Methods of Functional Analysis and Topology
Vol. 31 (2025), no. 4, pp. 336–343
doi.org/10.31392/MFAT-npu26_4.2025.04

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

NEW DEFINITION OF $\mathcal{N}_F^{\alpha}$-SUMUDU CONFORMABLE TRANSFORM

BAHLOUL RACHID AND RACHAD HOUSSAME

Abstract. Using the new definition of the $\mathcal{N}_F^{\alpha}$-derivative function introduced by Juan E. Nápoles Valdés and al. (2020), we provide a new definition for the $\mathcal{N}_F^{\alpha}$-Sumudu transform, $\mathcal{N}_F^{\alpha}$-Sumudu conformable transform. Additionally, we establish several important results related to these new transforms. We also give a new definition of convolution related to this $\mathcal{N}_F^{\alpha}$-derivative and we show that it is commutative and associative.

1. Introduction

The Sumudu transform is a fundamental tool in mathematical analysis, widely used for solving differential equations, control theory, signal processing, and various areas of physics and engineering. However, in recent years, the classical Laplace transform has been extended to accommodate more complex behaviors in systems, particularly those involving memory effects, fractional-order dynamics, and nonlocal operators. One such extension is the conformable fractional Laplace transform, which has proven to be an effective generalization for analyzing systems with fractional dynamics and anomalous diffusion. The conformable fractional Sumudu transform introduces additional flexibility by incorporating a fractional parameter α and a modulating function $F(t, \alpha)$, providing a broader framework for solving problems that go beyond classical integer-order systems. This generalization retains many of the essential properties of the classical Laplace transform while extending its applicability to fractional calculus, which plays a critical role in modeling phenomena such as viscoelasticity, fluid dynamics, and complex networks. In this article, we examine a number of findings from the conformable fractional Laplace transform inthis article. Within the framework of this novel transform, the properties and theorems offered crucial tools of managing integrals, convolutions and differential equations of fractional order. By extending these classical results into the realm of conformable fractional calculus, this work provides a comprehensive overview of the capabilities of the conformable fractional Laplace transform. The results presented not only generalize

classical Sumudu theory but also provide powerful tools for researchers and engineers working with systems characterized by fractional dynamics. This study serves as a foundation for further explorations into the application of conformable fractional calculus to real-world problems where traditional integer-order models are insufficient to capture complex behaviors.

The paper is organized as follows: Section 2 provides definitions and preliminary results, including a review of the $\mathcal{N}_F^{\alpha}$-derivative and its properties. Section 3 presents the main results concerning the Sumudu transform of exponential functions and integrals. In Section 4, we presents the main results concerning the conformable Sumudu transform.

2020 *Mathematics Subject Classification.* 43A15, 44A35, 43A25, 43A50, 45D05.
Keywords. Conformable derivative, Sumudu transform.

2. Basic notions

Definition 2.1. [1] The function f is $\mathcal{N}_F^\alpha$-derivative at t if the quotient

$$\frac{f(t+\frac{h}{F(t,\alpha)})-f(t)}{h}$$

has a limit when h tends to 0. In this case, the limit is denoted

$$\mathcal{N}_F^\alpha f(t) := \lim_{h\to 0}\frac{f(t+\frac{h}{F(t,\alpha)})-f(t)}{h},$$

with $\alpha \in (0,1], F(t,\alpha)\neq 0$, for all $t\in[0,+\infty[$.

Definition 2.2. Let $0<\alpha\leq 1$ and $f:[0,+\infty[\to\mathbb{R}$.

(1) We say that f is $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty[$ if f is $\mathcal{N}_F^\alpha$-differentiable at every point of $[0,+\infty[$.
(2) We say that f is n times $\mathcal{N}_F^\alpha$-differentiable on $[0,+\infty[$ if f is continuous, $\forall j\in\{0,...n\}$ $\mathcal{N}_F^{(j\alpha)}f(t)=\mathcal{N}_F^\alpha(\mathcal{N}_F^\alpha...(\mathcal{N}_F^\alpha(f)))(t), j$ times, exist for all $t\in]0,+\infty[$ and $\mathcal{N}_F^{(j\alpha)}f(0)=\lim_{t\to 0^+}\mathcal{N}_F^{(j\alpha)}f(t)$ exists.

