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LAGUERRE-BESSEL WAVELET PACKETS TRANSFORM

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Abstract. In this paper, the Laguerre-Bessel wavelet packets transform is defined and studied. The scale discrete scaling function and the associated Plancherel and inversion formulas are given and established. Furthermore, the Calderón reproducing formula is given and proved for the proposed transform.

1. Introduction

Let $L^p_\alpha(\mathbb{K})$, $1 \le p \le \infty$, denote the space of measurable functions f on $\mathbb{K} = \mathbb{R}_+ \times \mathbb{R}_+$ equipped with the following norm

$$\|f\|_{p,\alpha} := \left[\int_{\mathbb{K}} |f(x,t)|^p \, dm_\alpha(x,t)\right]^{1/p} < +\infty, \quad \text{if } 1 \le p < \infty,$$
$$\|f\|_{\infty,\alpha} := \operatorname{ess}\sup_{(x,t)\in\mathbb{K}} |f(x,t)| < \infty,$$

where m_α is the weighted Lebesgue measure on $\mathbb{K}$ defined by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}t^{2\alpha}}{\Gamma(\alpha+1/2)\,\Gamma(\alpha+1)}\,dxdt.$$

The Laguerre-Bessel transform $\mathcal{F}_{LB}$ of a function f is defined on $L^1_\alpha(\mathbb{K})$ by

$$\mathcal{F}_{LB}(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{\lambda,m}(x,t) f(x,t)\, dm_\alpha(x,t), \quad \forall(\lambda,m) \in \widehat{\mathbb{K}} = \mathbb{R}_+ \times \mathbb{N}.$$

The harmonic analysis associated with the Laguerre-Bessel transform is discussed and studied in [11]. This analysis is generated by the two differential operators D_1 and D_2 defined on $\mathbb{K}$ by

$$D_1 = \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t}\frac{\partial}{\partial t}, \quad t \ge 0,$$
$$D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2 D_1, \quad x > 0.$$

In the same paper, the authors have extended this integral operator to the wavelet theory. The basic properties and Calderón's formula have been given and established. Nowadays, the applications of wavelets are divers and large. Signal and image processing [2] as well as biomedical engineering [3] and statistics [1] are some of these applications. We refer the reader to [4, 5, 9, 10] for a detailed theory of wavelets and its applications. To obtain a good frequency localization of the signal in a wavelet basis, it is much more appropriate to use so-called wavelets packet. This new concept which was introduced by R. Coifman, Y. Meyer, and M. V. Wickerhauser (see [6]) divide the frequency space into several parts. Therefore, there is no redundant informations in the decomposed frequency bands. Several works have been published recently dealing with the wavelet packets associated with various integral transforms. We refer the reader to the following references [8, 12, 13, 14, 15, 16, 17, 19, 20] for a very good understanding of this concept. The

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main objective of this work is to extend the notion of wavelets packet to the LB-wavelet transform. In particular, we give a general construction allowing the development of LB-wavelets packet. In our construction, we follow the same schemes as that given in Trimèche's book [18]. For other construction schemes we refer to [7].

This work is organized as follows: In the second section, we recall some results of harmonic analysis associated with the LB-transform. Then we give our construction of wavelet packets for the proposed transform. More specifically, we define the LB-wavelet packet transform and we prove its Plancherel and inversion formulas. In the third section, we introduce the scale discrete scaling function, then we give its properties. The last section is devoted to Caldéron's formula for the LB-wavelet packet.

2. The LB-wavelet packets Transform

In this section, we first recall some results of harmonic analysis associated with the LB-transform. Then, we define the LB-wavelet packet transform. We prove then Plancherel and inversion formulas for the proposed transform.

Definition 2.1. The Laguerre-Bessel transform $\mathcal{F}_{LB}$ is defined on $L^1_\alpha(\mathbb{K})$ by

$$\mathcal{F}_{LB}(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{\lambda,m}(x,t) f(x,t)\ dm_\alpha(x,t), \quad \forall(\lambda,m)\in\widehat{\mathbb{K}}. \tag{2.1}$$

The function $\varphi_{\lambda,m}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable and we have

$$\sup_{(x,t)\in\mathbb{K}} |\varphi_{\lambda,m}(x,t)| = 1. \tag{2.2}$$

The Laguerre transform satisfies the following properties [11]:

(1) Plancherel formula:

$$\int_{\mathbb{K}} |f(x,t)|^2 dm_\alpha(x,t) = \int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(f)(\lambda,m)|^2\ d\gamma_\alpha(\lambda,m). \tag{2.3}$$

