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ON FAITHFULNESS, DP-TRANSFORMATIONS AND CANTOR SERIES EXPANSIONS

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This paper is dedicated to the memory of academician Yu.M. Berezansky

Abstract. The paper is devoted to the study of conditions for the Hausdorff-Besicovitch faithfulness of the family of cylinders generated by Cantor series expansions. We show that there exist subgeometric Cantor series expansions for which the corresponding families of cylinders are not faithful for the Hausdorff-Besicovitch dimension on the unit interval. On the other hand we found a rather wide subfamily of subgeometric Cantor series expansions generating faithful families of cylinders.

We also study conditions for the Hausdorff-Besicovitch dimension preservation on [0;1] by probability distribution functions of random variables with independent symbols of arithmetic Cantor series expansions

1. Introduction

The notion of Hausdorff-Besicovitch dimension is widely known and plays an important role in both mathematics and applied research (see, e.g., [11, 18, 26, 28, 33]). However, its calculation or even estimation is a rather non-trivial problem ([3, 6, 12, 14, 18, 27, 29]).

Various approaches and special techniques for computing the Hausdorff-Besicovitch dimension are described in detail in [18, 19, 26]. In particular, an approach based on the theory of DP-transformations was presented in [8, 9] and developed in [4, 20, 22].

Definition 1.1. A bijective function $f(x)$: $[0;1] \to [0;1]$ is called a DP-transformation on $[0;1]$ if

$$\forall E \subset [0;1]: \quad \dim_H(E) = \dim_H(f(E)).$$

Later in the work [5], an alternative approach was presented that is closely related to the theory of DP-transformations and is also based on the notion of faithful families of coverings, which significantly simplifies the calculation of the Hausdorff-Besicovitch dimension for a given set.

The study of DP-transformations is important for two main reasons [8]: if a DP-transformation maps a set E to a set E' and preserves the Hausdorff-Besicovitch dimension, then it suffices to calculate the dimension of a simpler set. Fractal geometry can be considered as the study of invariants of the group of DP-transformations of a space. So, fractal geometry can be considered as a generalization of affine geometry, as the latter discipline investigates the invariants of the affine transformation group, which forms a subgroup of the group of DP-transformations.

The paper is devoted to the study the above mentioned problems (faithfulness and DP-transformations) related to the Cantor series expansions. Let us recall ([15]) that for a given sequence $\{n_k\}$ of positive intergers $n_k \geq 2$ any real number x from the unit

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interval $[0, 1]$ can be represented in the following form:

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k(x)}{n_1 \cdot n_2 \cdot \ldots \cdot n_k}, \tag{1.1}$$

where $\alpha_k = \alpha_k(x) \in \{0, 1, ..., n_k - 1\}$. Expansion (1.1) is said to be Cantor series expansion of x. We shall also use the notation $\Delta^C_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)...}$ for such an expansion. The above $\alpha_1(x), \alpha_2(x), ..., \alpha_k(x), ..$ are said to be symbols (digits) of the Cantor series expansion of x. Closed interval

$$\Delta^C_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)} := \left[\sum_{j=1}^{k} \frac{\alpha_j(x)}{n_1 \cdot n_2 \cdot \ldots \cdot n_j}, \frac{1}{n_k} + \sum_{j=1}^{k} \frac{\alpha_j(x)}{n_1 \cdot n_2 \cdot \ldots \cdot n_j}\right]$$

is said to be the cylinder of rank k containing x. Let us remark that for the case $n_k = s, \forall k \in N$, we get the clssical s-adic expansion of x. The Cantor series expansion can also be considered as a special case of $\widetilde{Q}$-expansion([7]). A series of papers [2, 1, 17, 25, 30]) is devoted to normal properties of digits in the Cantor series expansions. In the paper [5] necessary and sufficient conditions for the family of cylinders of Cantor series expansion to be faithful for the Hausdorff-Besicovitch dimension calculation has been proven, and fractal properies of the corresponding probability measures are studied.

This work is devoted to the study of DP-properties of probability distribution function of random variables with independent symbols of Cantor arithmetic expansions and to the problem of faithfulness for subgeometric Cantor series expansions. In Section 2 we show that for the subgeometric case (i.e., if the basic sequence $\{n_k\}$ grows at most geometrically), the corresponding family of cylinders can be non-faithful. We also prove sufficient conditions for the family of cylinders generated by subgeometric Cantor expansions to be faithful. In particular, if the basic sequence $\{n_k\}$ satisfies the following condition

$$a_k \le n_k \le b_k, \quad \forall k \in N,$$

with $\{a_k\}$ being an arithmetic progression ($a_1 \ge 2$, $d \ge 1$), and $\{b_k\}$ being a geometric progression ($b_1 \ge 2$, $q \ge 1$), then the corresponding family of cylinders is faithful.