Theorem 2.3. [1] *Let α be in $(0,1]$ and $f,g:[0,+\infty)\to\mathbb{R}$ $\mathcal{N}_F^\alpha$-differentiable. Then for all $t>0$*

(1) $\mathcal{N}_F^\alpha(af+bg)(t)=a\mathcal{N}_F^\alpha(f)(t)+b\mathcal{N}_F^\alpha(g)(t),\ a,b\in\mathbb{R}.$
(2) $\mathcal{N}_F^\alpha(\lambda)=0,\lambda\in\mathbb{R}.$
(3) $\mathcal{N}_F^\alpha(fg)(t)=\mathcal{N}_F^\alpha(f)(t)g(t)+f(t)\mathcal{N}_F^\alpha(g)(t).$
(4) $\mathcal{N}_F^\alpha(\frac{f}{g})(t)=\frac{g(t)\mathcal{N}_F^\alpha(f)(t)-f(t)\mathcal{N}_F^\alpha(g)(t)}{g^2(t)}.$
(5) *If, in addition, f is differentiable then* $\mathcal{N}_F^\alpha(f)(t)=\frac{f'(t)}{F(t,\alpha)}.$

Example 2.4. :
Let f and $F(t,\alpha)$ be two real functions defined on $[0,+\infty[$ by $f(t)=t^3+\frac{7}{2}t+2t$ and $F(t,\alpha)=\alpha(t+2)$, where $\alpha\in(0,1]$. Then

$$\mathcal{N}_F^\alpha f(t) := \frac{3t+1}{\alpha}$$

for all $t>0$.

Definition 2.5. ($\mathcal{N}_F^\alpha$-integral)
Let f be a real function taking its values in a segment $[0,t]$ and $\alpha\in(0,1]$, then the $\mathcal{N}_F^\alpha$-integral of f on $[0,t]$, defined and denoted

$$\mathcal{I}_F^\alpha f(t)=\int_0^t F(\nu,\alpha)f(\nu)d\nu,\quad t\in[0,+\infty[.$$

Example 2.6. Let $F(t,\alpha)=\alpha t+3$. Then

$$\mathcal{I}_F^\alpha(e^{-t})=-\alpha te^{-t}-(3+\alpha)e^{-t}+3+\alpha.$$

Lemma 2.7. [1] *Let us consider $\alpha\in(0,1]$ and the continuous function $f:[0,+\infty)\to\mathbb{R}$. Then, for all $t\in(0,+\infty)$*

$$\mathcal{N}_F^\alpha(\mathcal{I}_F^\alpha(f))(t)=f(t).$$

Example 2.8. Let F be a function defined by $F(t,\alpha)=\alpha t+3$. Then by Example 2.6 and Theorem 2.3

$$\mathcal{N}_F^\alpha(\mathcal{I}_F^\alpha(e^{-t}))=\frac{[-\alpha te^{-t}-(3+\alpha)e^{-t}+3+\alpha]'}{\alpha t+3}=e^{-t}.$$

Lemma 2.9. [1] *Let us consider* $\alpha \in (0,1]$ *and the* $\mathcal{N}_F^\alpha$*-differentiable function* $f : [0,+\infty) \to \mathbb{R}$*. Then, for all* $t \in (0,+\infty)$

$$\mathcal{I}_F^\alpha(\mathcal{N}_F^\alpha f)(t) = f(t) - f(0).$$

Consider the following continuous function $F(t,\alpha)$ such that $F(t,\alpha) > 0$ for all $t > 0$ and $G_\alpha(t)$ its primitive function verifies $G_\alpha(0) = 0$ and $\lim_{t\to+\infty} G_\alpha(t) = +\infty$, where $0 < \alpha \leq 1$. For example

$$F(t,\alpha) = 8\alpha t + 2 \text{ and } G_\alpha(t) = 4\alpha t^2 + 2t.$$

Now, we will present some results and proofs of $\mathcal{N}_F^\alpha$-Sumudu transform.

3. $\mathcal{N}_F^\alpha$-Sumudu transform

Definition 3.1. Across the subsequent collection of functions:

$$A_F^\alpha = \{f(t) : \exists k, r_1, r_2, |f(t)| < ke^{\frac{|G_\alpha(t)|}{r_j}}; if\ G_\alpha(t) \in (-1)^j \times [0,+\infty), j = 1,2\}$$

The definition of the $\mathcal{N}_F^\alpha$-Sumudu transform of f is

$$\mathcal{S}_F^\alpha\{f(t)\}(u) = \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_\alpha(t)} f(t)F(t,\alpha)dt. \tag{3.1}$$

assuming the integral converges .