(2) Inversion formula:

$$f(x,t) = \int_{\widehat{\mathbb{K}}} \varphi_{\lambda,m}(x,t)\mathcal{F}_{LB}(f)(\lambda,m)\ d\gamma_\alpha(\lambda,m), \quad a.e \quad (x,t)\in\mathbb{K}, \tag{2.4}$$

where γ_α is the positive measure defined on Γ by

$$\int_{\mathbb{K}} f(\lambda,m)d\gamma_\alpha(\lambda,m) := \frac{1}{2^{2\alpha-1}\Gamma(\alpha+1/2)} \sum_{m=0}^{\infty} L^\alpha_m(0) \int_0^\infty f(\lambda,m)\lambda^{3\alpha+1} d\lambda.$$

Definition 2.2. The LB-translation operator associated with the operators D_1 and D_2 is defined by

$$T^\alpha_{(x,t)} f(y,s) := \frac{1}{4\pi} \sum_{i,j=0}^{1} \int_0^\pi f\left(\Delta_\theta(x,y), Y + (-1)^i t + (-1)^j s\right) d\theta \quad \text{with} \quad \alpha = 0, \tag{2.5}$$

and

$$T^\alpha_{(x,t)} f(y,s) := b_\alpha \int_{[0,\pi]^3} f\left(\Delta_\theta(x,y), \Delta_\xi(X,Y)\right) d\mu_\alpha(\xi,\psi,\theta) \quad \text{with} \quad \alpha > 0. \tag{2.6}$$

For more properties about LB-translation operator, we refer to [11]. We'll just mention the necessary results that we'll use to prove our main results in this paper.

Definition 2.3. The generalized convolution product of $f,g\in S_*(\mathbb{K})$ is defined by

$$f *_\alpha g(x,t) = \int_{\mathbb{K}} T^\alpha_{(x,t)}(f)(s,y) g(s,y) dm_\alpha(s,y), \quad \forall(x,t)\in\mathbb{K}. \tag{2.7}$$

Proposition 2.4. i) *Let $f \in L^1_\alpha(\mathbb{K})$. Then for all $(x,t) \in \mathbb{K}$ and $(\lambda,m) \in \widehat{\mathbb{K}}$, we have*

$$\mathcal{F}_{LB}\left(T^\alpha_{(x,t)}f\right)(\lambda,m) = \varphi_{\lambda,m}(x,t)\mathcal{F}_{LB}(f)(\lambda,m). \tag{2.8}$$

ii) *For all f and g in $L^1_\alpha(\mathbb{K})$, we have*

$$\mathcal{F}_{LB}(f *_\alpha g)(\lambda,m) = \mathcal{F}_{LB}(f)(\lambda,m)\mathcal{F}_{LB}(g)(\lambda,m). \tag{2.9}$$

By product formula (2.9), we have this following proposition:

Proposition 2.5. *For all $f, g \in L^2_\alpha(\mathbb{K})$, we have the identity*

$$\int_{\mathbb{K}} |f *_\alpha g(x)|^2 \, dm_\alpha(x,t) = \int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(f)(\lambda)|^2 |\mathcal{F}_{LB}(g)(\lambda)|^2 \, d\gamma_\alpha(\lambda,m). \tag{2.10}$$

Definition 2.6. Let $g \in L^2_\alpha(\mathbb{K})$, We say that g is a LB-wavelet on $\mathbb{K}$ if the following admissibility condition holds

$$0 < C_g = \int_0^{+\infty} |\mathcal{F}_{LB}(g)(\lambda,m)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{2.11}$$

Let $a \in]0,+\infty[$, we put

$$g_a(x,t) = \frac{1}{a^{3\alpha+2}} g\left(\frac{x}{\sqrt{a}}, \frac{t}{a}\right), \tag{2.12}$$

this function satisfies the following proposition:

Proposition 2.7. (i) *For $g \in L^2_\alpha(\mathbb{K})$, the function g_a belongs to $L^2_\alpha(\mathbb{K})$ and we have*

$$\|g_a\|_{2,\alpha} = \frac{1}{a^{\frac{3}{2}\alpha+\frac{7}{4}}} \|g\|_{2,\alpha}. \tag{2.13}$$

(ii) *For $g \in L^2_\alpha(\mathbb{K})$, we have*

$$\mathcal{F}_{LB}(g_a)(\lambda,m) = \mathcal{F}_{LB}(g)(a\lambda,m), \quad (\lambda,m) \in \widehat{\mathbb{K}}. \tag{2.14}$$

Proof. (i) A simple change of variable gives the desired result.
(ii) See [11].