Section 3 is devoted to the study of properties of random variables with independent symbols of Cantor series expansions, i.e., random variables of the following form

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{n_1 \cdot n_2 \cdot ... \cdot n_k} =: \Delta^C_{\xi_1 \xi_2 ... \xi_k ...}$$

where ξ_k is a sequence of independent random variables taking values $0, 1, ..., n_k - 1$ with probabilities $p_{0k}, p_{1k}, ..., p_{(n_k-1)k}$ correspondingly. Specifically, it provides necessary and sufficient conditions for the probability distribution functions of random variables with independent symbols of Cantor series expansions to be in to the DP-class, under the condition that $\{n_k\}$ is bounded and probabilities p_{ik} are separated from zero.

Section 4 shows that the well-known necessary conditions for the distribution function of a random variable with independent Cantor symbols to belong to DP-transformations are not sufficient, where the sequence $\{n_k\}$ forms an arithmetic progression.

2. On some faithful families of coverings for the Hausdorff-Besicovich dimension calculation generated by Cantor series expansions

Let $E \subset [0; 1]$ and let Φ be some family of subsets from this segment.

Definition 2.1. A family Φ of subsets of $[0; 1]$ is said to be locally fine if for any $E \subset [0; 1]$ there exists an at most countable ε-covering $\{E_j\}$ of E , $E_j \in \Phi$.

Recall that the Hausdorff-Besicovitch dimension of a set $E \subset [0;1]$ with respect to Φ is the number

$$\dim_H(E, \Phi) = \inf\{\alpha : H^\alpha(E, \Phi) = 0\},$$

where $H^\alpha(E, \Phi) := \lim_{\varepsilon \to 0} H^\alpha_\varepsilon(E, \Phi) = \lim_{\varepsilon \to 0}(\inf_{|E_k| \le \varepsilon}(\sum_k |E_k|^\alpha),$

where infimum is taken over all possible ε-covering of E by subsets E_k from Φ.

Definition 2.2. A locally fine covering family Φ is said to be faithful for the calculation of the Hausdorff-Besicovitch dimension on $[0;1]$, if

$$\dim_H(E) = \dim_H(E, \Phi), \quad \forall E \subset [0;1].$$

The problem on necessary and sufficient conditions for the faithfulness of certain locally fine systems of coverings has been the subject of research by many scientists (see, e.g., [10, 13, 16, 34] and references therein). In particular, an important contribution was made by A. S. Besicovitch, who first proved the faithfulness of systems of cylinders of binary expansion [13] . Later, the faithfulness of various covering systems was studied by: Patrick Billingsley (for families of s-adic cylinders [14]); Mykola Pratsiovytyi (for families of Q-cylinders [34]); S. Albeverio, M. Pratsiovytyi, G. Torbin, M. Ibrahim, V. Vasylenko (for families of cylinders of Q^*-expansion [23, 35]); S. Albeverio, Y. Kondratiev, R. Nikiforov, O. Smiyan, G. Torbin (for families of cylinders of Q_∞-expansion [6]); G. Torbin, V. Vasylenko (for families of cylinders of $\tilde{Q}$-expansion [36]). Necessary and sufficient conditions for the family of Cantor series cylinders to be faithful were found in [5].

Theorem 2.3. *The family $\Phi(C)$ of cylinders of the Cantor series expansion is faithful of the Hausdorff-Besicovitch dimension calculation on $[0;1]$ if and only if the following condition holds:*

$$\lim_{k \to \infty} \frac{\ln(n_k)}{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})} = 0$$

By using this result it is easy to produce examples of faithful as well as non-faithful families $\Phi(C)$ of Cantor series expansion cylinders:

a) if $n_k = 2^{2^k}$, then the family $\Phi(C)$ is non-faithful;

b) if $\{n_k\}$ is bounded, then the family $\Phi(C)$ is faithful.

During several years the following conjecture was dominated: if the sequence $\{n_k\}$ is subgeometric (i.e., there exists a positive integer q such that $n_k \le q^k, \forall k \in N$), then the family $\Phi(C)$ is faithful.