Theorem 3.2. *Let* $\mu, k, \alpha \in \mathbb{R}$ *and* $0 < \alpha \leq 1$*. Then we have:*

(i) $\mathcal{S}_F^\alpha[k] = k.$

(ii) $\mathcal{S}_F^\alpha\left[e^{\mu(G_\alpha(t))}\right] = \frac{1}{1-\mu u}, \quad u > \frac{1}{\mu}.$

(iii) $\mathcal{S}_F^\alpha[\sin(\mu(G_\alpha(t)))] = \frac{\mu u}{1+\mu^2u^2}, \quad u > \frac{1}{|\mu|}.$

(iv) $\mathcal{S}_F^\alpha[\cos(\mu(G_\alpha(t)))] = \frac{1}{1+\mu^2u^2}, \quad u > \frac{1}{|\mu|}.$

Proof. Follows by applying Definition 3.1 and integrating by parts. □

Theorem 3.3. *Consider the* $\mathcal{N}_F^\alpha$*-differentiable function* f *and* $\alpha \in (0,1]$*. Then*

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha(f(t)))(u) = \frac{\mathcal{S}_F^\alpha(f(t))(u)}{u} - \frac{f(0)}{u}, u > 0.$$

Proof. Let $u > 0$.

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha f(t))(u) = \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_\alpha(t)} f'(t)dt.$$

Through part-by-part integration, we have:

$$\int_0^\infty e^{-\frac{1}{u}G_\alpha(t)} f'(t)dt = -f(0) + \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_\alpha(t)} f(t)F(t,\alpha)dt$$

then

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha(f(t)))(u) = \frac{\mathcal{S}_F^\alpha(f(t))(u)}{u} - \frac{f(0)}{u}.$$

□

Example 3.4. Consider $F(t,\alpha) = \alpha, G_\alpha(t) = \alpha t$ and $f(t) = t^2$. We have

$$\mathcal{S}_F^\alpha(t^2)(u) = \frac{2}{\alpha}u^2$$

and

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha(t^2)(u) = \frac{2}{\alpha}u$$

thus

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha(f(t)))(u) = \frac{\mathcal{S}_F^\alpha(f(t))(u)}{u} - \frac{f(t)}{u}.$$

Theorem 3.5. *Consider the continuous function f defined on $[0, +\infty)$ and $0 < \alpha \leq 1$.*

$$\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha(f(t)))(u) = u\ \mathcal{S}_F^\alpha(f(t))(u), u > 0.$$

Proof. Let $u > 0$. By Theorem 2.7

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha \mathcal{I}_F^\alpha(f(t))(u) = \mathcal{S}_F^\alpha(f(t))(u)$$

and by Theorem 3.3

$$\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha \mathcal{I}_F^\alpha(f(t)))(u) = \frac{\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u)}{u} - \frac{\mathcal{I}_F^\alpha f(0)}{u} = \frac{\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u)}{u}$$

thus

$$\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha(f(t)))(u) = u\ \mathcal{S}_F^\alpha(f(t))(u).$$

□

Theorem 3.6. *Consider two differentiable functions $G_1^\alpha(t)$ and $G_2^\alpha(t)$ such that $G_1'^\alpha(t) = F_1(t, \alpha)$ and $G_2'^\alpha(t) = F_2(t, \alpha)$. Then*

$$\mathcal{S}_{F_1+F_2}^\alpha(f(t))(s) = \mathcal{S}_{F_1}^\alpha(e^{-sG_2^\alpha(t)} f(t))(s) + \mathcal{S}_{F_2}^\alpha(e^{-sG_1^\alpha(t)} f(t))(s).$$

where $\alpha \in (0, 1]$.