□

Proposition 2.8. (i) *Let $j \in \mathbb{Z}$, the function*

$$(\lambda,m) \longmapsto \left(\frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathcal{F}_{LB}(g)(a\lambda,m)|^2 \frac{da}{a}\right)^{\frac{1}{2}}, \tag{2.15}$$

belongs to $L^2_\alpha(\widehat{\mathbb{K}})$.
(ii) *There exist a function $g^p_j \in L^2_\alpha(\mathbb{K})$ such that*

$$\mathcal{F}_{LB}(g^p_j)(a\lambda,m) = \frac{1}{C_g}\int_{r_{j+1}}^{r_j} |\mathcal{F}_{LB}(g)(a\lambda,m)|^2 \frac{da}{a}. \tag{2.16}$$

Proof. By using the Fubini-Tonelli's theorem we get

$$\begin{aligned}
&\frac{1}{C_g}\int_{\widehat{\mathbb{K}}}\left[\int_{r_{j+1}}^{r_j}|\mathcal{F}_{LB}(g)(a\lambda,m)|^2\frac{da}{a}\right]d\gamma_\alpha(\lambda,m)\\
&=\frac{1}{C_g}\int_{r_{j+1}}^{r_j}\left[\int_{\widehat{\mathbb{K}}}\left|\mathcal{F}_{LB}\left(g_a\right)(\lambda,m)\right|^2d\gamma_\alpha\left(\lambda,m\right)\right]\frac{da}{a}\\
&=\frac{1}{C_g}\int_{r_{j+1}}^{r_j}\left\|\mathcal{F}_{LB}\left(g_a\right)\right\|_{\gamma,2}^2\frac{da}{a}\\
&=\frac{1}{C_g}\int_{r_{j+1}}^{r_j}\|g_a\|_{2,\alpha}^2\frac{da}{a}\\
&=\frac{\|g\|_{2,\alpha}^2}{C_g}\int_{r_{j+1}}^{r_j}\frac{da}{a^{3\alpha+9/2}}\\
&=\frac{\|g\|_{2,\alpha}^2}{\left(3\alpha+\frac{7}{2}\right)C_g}\left[\frac{1}{r_{j+1}^{3\alpha+\frac{9}{2}}}-\frac{1}{r_j^{3a+\frac{9}{2}}}\right]<\infty.
\end{aligned}$$

The result is proved. □

Definition 2.9. The sequence $\left\{g_j^p\right\}_{j\in\mathbb{Z}}$ is called LB-wavelet packet.

Remark 2.10. It is easy to check that

$$0\leqslant\mathcal{F}_{LB}\left(g_j^p\right)(\lambda,m)\leqslant 1 \text{ and } \sum_{j=-\infty}^{+\infty}\left[\mathcal{F}_{LB}\left(g_j^p\right)(\lambda,m)\right]^2=1. \tag{2.17}$$

Now, we consider the family of wavelet packet defined as follow:

$$g_{j,a,x,t}^p(y,s)=T_{(x,t)}^\alpha g_j^p(y,s). \tag{2.18}$$

Definition 2.11. Let $\left\{g_j^p\right\}_{j\in\mathbb{Z}}$ be a LB-wavelet packet. The LB-wavelet packet transform Ψ_g^p is defined for a function f in $L_\alpha^2(\mathbb{K})$ by

$$\begin{aligned}
\Psi_g^pf(j,a,x,t)&=\int_{\mathbb{K}}f(y,s)\overline{g_{j,a,x,t}^p}(y,s)dm_\alpha(y,s)\\
&=g_j^p*_\alpha f.
\end{aligned} \tag{2.19}$$

Theorem 2.12. *Let $f\in L_\alpha^2(\mathbb{K})$, we have*

$$\int_{\mathbb{K}}|\ f(x,t)\ |^2\ dm_\alpha(x,t)=\sum_{j=-\infty}^{+\infty}\int_{\mathbb{K}}\left|\Psi_g^pf(j,a,y,s)\right|^2\ dm_\alpha(y,s). \tag{2.20}$$

