

## ERGODIC THEOREM FOR A $C_0$ -SEMIGROUPS OF UNIVERSALLY BOUNDED OPERATORS

ABDELLAH AKRYM AND ABDESLAM EL BAKKALI

**ABSTRACT.** In this paper, we study uniform ergodicity for  $C_0$ -semigroups of universally bounded operators acting on locally convex spaces. Characterizations of uniform ergodic  $C_0$ -semigroups are given. Importantly, we give a  $C_0$ -semigroups version of F. Pater, T. Bînzar [14] theorem.

### 1. INTRODUCTION

During the last decades, the ergodicity has already developed in different directions for operators acting on Banach spaces at first and lately, more generally, on locally convex spaces. In the present paper, we study the ergodic theorem for  $C_0$ -semigroups of universally bounded operators acting on locally convex spaces, for more information on this class we refer to [7, 13]. This class extends the Banach spaces one [15].

The class of universally bounded operators acting on a locally convex space has been elaborated by R. T. Moore in 1969 [12], and G. Joseph in 1977 [6], and more recently by L. D. Lemle and L. Wu [8].

In 1943, N. Dunford [5] showed that for an operator  $T$  on a complex Banach space satisfying  $\frac{1}{n}\|T^n\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $T$  is uniformly ergodic if and only if  $(I - T)^2\mathcal{X}$  is closed. In 1974, M. Lin [9] improved this characterization by proving that if  $\frac{1}{n}\|T^n\| \rightarrow 0$  as  $n \rightarrow \infty$ , then  $T$  is uniformly ergodic if and only if  $(I - T)\mathcal{X}$  is closed. Hence  $(I - T)^k\mathcal{X}$  is closed for each integer  $k \geq 1$ . In 1993, Mbekhta and Zemánek [11], still assuming  $\frac{1}{n}\|T^n\| \rightarrow 0$ , showed that  $T$  is uniformly ergodic if (and only if)  $(I - T)^k\mathcal{X}$  is closed for some integer  $k$ . Recently, in 2010, F. Pater and T. Bînzar [14] extended the same equivalent properties of the uniform ergodic theorem for universally bounded operators acting on locally convex spaces. In [10], M. Lin obtained the analogous result to [9] for  $C_0$ -semigroups on Banach spaces. In this paper, we obtain the analogous result for  $C_0$ -semigroups of universally bounded operators. More precisely, we show that, under some hypotheses, a  $C_0$ -semigroups  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  such that  $\frac{1}{t}\|T(t)\|_{\mathcal{P}} \rightarrow 0$  as  $t \rightarrow \infty$ , is uniform ergodic (i.e. the Cesàro averages  $C(t)x = \frac{1}{t} \int_0^t T(s)x ds$ ,  $t > 0$  converges uniformly in  $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$ ) if and only if its infinitesimal generator  $A$  has a closed range in  $\mathcal{X}$ , if and only if  $(T(t))_{t \geq 0}$  is Abel ergodic (i.e. the Abel averages  $\lambda R(\lambda; A)x = \lambda \int_0^{\infty} e^{-\lambda t} T(t)x dt$ ,  $\lambda > 0$ , converges uniformly in  $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$  when  $r \rightarrow \infty$  and  $\lambda \rightarrow 0^+$ ). These characterizations are shown to remain valid in the class of quojection Fréchet spaces [3].

A characterization of when a complete, barrellled space with a basis is Montel, in terms of uniform ergodicity of  $C_0$ -semigroups acting in the space, is given in [2]. Sufficient conditions for some classes of  $C_0$ -semigroups of linear operators, in a sequentially complete locally convex Hausdorff space  $\mathcal{X}$  (briefly, lchS), to be mean ergodic have been given in [1].

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## 2. PRELIMINARIES

Let  $\mathcal{X}$  be a Hausdorff locally convex vector space over the complex field  $\mathbb{C}$ . A calibration for  $\mathcal{X}$  is a family  $\mathcal{P}$  of seminorms generating its topology (in the sense that the topology of  $\mathcal{X}$  is the coarsest with respect to which all seminorms of  $\mathcal{P}$  are continuous). We denote by  $(\mathcal{X}, \mathcal{P})$  a locally convex space  $\mathcal{X}$  with a calibration  $\mathcal{P}$ .

