_j)\big)+$$
$$\frac{(m-1)}{m}\sum_{1\le i<j\le m} f\big((1-n)(x_i-x_j)\big)-(m-1)\sum_{1\le i<j\le m} f\big(n(x_i-x_j)\big)\Bigg\},$$

for all $x_1,x_2,\dots,x_m\in E$. Hence

$$\begin{aligned}
&\left\|f\left(\sum_{i=1}^m x_i\right)+\sum_{1\le i<j\le m} f\big(x_i-x_j\big)-m\sum_{i=1}^m f(x_i)\right\|\\
&\quad=\lim_{n\longrightarrow\infty}\Bigg\|\frac{1}{m}f\left((1+mn-n)\sum_{i=1}^m x_i\right)-\frac{(m-1)}{m}f\left((1-n)\sum_{i=1}^m x_i\right)\\
&\qquad-(m-1)f\left(n\sum_{i=1}^m x_i\right)+\frac{1}{m}\sum_{1\le i<j\le m} f\big((1+mn-n)(x_i-x_j)\big)\\
&\qquad+\frac{(m-1)}{m}\sum_{1\le i<j\le m} f\big((1-n)(x_i-x_j)\big)-(m-1)\sum_{1\le i<j\le m} f\big(n(x_i-x_j)\big)\\
&\qquad-\sum_{i=1}^m f\big((1+mn-n)x_i\big)+(m-1)\sum_{i=1}^m f((1-n)x_i)+m(m-1)\sum_{i=1}^m f\big(nx_i\big)\Bigg\|\\
&\quad\le\lim_{n\longrightarrow\infty}\sup\Bigg\{\frac{1}{m}\Bigg\|f\left((1+mn-n)\sum_{i=1}^m x_i\right)+\sum_{1\le i<j\le m} f\big((1+mn-n)(x_i-x_j)\big)\\
&\qquad\qquad-m\sum_{i=1}^m f\big((1+mn-n)x_i\big)\Bigg\|\Bigg\}\\
&\qquad+\lim_{n\longrightarrow\infty}\sup\Bigg\{\frac{(m-1)}{m}\Bigg\|f\left((1-n)\sum_{i=1}^m x_i\right)+\sum_{1\le i<j\le m} f\big((1-n)(x_i-x_j)\big)\\
&\qquad\qquad-m\sum_{i=1}^m f((1-n)x_i)\Bigg\|\Bigg\}\\
&\qquad+\lim_{n\longrightarrow\infty}\sup\Bigg\{(m-1)\Bigg\|f\left(n\sum_{i=1}^m x_i\right)+\sum_{1\le i<j\le m} f\big(n(x_i-x_j)\big)-m\sum_{i=1}^m f\big(nx_i\big)\Bigg\|\Bigg\}\\
&\quad\le\lim_{n\longrightarrow\infty}\sup\frac{1}{m}\varphi\Big((1+mn-n)x_1,\dots,(1+mn-n)x_m\Big)\\
&\qquad+\lim_{n\longrightarrow\infty}\sup\frac{(m-1)}{m}\varphi\Big((1-n)x_1,\dots,(1-n)x_m\Big)\\
&\qquad+\lim_{n\longrightarrow\infty}\sup(m-1)\varphi\Big(nx_1,\dots,nx_m\Big)=0,
\end{aligned}$$

for all $x_1,\dots,x_m\in E$. Which means that equation (0.3) is hyperstable on E. □

Corollary 5.2. *Let θ and p be two real numbers such that $\theta\ge 0$ and $p<0$. Let $f:E\to Y$ be a mapping satisfying*

$$\left\|f\left(\sum_{i=1}^m x_i\right)+\sum_{1\le i<j\le m} f(x_i-x_j)-m\sum_{i=1}^m f(x_i)\right\|\le\theta\prod_{i=1}^m\|x_i\|^p,$$

for all $x_1,\dots,x_m\in X\setminus\{0\}$. Then f satisfies the functional equation (0.3) on E.

Proof. In Theorem 5.1, we suppose that

$$\varphi(x_1, \ldots, x_m) := \theta \prod_{i=1}^{m} \|x_i\|^p$$

for all $x_1, \ldots, x_m \in E$. We note that

$$\lim_{n\to\infty} \varphi\big(x, nx, \ldots, nx\big) = \lim_{n\to\infty} \theta n^{(m-1)p}\|x\|^p = 0$$

and

$$\lim_{n\to\infty} \varphi\big(nx, nx, \ldots, nx\big) = \lim_{n\to\infty} \theta n^{mp}\|x\|^p = 0,$$

for all $x \in E$, which means that f satisfies equation (0.3) on E. □

Conclusion

In this paper, we investigated the hyperstability of several functional equations in the context of convex modular spaces. Using the direct method, we established that approximate solutions of the quadratic functional equation, the general linear functional equation, and the n-dimensional quadratic functional equation necessarily coincide with their exact forms. We also derived corresponding hyperstability results in Banach spaces, thereby linking the modular space setting with classical normed structures. Our results complement the fixed point approaches developed by Brzdęk and others, and extend the scope of hyperstability theory beyond Banach spaces. Potential directions for future research include exploring hyperstability for other classes of functional equations in modular and Orlicz spaces.

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Authors' contributions
The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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