Proof. By the Proposition 2.4, we have

$$\begin{aligned}
\int_{\mathbb{K}}\left|\Psi_g^pf(j,a,y,s)\right|^2\ dm_\alpha(y,s)&=\int_{\mathbb{K}}\left|g_j^p*_\alpha f\right|^2\ dm_\alpha(y,s)\\
&=\int_{\widehat{\mathbb{K}}}\left|\mathcal{F}_{LB}(g_j^p)(\lambda,m)\right|^2\left|\mathcal{F}_{LB}(f)(\lambda,m)\right|^2d\gamma_\alpha(\lambda,m).
\end{aligned}$$

Now applying Fubini-Tonelli 's theorem and remark 2.10, we get

$$\begin{aligned}
&\sum_{j=-\infty}^{+\infty}\int_{\mathbb{K}}\left|\Psi_g^p f(j,a,y,s)\right|^2\ d\mu_\alpha(y,s)\\
&=\int_{\widehat{\mathbb{K}}}\sum_{j=-\infty}^{+\infty}\left|\mathcal{F}_{LB}(g_j^p)(\lambda,m)\right|^2\left|\mathcal{F}_{LB}f(\lambda,m)\right|^2 d\gamma_\alpha(\lambda,m)\\
&=\int_{\widehat{\mathbb{K}}}\left|\mathcal{F}_{LB}(f)(\lambda,m)\right|^2 d\gamma_\alpha(\lambda,m)\\
&=\int_{\mathbb{K}}|f(x,t)|^2 dm_\alpha(x,t).
\end{aligned}$$

The proof is complete. □

Theorem 2.13. *Let $\left\{g_j^p\right\}_{j\in\mathbb{Z}}$ be a LB-wavelet packet. For all $f\in L^2_\alpha(\mathbb{K})$, such that $\mathcal{F}_{LB}(f)\in L^1_\alpha(\widehat{\mathbb{K}})$ we have:*

$$f(y,s)=\sum_{j=-\infty}^{+\infty}\int_{\mathbb{K}}\Psi_j^p f(j,a,x,t)g^p_{j,a,x,t}(y,s)\ d\mu_\alpha(x,t). \tag{2.21}$$

Proof. Let $j\in\mathbb{Z}$ and let $f\in L^2_\alpha(\mathbb{K})$ such that $\mathcal{F}_{LB}(f)\in L^1_\alpha(\widehat{\mathbb{K}})$, we consider the function

$$\Upsilon(j,a,y,s)=\int_{\mathbb{K}}\Psi_j^p f(j,a,x,t)g^p_{j,a,x,t}(y,s)\ d\mu_\alpha(x,t). \tag{2.22}$$

Relations (2.3), (2.8) and (2.9) gives as

$$\begin{aligned}
\Upsilon(j,a,y,s)&=\int_{\widehat{\mathbb{K}}}\mathcal{F}_{LB}\left(g_j^p\right)(\lambda,m)\mathcal{F}_{LB}(f)(\lambda,m)\varphi_{\lambda,m}(x,t)\mathcal{F}_{LB}\left(g_j^p\right)(\lambda,m)d\gamma_\alpha(\lambda,m)\\
&=\int_{\widehat{\mathbb{K}}}\mathcal{F}_{LB}\left(f\right)(\lambda,m)\varphi_{\lambda,m}(x,t)\left[\mathcal{F}_{LB}(g_j^p)(\lambda,m)\right]^2 d\gamma_\alpha(\lambda,m).
\end{aligned}$$

Using now (2.2), we obtain

$$\begin{aligned}
\sum_j\Upsilon(j,a,y,s)&\le\sum_{j=-\infty}^{+\infty}\int_{\widehat{\mathbb{K}}}\left|\mathcal{F}_{LB}(f)(\lambda,m)\right|\mid\varphi_{\lambda,m}(x,t)\mid\left[\mathcal{F}_{LB}\left(g_j^P\right)(\lambda,m)\right]^2 d\gamma_\alpha(\lambda,m)\\
&=\int_{\widehat{\mathbb{K}}}\left|\mathcal{F}_{LB}(f)(\lambda,m)\right|\sum_j\left[\mathcal{F}_{LB}\left(g_j^p\right)(\lambda,m)\right]^2 d\gamma_\alpha(\lambda,m)\\
&=\int_{\widehat{\mathbb{K}}}\left|\mathcal{F}_{LB}(f)(\lambda,m)\right|d\gamma_\alpha(\lambda,m)\\
&=\|\mathcal{F}_{LB}f\|_{1,\gamma_\alpha}\\
&<\infty.
\end{aligned}$$