Unfortunately this conjecture fails to be true. The simplest counterexample can be produced as follows: let

$$n_k = \begin{cases} 2, \text{ for } k \neq 10^s; \\ 10^k, \text{ for } k = 10^s, \ s \in N. \end{cases}$$

In such a case $\{n_k\}$ is subgeometric ($n_k \le 10^k$), but the limit $\lim_{k \to \infty} \frac{\ln(n_k)}{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})}$ does not equal 0.

The following theorem gives sufficient conditions for subgeometric families of Cantor series cylinders to be faithful.

Theorem 2.4. *Let the basic sequence $\{n_k\}$ satisfies the following condition:*

$$a_k \le n_k \le b_k, \quad \forall k \in N,$$

where $\{a_k\}$ forms an arithmetic progression with $a_1 \ge 2, d \ge 1$,

and $\{b_k\}$ forms a geometric progression with $b_1 \ge 2, q \ge 1$,

then the family $\Phi(C)$ of Cantor series cylinders is faithful for the Hausdorff-Besicovitch dimension calculation on $[0;1]$.

Proof. Consider the expression

$$\frac{\ln(n_k)}{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})} \leq \frac{\ln(b_k)}{\ln(a_1 \cdot a_2 \cdot \ldots \cdot a_{k-1})} =$$

$$= \frac{\ln(b_1 q^{k-1})}{\ln(a_1 \cdot (a_1+d) \cdot \ldots \cdot (a_1+(k-2)d)} \leq \frac{\ln(b_1)+(k-1)\ln(q)}{\ln(2 \cdot (2+1) \cdot \ldots \cdot (k-2))} =$$

$$= \frac{\ln(b_1)+(k-1)\ln(q)}{\ln(k-2)!} \leq \frac{\ln(b_1)+(k-1)\ln(q)}{\ln(\sqrt{2\pi(k-2)} \cdot (\frac{k-2}{e})^{k-2} \cdot e^{\theta_{k-2}})},$$

where $|\theta_{k-2}| \leq \frac{1}{12(k-2)}$.

So,

$$\frac{\ln(n_k)}{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})} \leq \frac{\ln(b_1)+(k-1)\ln(q)}{\ln(\sqrt{2\pi(k-2)}+(k-2)\ln(\frac{k-2}{e})+\theta_{k-2})} \to 0 \ (\text{as } k \to \infty)$$

Therefore,

$$\lim_{k\to\infty} \frac{\ln(n_k)}{\ln(n_1 \cdot n_2 \cdot \ldots \cdot n_{k-1})} = 0$$

Taking into account results from [5], we get the faithfulness of $\Phi(C)$. □

Corollary 2.5. *If $\{n_k\}$ is strictly increasing and subgeometric (i.e., there exists a constant q such that $n_k \leq q^k, \forall k \in N$), then $\Phi(C)$ is faithful for the Hausdorff-Besicovitch dimension calculation on* $[0;1]$.

Proof. Since $n_1 \geq 2$ and $\{n_k\}$ is increasing, we have $n_k \geq k+1$. Therefore,

$$a_k \leq n_k \leq b_k, \quad \forall k \in N,$$

where $a_k = k+1$, $b_k = q^k$.

The faithfulness of Φ follows from the previous theorem. □

3. DP-transformations generated by Cantor series expansions

Despite the fact that Cantor series expansions are natural generalizations of s-adic representations, the vast majority of problems that are completely solved for s-adic representations are still very far from being solved for Cantor series expansions. In particular, the problem of finding necessary and sufficient conditions for the distribution function F_ξ to belong to the DP-class, i.e. to preserve the Hausdorff-Besicovitch dimension of an arbitrary subset on $[0;1]$. Important steps in the study of this problem have been made in the works of M. V. Pratsiovytyi, G. M. Torbin and their students.

Before presenting the new results of our study, let us recall the following definitions.

Definition 3.1. Let $\{\xi_k\}$ be a sequence of independent random variables taking values $0, 1, \ldots, n_k - 1$ with probabilities $p_{0k}, p_{1k}, \ldots, p_{(n_k-1)k}$ correspondingly. The random variable

$$\xi = \sum_{k=1}^{\infty} \frac{\xi_k}{n_1 \cdot n_2 \cdot \ldots \cdot n_k} =: \Delta^C_{\xi_1\xi_2\ldots\xi_k\ldots}$$

is said to be the random variable with independent symbols of Cantor series expansion.

Definition 3.2. A number

$$\dim_H \mu_\psi = \inf_{\mu_\psi(E)=1} \{\dim_H E\}$$

is said to be the Hausdorff dimension of the measure μ_ψ.