Proof. We have

$$\begin{aligned}
\mathcal{S}_{F_1+F_2}^\alpha(f(t))(u) &= \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}(G_1^\alpha(t)+G_2^\alpha(t))} f(t)(F_1(t,\alpha) + F_2(t,\alpha))dt \\
&= \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_1^\alpha(t)}(e^{-\frac{1}{u}G_2^\alpha(t)} f(t))F_1(t,\alpha)dt + \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_2^\alpha(t)}(e^{-\frac{1}{u}G_1^\alpha(t)} f(t))F_2(t,\alpha)dt \\
&= \mathcal{S}_{F_1}^\alpha(e^{-\frac{1}{u}G_2^\alpha(t)} f(t))(u) + \mathcal{S}_{F_2}^\alpha(e^{-\frac{1}{u}G_1^\alpha(t)} f(t))(u).
\end{aligned}$$

□

Theorem 3.7. *Let $n \geq 2$ and $i \in \{1, 2, ..., n\}$. Asume that $G_i^\alpha(t)$ derivables functions, we have*

$$\mathcal{S}_{(\sum_{i=1}^n F_i)}^\alpha(f(t))(s) = \sum_{i=1}^n \mathcal{S}_{F_i}^\alpha(e^{-s[\sum_{j=1,j\neq i}^n G_j^\alpha(t)]} f(t))(s)$$

where $\alpha \in (0, 1]$ and $G_i'^\alpha(t) = F_i(t, \alpha)$.

Proof. demonstration by recurrence.
For $n = 2$, see Theorem 3.6.
suppose that

$$\mathcal{S}_{(\sum_{i=1}^n F_i)}^\alpha(f(t))(u) = \sum_{i=1}^n \mathcal{S}_{F_i}^\alpha(e^{-\frac{1}{u}[\sum_{j=1,j\neq i}^n G_j^\alpha(t)]} f(t))(u),$$

and prove that

$$\mathcal{S}_{(\sum_{i=1}^{n+1} F_i)}^\alpha(f(t))(u) = \sum_{i=1}^{n+1} \mathcal{S}_{F_i}^\alpha(e^{-\frac{1}{u}[\sum_{j=1,j\neq i}^{n+1} G_j^\alpha(t)]} f(t))(u).$$

We have, by Theorem 3.6 and recurrence hypotheses

$$
\begin{aligned}
&\mathcal{S}^{\alpha}_{(\sum_{i=1}^{n+1} F_i)}(f(t))(u) = \mathcal{S}^{\alpha}_{(F_{n+1}+\sum_{i=1}^{n} F_i)}(f(t))(u)\\
&= \mathcal{S}^{\alpha}_{F_{n+1}}(e^{-\frac{1}{u}[\sum_{j=1}^{n} G_j^{\alpha}(t)]}f(t))(u) + \mathcal{S}^{\alpha}_{(\sum_{i=1}^{n} F_i)}(e^{-\frac{1}{u}G^{\alpha}_{n+1}(t)}f(t))(u)\\
&= \mathcal{S}^{\alpha}_{F_{n+1}}(e^{-\frac{1}{u}[\sum_{j=1,j\neq n+1}^{n+1} G_j^{\alpha}(t)]}f(t))(u) + \sum_{i=1}^{n}\mathcal{S}^{\alpha}_{F_i}(e^{-\frac{1}{u}[\sum_{j=1,j\neq i}^{n} G_j^{\alpha}(t)]}(e^{-\frac{1}{u}G^{\alpha}_{n+1}(t)}f(t))(u)\\
&= \mathcal{S}^{\alpha}_{F_{n+1}}(e^{-\frac{1}{u}[\sum_{j=1,j\neq n+1}^{n+1} G_j^{\alpha}(t)]}f(t))(u) + \sum_{i=1}^{n}\mathcal{S}^{\alpha}_{F_i}(e^{-\frac{1}{u}[\sum_{j=1,j\neq i}^{n+1} G_j^{\alpha}(t)]}f(t)(u)\\
&= \sum_{i=1}^{n+1}\mathcal{S}^{\alpha}_{F_i}(e^{-\frac{1}{u}[\sum_{j=1,j\neq i}^{n+1} G_j^{\alpha}(t)]}f(t))(u).
\end{aligned}
$$

□

Theorem 3.8. *The $\mathcal{N}_F^{\alpha}$-Sumudu transform changes the $\mathcal{N}_F^{\alpha}$-convolution product into a product:*

$$\mathcal{S}^{\alpha}_F[(f *^{\alpha}_F g)(t)](u) = u\mathcal{S}^{\alpha}_F(f(t))(u).\mathcal{S}^{\alpha}_F(g(t))(u).$$