So,

$$\begin{aligned}
\sum_j\Upsilon(j,a,y,s)&=\int_{\widehat{\mathbb{K}}}\mathcal{F}_{LB}(f)(\lambda,m)\varphi_{\lambda,m}(x,t)\left(\sum_j\left[\mathcal{F}_{LB}\left(g_j^p\right)(\lambda,m)\right]^2\right)d\gamma_\alpha(\lambda,m)\\
&=\int_{\widehat{\mathbb{K}}}\mathcal{F}_{LB}(f)(\lambda,m)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m)\\
&=f(x,t).
\end{aligned}$$

The result is then proved. □

3. Scale discrete scaling function

In the following section, we define and study the scale discrete scaling function associated to the LB-wavelet transform. We give the associated inversion and Plancherel formulas. As the result given in the previous section, we have the following proposition.

Proposition 3.1. *Let $\left\{g_j^P\right\}_{j\in\mathbb{Z}}$ be a LB-wavelet packet. The following points are holds:*

(i) *For all $J\in\mathbb{Z}$,*

$$\sum_{j=-\infty}^{J-1}\left(\mathcal{F}_{LB}(g_j^P)(\lambda,m)\right)^2=\frac{1}{C_g}\int_{r_J}^{\infty}\mid\mathcal{F}_{LB}(g)(a\lambda,m)\mid^2\frac{da}{a}. \tag{3.23}$$

(ii) *For all $J\in\mathbb{Z}$, there exists a unique function $G_J^P\in L^2_\alpha(\mathbb{K})$ satisfying*

$$\mathcal{F}_{LB}(G_J^P)(\lambda,m)=\left(\sum_{j=-\infty}^{J-1}\left((\mathcal{F}_{LB}(g_j^P)(\lambda,m)\right)^2\right)^{\frac{1}{2}},\qquad(\lambda,m)\in\widehat{\mathbb{K}}. \tag{3.24}$$

Proof. The proof is the same as Proposition 2.8. □

Definition 3.2. The sequence $\left\{G_J^P\right\}_{J\in\mathbb{Z}}$ is called a scale discrete scaling function.

Remark 3.3. From (3.23) and (3.24) we have $\forall J\in\mathbb{Z}$,

$$0\leq\mathcal{F}_{LB}(G_J^P)(\lambda,m)\leq 1\qquad;\qquad\lim_{J\longrightarrow\infty}\mathcal{F}_{LB}(G_J^P)(\lambda,m)=1; \tag{3.25}$$

$$\left(\mathcal{F}_{LB}(g_J^P)(\lambda,m)\right)^2=\left(\mathcal{F}_{LB}(G_{J+1}^P)(\lambda,m)\right)^2-\left(\mathcal{F}_{LB}(G_J^P)(\lambda,m)\right)^2; \tag{3.26}$$

$$\sum_{j=-\infty}^{\infty}\left[\left(\mathcal{F}_{LB}(G_{j+1}^P)(\lambda,m)\right)^2-\left(\mathcal{F}_{LB}(G_j^P)(\lambda,m)\right)^2\right]=1. \tag{3.27}$$

Theorem 3.4. *Let $\left\{G_J^P\right\}_{J\in\mathbb{Z}}$ be a discrete scaling function. Then $\forall f\in L^2_\alpha(\mathbb{K})$ we have the Plancherel formula*

$$\int_{\mathbb{K}}|f(y,s)|^2\,dm_\alpha(y,s)=\lim_{J\longrightarrow\infty}\int_1^{\infty}|\langle f,G^P_{J,a,x,t}\rangle|^2\,dm_\alpha(x,t),$$

where

$$G^P_{J,a,x,t}(y,s)=T^\alpha_{(x,t)}G_J^P(y,s),\quad(x,t)\in\mathbb{K}. \tag{3.28}$$

Proof. We have

$$\langle f,G^P_{J,a,x,t}\rangle=\int_{\mathbb{K}}f(y,s)T^\alpha_{(x,t)}G_J^P(y,s)\,dm_\alpha(y,s)=f*_\alpha G_J^P(x,t). \tag{3.29}$$

Theorem 2.12 gives then the desired result. □

Theorem 3.5. *Let $f\in L^2_\alpha(\mathbb{K})$. For all $J\in\mathbb{Z}$ we have*

$$\int_{\mathbb{K}}|f(y,s)|^2\,dm_\alpha(y,s)$$
$$=\int_{\mathbb{K}}|\langle f,G^P_{j,a,x,t}\rangle|^2\quad dm_\alpha(x,t)+\sum_{j=J}^{\infty}\int_{\mathbb{K}}|(\Psi_g^P f)(j,a,x,t))|^2\,dm_\alpha(x,t).$$