Definition 3.3. The spectrum of a random variable ψ is the set

$$S_\psi := \{x : F_\psi(x+\varepsilon) - F_\psi(x-\varepsilon) > 0, \forall \varepsilon > 0\}$$

i.e. , S_ψ is the minimal closed support of the measure μ_ψ.

Properties of Cantor expansions and properties of random variable ξ were studied by M. Pratsiovytyi, G. Torbin, M. Lebid, B. Mance, R. Nikiforov and other authors.

A fundamentally important breakthrough in the development of the metric and dimensional theory of Cantor series expansion was made in [5], where, in addition to the criterion for the faithfulness of the system of cylinders of the Cantor series expansions for the calculating of the Hausdorff-Besicovitch dimension on $[0;1]$, authors also proved formulae for the calculating the Hausdorff dimension of the measure μ_ξ under the following restriction:

$$\sum_{k=1}^{\infty}\left(\frac{\ln n_k}{\ln(n_1\cdot n_2\cdot\ldots\cdot n_{k-1})}\right)^2<+\infty,$$

where $\{n_k\}$ is the basic sequence that determines the Cantor series expansion.

The study of DP-properties of distribution functions F_ξ at this moment is limited only to cases when the sequence $\{n_k\}$ is bounded [24]. In particular, the following fact has been proven.

Theorem 3.4. *If $\{n_k\}$ is bounded and there exists a constant $p_0>0: p_{ik}\geq p_0$, then the probability distribution function of random variable ξ with independent symbols of Cantor series expansion is DP-transformation if and only if*

$$\dim_H\mu_\xi=1.$$

4. Counterexample related to DP-transformations generated by arithmetic Cantor series expansions

If the sequence $\{n_k\}$ is unbounded (for example, when $\{n_k\}$ forms an arithmetic progression), the condition of separation of p_{ik} from zero is impossible to fulfill since $\min_i p_{ik}\leq\frac{1}{n_k}$ and if $\{n_k\}$ is unbounded, the sequence $\{\frac{1}{n_k}\}$ has a subsequence tending to 0. Therefore, the previous theorem cannot be applied to the class of unbounded sequences $\{n_k\}$.

At the same time, we note that the general necessary conditions for F_ξ to belong to the DP-class are the following ones:

1)$p_{ik}>0,\quad\forall i\in\{0,...,n_k-1\}$;

2) $\dim_H\mu_\xi=1$.

Let us construct a counterexample that demonstrates that for arithmetic Cantor series expansions, even the simultaneous fulfillment of the above two conditions is not sufficient for F_ξ to belong to the DP-class.

Example 4.1. Let $n_k=k+1$ and random variable ξ:

$$\xi=\sum_{k=1}^{\infty}\frac{\xi_k}{(k+1)!},$$

where the random variable ξ_k takes values $0,1,...,k$ with probabilities $\frac{1}{k+1}$ for any $k\in A:=\{n:n\neq 10^s, s\in N\}$; and for any $k\in\bar{A}$ the random varialbe ξ_k takes value 0 with probability $\frac{1}{10^{10^{10^k}}}$, and takes values $1,2,...,k$ with probabilities $\frac{1-\frac{1}{10^{10^{10^k}}}}{k}$.

Lets us check whether $\dim_H\mu_\xi=1$.

Since

$$\sum_{k=1}^{\infty}\left(\frac{\ln n_k}{\ln(n_1\cdot n_2\cdot\ldots\cdot n_{k-1})}\right)^2=\sum_{k=1}^{\infty}\left(\frac{\ln(k+1)}{\ln(k!)}\right)^2<+\infty,$$

we can calculate the Hausdorff dimension of the measure by the following formula [5]:

$$\dim_H\mu_\xi=\varliminf_{k\to\infty}\frac{h_1+h_2+\ldots+h_k}{\ln(n_1\cdot n_2\cdot\ldots\cdot n_k)},$$

where h_k is the entropy of the random variable ξ_k, i.e. $h_k = -\sum\limits_{i=0}^{n_k-1} p_{ik}\ln p_{ik}$

If $k \in A$, then $h_k = \ln(k+1)$.

If $k \in \bar{A} = B$, then $h_k = -\left(\frac{1}{10^{10^{10^k}}}\ln\frac{1}{10^{10^{10^k}}} + k\cdot\frac{1-\frac{1}{10^{10^{10^k}}}}{k}\ln\frac{1-\frac{1}{10^{10^{10^k}}}}{k}\right) \sim \ln k$

because $\lim\limits_{x\to 0+} x\ln x = 0$.