Proof. This is easily seen by operating the following change of variable: $v = G_{\alpha}^{-1}(G_{\alpha}(t) - G_{\alpha}(s))$,

$$
\begin{aligned}
(f *^{\alpha}_F g)(t) &= \int_0^t f(s)g[G_{\alpha}^{-1}(G_{\alpha}(t) - G_{\alpha}(s))]F(s,\alpha)ds\\
&= \int_0^t f[G_{\alpha}^{-1}(G_{\alpha}(t) - G_{\alpha}(v))]g(v)F(v,\alpha)dv\\
&= (g *^{\alpha}_F f)(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\mathcal{S}^{\alpha}_F[(f *^{\alpha}_F g)(t)](u)\\
&= \frac{1}{u}\int_0^{+\infty} e^{-\frac{1}{u}G_{\alpha}(t)}\left(\int_0^t f(s)g[G_{\alpha}^{-1}(G_{\alpha}(t) - G_{\alpha}(s))]F(s,\alpha)ds\right)F(t,\alpha)dt\\
&= \frac{1}{u}\int_0^{+\infty} f(s)\left(\int_s^{+\infty} e^{-\frac{1}{u}G_{\alpha}(t)}g[G_{\alpha}^{-1}(G_{\alpha}(t) - G_{\alpha}(s))]F(t,\alpha)dt\right)F(s,\alpha)ds\\
&= \frac{1}{u}\int_0^{+\infty} f(s)\left(\int_0^{+\infty} e^{-\frac{1}{u}[G_{\alpha}(v)+G_{\alpha}(s)]}g(v)F(v,\alpha)dv\right)F(s,\alpha)ds\\
&= \left(\frac{1}{u}\int_0^{+\infty} e^{-\frac{1}{u}G_{\alpha}(v)}g(v)F(v,\alpha)dv\right)\left(\int_0^{+\infty} e^{-\frac{1}{s}G_{\alpha}(s)}g(s)F(s,\alpha)ds\right)\\
&= u\mathcal{S}^{\alpha}_F(f(t))(u).\mathcal{S}^{\alpha}_F(g(t))(u).
\end{aligned}
$$

□

The $\mathcal{N}_F^{\alpha}$-Sumudu Conformable transform will now be defined and some findings will be shown.

4. $\mathcal{N}_F^{\alpha}$-Sumudu Conformable transform

Definition 4.1. Across the subsequent collection of functions:

$$_cA^{\alpha}_F = \{f(t) : \exists k, r_1, r_2, |f(t)| < ke^{\frac{|G_{\alpha}(\frac{t^{\alpha}}{\alpha})|}{r_j}}; if\ G_{\alpha}(\frac{t^{\alpha}}{\alpha}) \in (-1)^j \times [0, +\infty), j = 1, 2\}$$

The definition of the $\mathcal{N}_F^\alpha$-Sumudu transform of f is

$$ {}_c\mathcal{S}_F^\alpha\{f(t)\}(u) = \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_\alpha(\frac{t^\alpha}{\alpha})} f(t)F(\frac{t^\alpha}{\alpha},\alpha)t^{\alpha-1}dt. \tag{4.2}$$

Theorem 4.2. *Consider the function f defined on $[0,+\infty)$. Then*

$$ {}_c\mathcal{S}_F^\alpha\{f(t)\}(s) = \mathcal{S}_F^\alpha\{f((\alpha t)^{\frac{1}{\alpha}})\}(s).$$

Proof. By using Definition 4.1 and letting $v = \frac{t^\alpha}{\alpha}$, we have:

$$ {}_c\mathcal{S}_F^\alpha\{f(t)\}(s) = \frac{1}{u}\int_0^\infty e^{-\frac{1}{u}G_\alpha(v)} f((\alpha v)^{\frac{1}{\alpha}})F(v,\alpha)dv$$

then, by Definition of $\mathcal{N}_F^\alpha$-Sumudu transform of f

$$ {}_c\mathcal{S}_F^\alpha\{f(t)\}(s) = \mathcal{S}_F^\alpha\{f((\alpha t)^{\frac{1}{\alpha}})\}(s).$$

□

Theorem 4.3. *Consider the $\mathcal{N}_F^\alpha$-differentiable function f defined and $\alpha \in (0,1]$. Then*

$$ {}_c\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha f(t))(u) = \frac{{}_c\mathcal{S}_F^\alpha(f(t))(u)}{u} - \frac{f(0)}{u}, s>0.$$