Proof. From (2.3), (3.24) and (3.29) we have

$$\int_{\mathbb{K}} |\langle f, G^P_{j,a,x,t}\rangle|^2 \ \ dm_\alpha(x,t) = \int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(f)(\lambda,m)|^2 \left(\sum_{j=-\infty}^{J-1} \left[\mathcal{F}_{LB}(g^P_j)(\lambda,m)\right]^2\right) d\gamma_\alpha(\lambda,m).$$

Now, Fubini-Tonelli's theorem implies

$$\sum_{j=J}^{\infty} \int_{\mathbb{K}} |(\Psi^P_g f)(j,a,x,t))|^2 \ dm_\alpha(x,t) = \int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(f)(\lambda,m)|^2 \left(\sum_{j=J}^{\infty} \left[\mathcal{F}_{LB}(g^P_j)(\lambda,m)\right]^2\right) d\gamma_\alpha(\lambda,m).$$

Therefore,

$$\begin{aligned} &\int_{\mathbb{K}} |\langle f, G^P_{j,a,x,t}\rangle|^2 \ \ dm_\alpha(x,t) + \sum_{j=J}^{\infty} \int_{\mathbb{K}} |(\Psi^P_g f)(j,a,x,t))|^2 \ dm_\alpha(x,t) \\ &= \int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(f)(\lambda,m)|^2 \left(\sum_{j=-\infty}^{\infty} \left[\mathcal{F}_{LB}(g^P_j)(\lambda,m)\right]^2\right) d\gamma_\alpha(\lambda,m) \\ &= \|f\|^2_{L^2_\alpha(\mathbb{K})}. \end{aligned}$$

□

Theorem 3.6. *Let $\left\{G^P_j\right\}_{j\in\mathbb{Z}}$ be a scale discrete scaling function. For all $f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})$ such that $\mathcal{F}_{LB}(f) \in L^1_\alpha(\widehat{\mathbb{K}})$, we have:*

(i) *For almost all $(x,t) \in \mathbb{K}$,*

$$f(x,t) = \lim_{J\longrightarrow\infty} \int_{\mathbb{K}} \langle f, G^P_{J,a,y,s}\rangle \, G^P_{J,a,y,s}(x,t) \ dm_\alpha(y,s). \tag{3.30}$$

(ii) *For almost all $(x,t) \in \mathbb{K}$ and all $J \in \mathbb{Z}$,*

$$f(x,t) = \int_{\mathbb{K}} \langle f, G^P_{J,a,y,s}\rangle \, G^P_{J,a,y,s}(x,t) \ dm_\alpha(y,s) + \sum_{j=J}^{\infty} \int_{\widehat{\mathbb{K}}} (\Psi^P_g f)(j,a,y,s) g^P_{j,a,y,s}(x) \ dm_\alpha(y,s).$$

Proof. (i) By (3.28) and (3.29) we obtain

$$\langle f, G^P_{j,a,y,s}\rangle \, G^P_{J,s}(x,t) = f *_\alpha G^P_J(y,s) T^\alpha_{(y,s)} G^p_j(x,t).$$

Plancherel formula (2.3) gives

$$\int_{\mathbb{K}} \langle f, G^P_{J,a,y,s}\rangle \, G^P_{J,a,y,s}(x,t) \ dm_\alpha(y,s) = \mathcal{F}_{LB}(f)(\lambda,m) \left[\mathcal{F}_{LB}(G^P_J)(\lambda,m)\right]^2 \varphi_{\lambda,m}(x,t) d\gamma_\alpha(\lambda,m).$$

The relation (3.30) can be obtained by using (3.25) and the dominated convergence theorem.

(ii) We have

$$\int_{\mathbb{K}} \langle f, G^P_{J,a,y,s}\rangle \, G^P_{J,a,y,s}(x,t)\; dm_\alpha(y,s) =$$

$$\int_{\mathbb{K}} \mathcal{F}_{LB}(f)(\lambda,m) \left(\sum_{j=-\infty}^{J-1} \left[\mathcal{F}_{LB}(G^P_J)(\lambda,m)\right]^2 \right) \varphi_{\lambda,m}(x,t)\; d\gamma_\alpha(\lambda,m)$$