Therefore,

$$\dim_H \mu_\xi = \lim_{k\to\infty}\frac{h_1+h_2+...+h_k}{\ln(n_1\cdot n_2\cdot...\cdot n_k)} =$$

$$= \lim_{k\to\infty}\frac{\ln(2\cdot 3\cdot 4\cdot...\cdot 9\cdot 10\cdot 10\cdot 12\cdot...\cdot 100\cdot 100\cdot 102\cdot...\cdot 10^k\cdot 10^k)}{\ln(2\cdot 3\cdot 4\cdot...\cdot 9\cdot 10\cdot 11\cdot 12\cdot...\cdot 100\cdot 101\cdot 102\cdot...\cdot 10^k\cdot(10^k+1))} =$$

$$= \lim_{k\to\infty}\frac{\ln(2\cdot 3\cdot 4\cdot...\cdot 10^k\cdot(10^k+1)) - \ln(\frac{11}{10}\cdot\frac{101}{100}\cdot...\cdot\frac{10^k+1}{10^k})}{\ln(2\cdot 3\cdot 4\cdot...\cdot 10^k\cdot(10^k+1))} =$$

$$= \lim_{k\to\infty} 1 - \frac{\ln((1+\frac{1}{10})\cdot(1+\frac{1}{10^2})\cdot...\cdot(1+\frac{1}{10^k}))}{\ln(2\cdot 3\cdot 4\cdot...\cdot 10^k\cdot(10^k+1))} = 1,$$

because $\prod\limits_{k=1}^{\infty}(1+\frac{1}{10^k})$ is convergent.

Hence, $\dim_H \mu_\xi = 1$.

Consider the set V:

$$V = \{x : x = \Delta^C_{\alpha_1...\alpha_9 0\alpha_{11}...\alpha_{99}0\alpha_{101}...}\} =$$

$$= \{x : x = \Delta^C_{\alpha_1\alpha_2...\alpha_k...}, \alpha_i \in \{0,1,...,i\} \quad i \in A, \quad \alpha_i = 0 \quad i \in B\}$$

Let us show that $\dim_H(V) \neq \dim_H(F_\xi(V))$.

2. Let's calculate $\dim_H(V)$.

Consider the random variable $\psi = \Delta^C_{\psi_1\psi_2...\psi_k...}$, where the random variables ψ_k are independent with the following distributions

ψ_k	0	1	...	k
	$\frac{1}{k+1}$	$\frac{1}{k+1}$	...	$\frac{1}{k+1}$

$\forall k \in A := \{n : n \neq 10^s, s \in N\}$

ψ_k	0	1	...	k
	1	0	...	0

$\forall k \in \bar{A} = B$

It is easy to see that the set V is the spectrum of the random variable ψ.

If

$$\sum_{k=1}^{\infty}\left(\frac{\ln n_k}{\ln(n_1\cdot n_2\cdot...\cdot n_{k-1})}\right)^2 = \sum_{k=1}^{\infty}\left(\frac{\ln(k+1)}{\ln(k!)}\right)^2 < +\infty,$$

then the Hausdorff-Besicovitch dimension of the spectrum can be calculated by the following formula [37]:

$$\dim_H S_\psi = \lim_{k\to\infty}\frac{\ln(m_1\cdot m_2\cdot...\cdot m_k)}{\ln(n_1\cdot n_2\cdot...\cdot n_k)},$$

where m_k is the number of non-zero probabilities among $p_{0k}, p_{1k}, ..., p_{(n_k-1)k}$.

$$\dim_H S_\psi = \lim_{k\to\infty}\frac{\ln(m_1\cdot m_2\cdot...\cdot m_k)}{\ln(n_1\cdot n_2\cdot...\cdot n_k)} =$$

$$= \lim_{k\to\infty}\frac{\ln(2\cdot 3\cdot...\cdot 9\cdot 10\cdot 1\cdot 12\cdot...\cdot 100\cdot 1\cdot 102\cdot...\cdot 10^k\cdot 1)}{\ln(n_1\cdot n_2\cdot...\cdot n_{10^k})} =$$

$$= \lim_{k\to\infty} \frac{\ln(2\cdot 3\cdot ...\cdot 10^k\cdot(10^k+1)) - \ln(11\cdot\ 101\cdot ...\cdot(10^k+1))}{\ln(n_1\cdot n_2\cdot ...\cdot n_{10^k})} =$$

$$= \varliminf_{k\to\infty}\left(1 - \frac{\ln(11\cdot\ 101\cdot ...\cdot(10^k+1))}{\ln(n_1\cdot n_2\cdot ...\cdot n_{10^k})}\right)$$