Proof. Let's apply Theorems 4.2 and 3.3,

$$\begin{aligned} {}_c\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha f(x))(u) &= \mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha f((\alpha t)^{\frac{1}{\alpha}}))(u) \\ &= \frac{\mathcal{S}_F^\alpha(f((\alpha t)^{\frac{1}{\alpha}}))(u)}{u} - \frac{f(0)}{u} \\ &= \frac{{}_c\mathcal{S}_F^\alpha(f(t))(u)}{u} - \frac{f(0)}{u}. \end{aligned}$$

□

Theorem 4.4. *Let $f:[0,\infty)\to\mathbb{R}$ be a given continuous function and $0<\alpha\le 1$. Then*

$$ {}_c\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u) = u\ {}_c\mathcal{S}_F^\alpha(f(t))(u), s>0.$$

Proof. According to the previous Theorem 4.3

$$ {}_c\mathcal{S}_F^\alpha(\mathcal{N}_F^\alpha\mathcal{I}_F^\alpha f(x))(u) = \frac{{}_c\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u)}{u} - \frac{\mathcal{I}_F^\alpha f(0)}{u} = \frac{{}_c\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u)}{u}$$

and by Lemma 2.7

$$ \frac{{}_c\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u)}{u} = {}_c\mathcal{S}_F^\alpha(f(t))(u)$$

then

$$ {}_c\mathcal{S}_F^\alpha(\mathcal{I}_F^\alpha f(t))(u) = u\ {}_c\mathcal{S}_F^\alpha(f(t))(u).$$

□

Theorem 4.5. *Consider two differentiable functions $G_1^\alpha(t)$ and $G_2^\alpha(t)$ such that $G_1'^\alpha(t) = F_1(t,\alpha)$ and $G_2'^\alpha(t) = F_2(t,\alpha)$. Then*

$$ {}_c\mathcal{S}_{F_1+F_2}^\alpha(f(t))(s) = {}_c\mathcal{S}_{F_1}^\alpha(e^{-sG_2^\alpha(t)}f(t))(s) + {}_c\mathcal{S}_{F_2}^\alpha(e^{-sG_1^\alpha(t)}f(t))(s)$$

where $\alpha\in(0,1]$.

Proof. By direct application of Theorem 3.6 and Theorem 4.2. □

Theorem 4.6. *Let $n\ge 2$ and $i\in\{1,2,...,n\}$. Asume that $G_i^\alpha(t)$ derivables functions, we have*

$$ {}_c\mathcal{S}_{(\sum_{i=1}^n F_i)}^\alpha(f(t))(s) = \sum_{i=1}^n {}_c\mathcal{S}_{F_i}^\alpha(e^{-s[\sum_{j=1,j\neq i}^n G_j^\alpha(t)]}f(t))(s).$$

where $\alpha\in(0,1]$ and $G_i'^\alpha(t) = F_i(t,\alpha)$.

Proof. To prove this Theorem, we can use Theorem 4.2 and Theorem 3.7. ☐

Theorem 4.7. *The* ${}_c\mathcal{N}_F^{\alpha}$*-Sumudu transform changes the* $\mathcal{N}_F^{\alpha}$*-convolution product into a product:*

$$ {}_c\mathcal{S}_F^{\alpha}((f *_F^{\alpha} g)(t))(u) = u \ {}_c\mathcal{S}_F^{\alpha}(f(t))(u). \ {}_c\mathcal{S}_F^{\alpha}(g(t))(u). $$

Proof. By direct application of Theorems 4.2 and Theorem 3.8. ☐

5. Conclusion

The definition of $\mathcal{N}_F^{\alpha}$-derivative introduced by Juan E. Nápoles Valdés et al [1] has been investigated. Many results and examples related to this definition have been given and proved. A new definition of the conformable $\mathcal{N}_F^{\alpha}$-Sumudu transform has been given. A relationship between the conformable $\mathcal{N}_F^{\alpha}$-Sumudu transform and the classical $\mathcal{N}_F^{\alpha}$-Sumudu transform have been established. Many results relating to the classical $\mathcal{N}_F^{\alpha}$-Sumudu transform case have been obtained and demonstrated in the conformable $\mathcal{N}_F^{\alpha}$-Sumudu case. Our interest for future work is to apply this results to solve some conformable partial differential equations.