Applyin remark 2.10 and (2.3), we get

$$\int_{\mathbb{K}} \langle f, G^P_{J,a,y,s}\rangle \, G^P_{J,a,y,s}(x,t)\; dm_\alpha(y,s) + \sum_{j=J}^{\infty} \int_{\widehat{\mathbb{K}}} (\Psi^P_g f)(j,a,y,s) g^P_{j,a,y,s}(x)\; dm_\alpha(y,s)$$

$$\begin{aligned}
&= \int_{\mathbb{K}} \mathcal{F}_{LB}(f)(\lambda,m) \left(\sum_{j=-\infty}^{J-1} \left[\mathcal{F}_{LB}(g^P_J)(\lambda,m)\right]^2 + \sum_{j=J}^{\infty} \left[\mathcal{F}_{LB}(g^P_J)(\lambda,m)\right]^2 \right) \\
&\times \varphi_{\lambda,m}(x,t)\; d\gamma_\alpha(\lambda,m) \\
&= \int_{\Gamma} \mathcal{F}_{LB}(f)(\lambda,m)\; \varphi_{\lambda,m}(x,t)\; d\gamma_\alpha(\lambda,m) \\
&= f(x,t).
\end{aligned}$$

□

4. Calderón's reproducing formula

In this section, we prove Calderón's formula for the LB-wavelet packets transform. Let us first prove the following proposition which we'll use the prove our main result in this section.

Proposition 4.1. *Let* $\{g^P_j\}_{j\in\mathbb{Z}}$ *be a LB-wavelet packet. For* $p, q \in \mathbb{Z}$ *with* $p < q$*, we consider the following functions* $\mathcal{M}_{p,q}$ *and* $\mathcal{N}_{p,q}$ *defined by*

$$\mathcal{M}_{p,q}(x,t) = \sum_{j=p}^{q-1} g^P_j * \overline{g^P_j}(x,t), \quad (x,t) \in \mathbb{K}.$$

For all $(\lambda, m) \in \widehat{\mathbb{K}}$:

$$\mathcal{N}_{p,q}(\lambda,m) = \frac{1}{C_g} \int_{r_q}^{r_p} |\mathcal{F}_{LB}(g)(a\lambda,m)|^2 \frac{da}{a}.$$

Then we have

(a) $\mathcal{M}_{p,q} \in L^2_\alpha(\mathbb{K})$.
(b) $\mathcal{N}_{p,q} \in L^2_\alpha(\mathbb{K}) \cap L^\infty_\alpha(\mathbb{K})$.
(c) $\mathcal{F}_{LB}(\mathcal{M}_{p,q}) = \mathcal{N}_{p,q}$.
(d) *For all* $(\lambda, m) \in \widehat{\mathbb{K}}$,

$$\lim_{\substack{p\to-\infty \\ q\to+\infty}} \mathcal{N}_{p,q}(\lambda,m) = 1. \tag{4.31}$$

Proof. We apply the Hölder's inequality (2.9), we obtain then

$$\begin{aligned}
\|\mathcal{M}_{p,q}\|_2^2 &= \int_{\mathbb{K}} |\sum_{j=p}^{q-1} g_j^P *_\alpha \overline{g_j^P}(x)|^2 \ dm_\alpha(x,t) \\
&\leq (p-q) \int_{\mathbb{K}} \sum_{j=p}^{q-1} |g_j^P *_\alpha \overline{g_j^P}|^2 \ dm_\alpha(x,t) \\
&= (p-q) \sum_{j=p}^{q-1} \left(\int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(g_j^P)(\lambda,m)|^4 d\gamma_\alpha(\lambda,m) \right) \\
&= (p-q) \|\mathcal{F}_{LB}(g_j^P)\|_{\infty,\alpha}^2 \sum_{j=p}^{q-1} \int_{\widehat{\mathbb{K}}} |\mathcal{F}_{LB}(g_j^P)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m).
\end{aligned}$$

Taking into account that $\|\mathcal{F}_{LB}(g_j^P)\|_{\infty,\alpha} \leq \|g_j^P\|_{1,\alpha}$, we obtain

$$\begin{aligned}
\|\mathcal{M}_{p,q}\|_{2,\alpha}^2 &\leq (p-q) \|g_j^P\|_1^2 \sum_{j=p}^{q-1} \|\mathcal{F}_{LB}(g_j^P)\|_{2,\alpha}^2 \\
&= (p-q) \|g_j^P\|_{1,\alpha}^2 \sum_{j=p}^{q-1} \|g_j^P\|_{2,\alpha}^2 < \infty.
\end{aligned}$$