Since

$$\frac{\ln(11\cdot\ 101\cdot 1001\cdot ...\cdot(10^k+1))}{\ln(n_1\cdot n_2\cdot ...\cdot n_{10^k})} \leq \frac{\ln(10\cdot\ 100\cdot 1000\cdot ...\cdot 10^{k+1})}{\ln(2\cdot 3\cdot 4\cdot ...\cdot(10^k+1))} =$$

$$= \frac{\ln(10\cdot\ 10^2\cdot 10^3\cdot ...\cdot 10^{k+1})}{\ln((10^k+1)!)} =$$

$$= \frac{\frac{k(k+1)}{2}\ln 10}{\ln(\sqrt{2\pi(10^k+1)}) + (10^k+1)\ln(\frac{10^k+1}{e}) + \theta_k} \to 0 (k\to\infty),$$

because $0 < \theta_k < \frac{1}{12k}$.
Therefore,

$$= \varliminf_{k\to\infty}\left(1 - \frac{\ln(11\cdot\ 101\cdot ...\cdot(10^k+1))}{\ln(n_1\cdot n_2\cdot ...\cdot n_{10^k})}\right) = 1.$$

$$\dim_H(V) = \dim_H S_\psi = 1$$

3. Now let's calculate $\dim_H(F_\xi(V))$.
For any $x\in V$ consider the limit

$$\lim_{k\to\infty}\frac{\ln\lambda(\Delta_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)})}{\ln\mu_\xi(\Delta_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)})} = \lim_{k\to\infty}\frac{-\ln(n_1\cdot n_2\cdot ...\cdot n_k)}{\ln(p_{\alpha_1(x)1}\cdot p_{\alpha_2(x)2}\cdot ...\cdot p_{\alpha_k(x)k})}$$

Let us show that this limit exists for any $x\in V$.
Let

$$b_k(x) := \frac{\ln\lambda(\Delta_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)})}{\ln\mu_\xi(\Delta_{\alpha_1(x)\alpha_2(x)...\alpha_k(x)})} = \frac{-\ln(n_1\cdot n_2\cdot ...\cdot n_k)}{\ln(p_{\alpha_1(x)1}\cdot p_{\alpha_2(x)2}\cdot ...\cdot p_{\alpha_k(x)k})}.$$

Let's consider $\{b_k(x)\}$:

$$b_1(x) = \frac{-\ln(n_1)}{\ln(p_{\alpha_1(x)1})} = 1, \quad \forall x\in V$$

$$b_2(x) = \frac{-\ln(n_1\cdot n_2)}{\ln(p_{\alpha_1(x)1}\cdot p_{\alpha_2(x)2})} = 1, \quad \forall x\in V$$

$$\vdots$$

$$b_9(x) = \frac{-\ln(n_1\cdot ...\cdot n_9)}{\ln(p_{\alpha_1(x)1}\cdot ...\cdot p_{\alpha_9(x)9})} = 1, \quad \forall x\in V$$

$$b_{10}(x) = \frac{-\ln(n_1\cdot ...\cdot n_9\cdot n_{10})}{\ln(p_{\alpha_1(x)1}\cdot ...\cdot p_{\alpha_9(x)9}\cdot p_{\alpha_{10}(x)10})} = \frac{\ln(2\cdot ...\cdot 10\cdot 11)}{\ln(2\cdot ...\cdot 10\cdot 10^{10^{10}})} < 1, \quad \forall x\in V$$

$$b_{11}(x) = \frac{-\ln(n_1\cdot ...\cdot n_{10}\cdot n_{11})}{\ln(p_{\alpha_1(x)1}\cdot ...\cdot p_{\alpha_{10}(x)10}\cdot p_{\alpha_{11}(x)11})} = \frac{\ln(2\cdot ...\cdot 11\cdot 12)}{\ln(2\cdot ...\cdot 10^{10^{10}}\cdot 12)} < 1, \quad \forall x\in V$$

Hence

$$b_1(x) = b_2(x) = ... = b_9(x) > b_{10}(x)$$

$$b_{10}(x) < b_{11}(x) < ... < b_{99}(x) > b_{100}(x)$$

$$b_{100}(x) < b_{101}(x) < ... < b_{999}(x) > b_{1000}(x)$$

$$\vdots$$

$$b_{1000}(x) < b_{1001}(x) < ... < b_{9999}(x) > b_{10000}(x)$$

$$\vdots$$

$$b_{10^k}(x) < b_{10^k+1}(x) < ... < b_{10^{k+1}-1}(x) > b_{10^{k+1}}(x), \quad \forall k \in N$$