References

[1] Juan E. Nápoles Valdés, Paulo M. Guzmán, Luciano M. Lugo, Artion Kashuri, the local generalized derivative and Mittag-Leffer function, *Sigma J Eng* & Nat Sci **38** (2), (2020), 1007-1017.

[2] Riesz M. L'integrale de Riemann-Liouville et le probleme de Cauchy. *Acta Mathematica*. (1949); **81**(1):1-222. https://doi.org/10.1007/BF02395016.

[3] T.Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.* **279** (2015) 57-66. https://doi.org/10.1016/j.cam.2014.10.016.

[4] Y. Ding, M, J. Wang; Conformable linear and nonlinear non-instantaneous impulsive differential equations, *Electronic Journal Differential Equations*, (2020), 1–19. https://doi.org/10.58997/ejde.2021.94.

[5] V.Stojiljkovic, A New Conformable Fractional Derivative and Applications. *Sel.Mat.* (2022), **9**, 370 - 380. https://doi.org/10.17268/sel.mat.2022.02.12.

[6] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.* **264** (2014) 65–70. https://doi.org/10.1016/j.cam.2014.01.002.

[7] Z. Al-Zhour, F. Alrawajeh, N. Al-Mutairi, R. Alkhasawneh, New results on the conformable fractional Sumudu transform: Theories and applications, *Int. J. Anal. Appl.*, **17** (6) (2019) 1019-1033. https://doi.org/10.28924/2291-8639-17-2019-1019.

[8] M. Ayata, O. Ozkan; A new application of conformable Laplace decomposition method for fractional Newell-Whitehead-Segel equation, *AIMS Mathematics*, **5** (2020),7402–7412. https://doi.org/10.3934/math.2020474.

[9] Bahloul Rachid, Rechdaoui My Soufiane, Thabet Abdeljawad, Bahaaeldin Abdalla, Some Results of Conformable Fourier Transform. *European Journal of Pure and Applied Mathematics*, Vol. **17**, No. 4, (2024), 2405-2430. https://doi.org/10.29020/nybg.ejpam.v17i4.5411.

[10] R. Bahloul and M. Sbabheh, Some results of the new definition of $\mathcal{N}_F^{\alpha}$-Fourier transform and their applications. Gulf Journal of Mathematics Vol 19, Issue 2 (2025) 228–246. https://doi.org/10.56947/gjom.v19i2.2647.

[11] R. Bahloul, R. Houssame and A. Thabet, New Definition of $\mathcal{N}_F^{\alpha}$-Laplace Conformable Transform and Their Applications. Journal of Computational Analysis and Applications Vol. 34, No. 4 (2025), 730–746.

[12] M. Bouziani, R. Houssame and R. Bahloul, Conformable ARA Transform Function and its Properties. Asia Pac. J.Math. 2025 12:58. https://doi.org/0.28924/APJM/12-58.

[13] M. Bouziani and R. Bahloul, Higher-order conformable derivatives and their applications. *Results in Nonlinear Analysis* 8 (2025) No. 2, 253–270 https://doi.org/10.31838/rna/2025.08.02.020.

[14] Juan E. Nápoles Valdés, Paulo M. Guzmán, Luciano M. Lugo, Some New Results on Non conformable Fractional Calculus, *Advances in Dynamical Systems and Applications*. ISSN 0973-5321, Volume **13**, Number 2, pp. 167–175 (2018).

[15] Vuk Stojiljkovic, A new conformable fractional derivative and applications, *Selecciones Matem áticas*. (2022); **9**(2):370-380. https://doi.org/10.17268/sel.mat.2022.02.12.

[16] D.R. Anderson, D.J. Ulness, Properties of the Katugampola fractional derivative with potential application in quantum mechanics, *J. Math. Phys.* **56** (6) (2015) 063502. https://doi.org/10.1063/1.4922018.

Bahloul Rachid: bahloulrachid363@gmail.com
Department of Mathematics, polidisciplinary faculty, Sultan Moulay Slimane University, Beni Mellal, Morocco.

Rachad Houssame: houssamer405@gmail.com
Department of Mathematics, polidisciplinary faculty, Sultan Moulay Slimane University, Beni Mellal, Morocco.

Received 17/02/2025; Revised 13/09/2025