Recall that a linear operator  $T$  on a locally convex space  $\mathcal{X}$  is quotient-bounded with respect to a calibration  $\mathcal{P}$  if for every seminorm  $p \in \mathcal{P}$  there exists some  $c_p > 0$  such that

$$p(Tx) \leq c_p p(x), \quad \forall x \in \mathcal{X}.$$

The class of quotient-bounded operators with respect to a calibration  $\mathcal{P}$ , which was introduced in [6, 12], will be denoted by  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ . For each  $p \in \mathcal{P}$  and  $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  we define

$$\hat{p}(T) = \inf\{r > 0 : p(Tx) \leq rp(x), \quad \forall x \in \mathcal{X}\}.$$

For each  $p \in \mathcal{P}$ ,  $\hat{p}$  is then a sub-multiplicative seminorm on  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  satisfying  $\hat{p}(I) = 1$ . The space  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  will be endowed with a topology  $\tau_{\hat{\mathcal{P}}}$  generated by  $\hat{\mathcal{P}} = \{\hat{p} : p \in \mathcal{P}\}$ . We note that  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  becomes a Hausdorff local multiplicative convex (l.m.c.) algebra with respect to the topology determined by  $\hat{\mathcal{P}}$ ; for more information see [8].

In [16], S. M. Stoian proved that if  $\mathcal{X}$  is a sequentially complete convex space, then  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  is a sequentially complete m-convex algebra for all calibration  $\mathcal{P}$ .

An operator  $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  is a bounded element of the algebra  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  if it is a bounded element in the sense of G. R. Allan [4], i.e some scalar multiple of it generates a bounded semigroup. By  $(\mathcal{Q}_{\mathcal{P}}(\mathcal{X}))_0$  we denote the algebra of all bounded elements in  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ . Let  $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$ , the  $\mathcal{P}$ -spectral radius of  $T$ , denoted by  $r_{\mathcal{P}}(T)$ , is defined as the boundedness radius in the sense of Allan [4]

$$r_{\mathcal{P}}(T) = \inf\{\lambda > 0 : \text{the sequence } \left( (\lambda^{-1}T)^n \right)_{n \in \mathbb{N}} \text{ is bounded in } \mathcal{Q}_{\mathcal{P}}(\mathcal{X})\}.$$

Let  $\mathcal{P}$  be a calibration on  $\mathcal{X}$ . Recall [12], that a linear operator  $T : \mathcal{X} \rightarrow \mathcal{X}$  is universally bounded on  $(\mathcal{X}, \mathcal{P})$  if there exists  $r > 0$  such that

$$p(Tx) \leq rp(x), \quad \text{for all } x \in \mathcal{X} \text{ and } p \in \mathcal{P}.$$

We will denote by  $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$  the collection of all universally bounded operators on  $(\mathcal{X}, \mathcal{P})$  which is an unital normed algebra with respect to the norm

$$\|T\|_{\mathcal{P}} = \inf\{r > 0 : p(Tx) \leq rp(x) \text{ for all } p \in \mathcal{P} \text{ and all } x \in \mathcal{X}\}.$$

Furthermore

$$\|T\|_{\mathcal{P}} = \sup\{\hat{p}(T), p \in \mathcal{P}\}, \quad \text{for all } T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X}).$$

Recall [6, Definition 1.2], that two families  $\mathcal{P}$  and  $\mathcal{P}'$  of seminorms on a linear space are called  $\mathcal{B}$ -equivalent (denoted  $\mathcal{P} \simeq \mathcal{P}'$ ) provided each seminorm in each is a positive number multiple of a seminorm in the other.

**Proposition 2.1.** [6] *Let  $(\mathcal{X}, \mathcal{P})$  be a locally convex space. Then, an operator  $T \in \mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  is bounded in the algebra  $\mathcal{Q}_{\mathcal{P}}(\mathcal{X})$  if and only if there exists some calibration  $\mathcal{P}'$  such that  $\mathcal{P} \simeq \mathcal{P}'$  and  $T \in \mathcal{B}_{\mathcal{P}'}(\mathcal{X})$ .*

In view of the previous Proposition, for each bounded operator  $T$  of  $\mathcal{Q}_{\mathcal{P}'}(\mathcal{X})$ , we are able to choose an equivalent family of seminorms  $\mathcal{P}$  such that  $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ . In the following, we consider a family of seminorms  $\mathcal{P}$  such that  $T \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ . We denote by  $L(\mathcal{X})$  the space of all continuous linear operators on  $\mathcal{X}$ , and by  $cs(\mathcal{X})$  the set of all continuous semi-norms on  $\mathcal{X}$ .