To show that $\lim\limits_{k\to\infty} b_k(x)$ exists and is equal to 0, we calculate:

$$1) \ \underline{\lim}_{k\to\infty} b_k(x) = \underline{\lim}_{k\to\infty} b_{10^k}(x) = \underline{\lim}_{k\to\infty} \frac{-\ln(n_1 \cdot n_2 \cdot ... \cdot n_{10^k})}{\ln\left(p_{\alpha_1(x)1} \cdot p_{\alpha_2(x)2} \cdot ... \cdot p_{\alpha_{10^k}(x)10^k}\right)}$$

Since

$$\ln\left(\frac{1}{p_{\alpha_1(x)1}} \cdot \frac{1}{p_{\alpha_2(x)2}} \cdot ... \cdot \frac{1}{p_{\alpha_{10^k}(x)10^k}}\right) =$$

$$= \ln 2 \cdot 3 \cdot ... \cdot 10 \cdot 10^{10^{10^{10^1}}} \cdot 12 \cdot ... \cdot 99 \cdot 100 \cdot 10^{10^{10^{10^2}}} \cdot ... \cdot 10^{10^{10^{10^k}}} >$$

$$> \ln 10^{10^{10^{10^k}}} = 10^{10^{10^k}} \ln 10$$

and

$$\ln(n_1 \cdot n_2 \cdot ... \cdot n_{10^k}) = \ln(10^k + 1)! < \ln(10^k)! =$$

$$= \ln\left(\sqrt{2\pi 10^k} \cdot \left(\frac{10^k}{e}\right)^{10^k} \cdot \theta_k\right) = \ln\left(\sqrt{2\pi 10^k}\right) + 10^k \ln\left(\frac{10^k}{e}\right) + \ln\theta_k,$$

where $0 < \theta_k < \frac{1}{12k}$,
we have

$$\frac{\ln(n_1 \cdot n_2 \cdot ... \cdot n_{10^k})}{\ln\left(\frac{1}{p_{\alpha_1(x)1}} \cdot \frac{1}{p_{\alpha_2(x)2}} \cdot ... \cdot \frac{1}{p_{\alpha_{10^k}(x)10^k}}\right)}$$

$$< \frac{\ln\left(\sqrt{2\pi 10^k}\right) + 10^k \ln\left(\frac{10^k}{e}\right) + \ln\theta_k}{10^{10^{10^k}} \ln 10} \to 0 \ (k \to \infty)$$

Therefore

$$\underline{\lim}_{k\to\infty} \frac{\ln(n_1 \cdot n_2 \cdot ... \cdot n_{10^k})}{\ln\left(\frac{1}{p_{\alpha_1(x)1}} \cdot \frac{1}{p_{\alpha_2(x)2}} \cdot ... \cdot \frac{1}{p_{\alpha_{10^k}(x)10^k}}\right)} = 0.$$

$$2) \ \overline{\lim}_{k\to\infty} b_k(x) = \overline{\lim}_{k\to\infty} b_{(10^k-1)}(x) = \overline{\lim}_{k\to\infty} \frac{-\ln(n_1 \cdot n_2 \cdot ... \cdot n_{(10^k-1)})}{\ln\left(p_{\alpha_1(x)1} \cdot p_{\alpha_2(x)2} \cdot ... \cdot p_{\alpha_{(10^k-1)}(x)(10^k-1)}\right)}$$

Since

$$\ln\left(\frac{1}{p_{\alpha_1(x)1}} \cdot \frac{1}{p_{\alpha_2(x)2}} \cdot ... \cdot \frac{1}{p_{\alpha_{(10^k-1)}(x)(10^k-1)}}\right) =$$

$$= \ln(2 \cdot ... \cdot 10 \cdot 10^{10^{10^{10^1}}} \cdot 12 \cdot ... \cdot 10^2 \cdot 10^{10^{10^{10^2}}} \cdot ... \cdot 10^{k-1} \cdot 10^{10^{10^{10^{k-1}}}} \cdot ... \cdot 10^k) >$$