From (2.9) and Proposition 2.8, we have

$$\begin{aligned}
\mathcal{F}_{LB}(\mathcal{M}_{p,q})(\lambda,m) &= \sum_{j=p}^{q-1} \mathcal{F}_{LB}(g_j^P * \overline{g_j^P})(\lambda,m) \\
&= \sum_{j=p}^{q-1} \mathcal{F}_{LB}(g_j^P)(\lambda,m) \overline{\mathcal{F}_{LB}(g_j^P)(\lambda,m)} \\
&= \sum_{j=p}^{q-1} (\mathcal{F}_{LB}(g_j^P)(\lambda,m))^2 \qquad (4.32) \\
&= \frac{1}{C_g} \int_{r_q}^{r_p} |\mathcal{F}_{LB}(g)(a\lambda,m)|^2 \frac{da}{a} \\
&= \mathcal{N}_{p,q}(\lambda,m),
\end{aligned}$$

The assertions (b) and (d) are immediate. □

Proposition 4.2. *Let $\{g_j^P\}_{j\in\mathbb{Z}}$ be a LB-wavelet packet and let $f \in L_\alpha^2(\mathbb{K})$ and $p, q \in \mathbb{Z}$, with $p < q$, consider*

$$f_{p,q}^P(x,t) = \sum_{j=p}^{q-1} \int_{\mathbb{K}} \Psi_g^P(f)(j,a,y,s) g_{j,a,y,s}^P(x,t) \ dm_\alpha(y,s). \qquad (4.33)$$

The function $f_{p,q}^P$ belongs to $L_\alpha^2(\mathbb{K})$ and satisfies

$$f_{p,q}^P = f *_\alpha \mathcal{M}_{p,q},$$

$$\mathcal{F}_{LB}(f_{p,q}^P) = \mathcal{F}_{LB}(f)\mathcal{N}_{p,q}.$$

Proof. First, we have

$$\int_{\mathbb{K}} \Psi_g^P(f)(j,a,y,s) g_{j,a,y,s}^P(x,t)\ dm_\alpha(j,a,y,s)$$
$$= \int_{\widehat{\mathbb{K}}} \mathcal{F}_{LB}(f)(\lambda,m)\varphi_{\lambda,m}(x,t)\left[\mathcal{F}_{LB}(g_j^P)(\lambda,m)\right]^2 d\gamma_\alpha(\lambda,m).$$

Hence,

$$f_{p,q}^P(x,t) = \int_\Gamma \mathcal{F}_{LB}(f)(\lambda,m)\left(\sum_{j=p}^{q-1}(\mathcal{F}_{LB}(g_j^P)(\lambda,m))^2\right)\varphi_{\lambda,m}(x,t)d\gamma_\alpha(\lambda,m).$$

Applying (2.4) and (4.32), we get

$$\begin{aligned} f_{p,q}^P(x,t) &= \int_{\widehat{\mathbb{K}}} \mathcal{F}_{LB}(f)(\lambda,m)\ \mathcal{F}_{LB}(\mathcal{M}_{p,q})(\lambda,m)\ \varphi_{\lambda,m}(x,t)\ d\gamma_\alpha(\lambda,m) \\ &= \mathcal{F}_{LB}^{-1}\left[\mathcal{F}_{LB}(f)\ \mathcal{F}_{LB}(\mathcal{M}_{p,q})\right](x,t). \end{aligned}$$

Therefore,

$$f_{p,q}^P(x,t) = f *_\alpha \mathcal{M}_{p,q}(x,t), \quad \forall (x,t)\in\mathbb{K}.$$

By the Proposition 4.1, $f_{p,q}^P \in L_\alpha^2(\mathbb{K})$, and

$$\mathcal{F}_{LB}(f_{p,q}^P) = \mathcal{F}_{LB}(f)\ \mathcal{N}_{p,q}. \tag{4.34}$$

□

Theorem 4.3. *The function $f_{p,q}^P$ defined by Proposition 4.2 satisfies*

$$\lim_{\substack{p\to-\infty \\ q\to+\infty}} \|f_{p,q}^P - f\|_{2,\alpha} = 0.$$

Proof. By 2.3 and (4.34), we get

$$\begin{aligned} \|f_{p,q}^P - f\|_{2,\alpha} &= \|\mathcal{F}_{LB}(f_{p,q}^P) - \mathcal{F}_{LB}(f)\| \\ &= \|\mathcal{F}_{LB}(f)(1-\mathcal{N}_{p,q})\|. \end{aligned}$$

The desired result follows then from (4.31) and the dominated convergence theorem. □

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