$$> \ln 10^{10^{10^{10^{k-1}}}} = 10^{10^{10^{k-1}}} \ln 10$$

and

$$\ln\left(n_1\cdot n_2\cdot\ldots\cdot n_{(10^k-1)}\right)=\ln(10^k)!=$$

$$=\ln\left(\sqrt{2\pi 10^k}\cdot\left(\frac{10^k}{e}\right)^{10^k}\cdot\theta_k\right)=\ln\left(\sqrt{2\pi 10^k}\right)+10^k\ln\left(\frac{10^k}{e}\right)+\ln\theta_k,$$

where $0<\theta_k<\frac{1}{12k}$,

we have

$$\overline{\lim_{k\to\infty}}\frac{-\ln\left(n_1\cdot n_2\cdot\ldots\cdot n_{(10^k-1)}\right)}{\ln\left(p_{\alpha_1(x)1}\cdot p_{\alpha_2(x)2}\cdot\ldots\cdot p_{\alpha_{(10^k-1)}(x)(10^k-1)}\right)}$$

$$\le\overline{\lim_{k\to\infty}}\frac{\ln\left(\sqrt{2\pi 10^k}\right)+10^k\ln\left(\frac{10^k}{e}\right)+\ln\theta_k}{10^{10^{10^k-1}}\ln 10}=0$$

If

$$0=\underline{\lim_{k\to\infty}}b_k(x)\le\overline{\lim_{k\to\infty}}b_k(x)\le 0,$$

then

$$\lim_{k\to\infty}b_k=0.$$

According to Billingsley's theorem[14], if

$$V=\left\{x:\lim_{k\to\infty}\frac{\ln\lambda(\Delta_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)})}{\ln\mu_\xi(\Delta_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)})}=\delta\right\},$$

then

$$\dim_H(V,\mu_\xi,\Phi)=\delta\dim_H(V,\lambda,\Phi).$$

1. $\dim_H(V,\lambda,\Phi)=\inf\{\alpha:H^\alpha(V,\lambda,\Phi(C))=0\}$,

where $\Phi(C)$ – family of cylinders of the Cantor expansion.

$$H^\alpha_\varepsilon((V,\lambda,\Phi(C))=\inf_{|E_j|\le\varepsilon}\sum_j\lambda(E_j)^\alpha,\quad E_j\in\Phi(C))$$

It is not difficult to prove that, if the sequence $\{n_k\}$ forms an arithmetic progression, then $\Phi(C)$ is a faithful family for computing the Hausdorff-Besicovitch dimension, i.e.

$$\dim_H(E)=\dim_H(E,\Phi(C)),\quad\forall E\subset[0;1].$$

Therefore

$$\dim_H(V,\lambda,\Phi(C))=\dim_H(V,\lambda)=\dim_H(V)$$

2. $\dim H(V,\mu_\xi,\Phi)=\inf\{\alpha:H^\alpha(V,\mu_\xi,\Phi(C))=0\}$

$$H^\alpha_\varepsilon(V,\mu_\xi,\Phi(C))=\inf_{|V_j|\le\varepsilon}\sum_j\mu_\xi(V_j)^\alpha,\quad V_j\in\Phi(C))$$

It is easy to see that

$$\mu_\xi(\Delta_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)})=p_{\alpha_1(x)1}\cdot p_{\alpha_2(x)2}\cdot\ldots\cdot p_{\alpha_k(x)k}=$$

$$=\left|\Delta^{\tilde{P}}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)}\right|=\left|F_\xi\left(\Delta^C_{\alpha_1(x)\alpha_2(x)\ldots\alpha_k(x)}\right)\right|.$$

Since V was covered by $\{V_j\}$, we conclude that $F_\xi(V)$ can be covered by $\{F_\xi(V_j)\}$.

Therefore

$$\dim_H(V, \mu_\xi, \Phi(C)) = \dim_H(F_\xi(V), \lambda, F_\xi(\Phi(C))) = \dim_H(F_\xi(V), \Phi'),$$

where $\Phi' = F_\xi(\Phi(C))$.

According to the article by V. Vasylenko, V. Misky, G. Torbin [36] at $n_k \le n_0$ and $p_{ik} \ge p_0 > 0 : \Phi' = \Phi'(\tilde{Q})$ is faithful.

Hence, $\dim_H(V, \mu_\xi) = \dim_H(F_\xi(V))$.

So,

$$\dim_H(F_\xi(V)) = 0 \cdot \dim_H(V)$$
$$\dim_H(F_\xi(V)) = 0.$$

That is $\dim_H(V) = 1 \ne 0 = \dim_H(V)$.

This means that F_ξ is not a DP-transformation.

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