**Definition 2.2.** Let  $\mathcal{X}$  be a locally convex Hausdorff space and  $(T(t))_{t \geq 0} \subset L(\mathcal{X})$  be a one-parameter family of operators. We say that  $(T(t))_{t \geq 0}$  is a  $C_0$ -semigroup, if it satisfies the following three conditions:

- (i)  $T(0) = I$ ;
- (ii)  $T(t + s) = T(t)T(s)$ .
- (iii)  $\lim_{t \rightarrow 0^+} T(t)x = x$ , for every  $x \in \mathcal{X}$ .

Furthermore, we say that  $(T(t))_{t \geq 0}$  is a strongly continuous  $C_0$ -semigroup if it satisfies conditions (i), (ii) and the additional condition

- (iv)  $\lim_{t \rightarrow s} T(t)x = T(s)x$ , for every  $s \geq 0$  and any  $x \in \mathcal{X}$ .

We call  $(T(t))_{t \geq 0}$  an exponentially equicontinuous semigroup of order  $\omega \in \mathbb{R}$  if the following property

$$\forall q \in \Gamma, \exists p \in \Gamma, M \geq 1, \forall t \geq 0, x \in \mathcal{X} : q(T(t)x) \leq Me^{\omega t}p(x),$$

holds for some, or equivalently, for every, fundamental system of semi-norms  $\Gamma \subseteq \text{cs}(\mathcal{X})$ . The  $C_0$ -semigroup is said to be equicontinuous if the latter condition holds with  $\omega = 0$ .

A semigroup  $(T(t))_{t \geq 0}$  is locally equicontinuous if, for fixed  $r > 0$ , the set  $\{T(t) : 0 \leq t \leq r\}$  is equicontinuous, i.e.,  $\forall p \in \Gamma$  there exists  $q \in \Gamma$  and  $M > 0$  (depending on  $p$  and  $r$ ) such that

$$p(T(t)x) \leq Mq(x), \quad x \in \mathcal{X}, t \in [0, r].$$

**Remark 2.3.** (1) A semigroup  $(T(t))_{t \geq 0}$  is exponentially equicontinuous of order  $\omega$  if and only if  $\{e^{-\omega t}T(t)\}_{t \geq 0}$  is equicontinuous.

- (2) Every exponentially equicontinuous semigroup is locally equicontinuous.
- (3) If  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  is a locally equicontinuous  $C_0$ -semigroup of universally bounded operators on  $\mathcal{X}$ , then it is strongly continuous [1].

**Definition 2.4.** Let  $(T(t))_{t \geq 0}$  be an exponentially equicontinuous  $C_0$ -semigroup. We denote by  $\omega_0(T)$  the infimum over all  $\omega \in \mathbb{R}$  for which  $\{e^{-\omega t}T(t)\}_{t \geq 0}$  is equicontinuous, and call this number the growth bound of  $(T(t))_{t \geq 0}$ .

The main ingredient for the following Lemma is the 'recalibration trick' of Moore [12, Theorem 4].

**Lemma 2.5.** Let  $(T(t))_{t \geq 0}$  be a  $C_0$ -semigroup and  $\omega \in \mathbb{R}$ . The following are equivalent:

- (i) The semigroup  $(T(t))_{t \geq 0}$  is exponentially equicontinuous of order  $\omega$ .
- (ii) There exists a fundamental system  $\Gamma$  and a constant  $M \geq 1$  such that

$$\forall q \in \Gamma, t \geq 0, x \in \mathcal{X} : q(T(t)x) \leq Me^{\omega t}q(x)$$

holds.

- (iii) There exists a fundamental system  $\Gamma$  such that every operator in the semigroup is universally bounded with respect to  $\Gamma$  and there exists a constant  $M \geq 1$  such that

$$\|T(t)\|_{\Gamma} = \sup_{q \in \Gamma} \sup_{q(x) \leq 1} q(T(t)x) \leq Me^{\omega t}$$

holds for all  $t \geq 0$ .

If  $\mathcal{X}$  is a sequentially complete lch and  $(T(t))_{t \geq 0}$  is a locally equicontinuous  $C_0$ -semigroup on  $\mathcal{X}$ , then the linear operator  $A$  defined by

$$Ax := \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t},$$

for  $x \in D(A) := \left\{ x \in \mathcal{X} : \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t} \text{ exists in } \mathcal{X} \right\}$ , is closed with  $\overline{D(A)} = \mathcal{X}$ , [7, Propositions 1.3 & 1.4]. The operator  $(A, D(A))$  is called the infinitesimal generator of  $(T(t))_{t \geq 0}$ . Moreover, we have the following:

- (1) for each  $t \geq 0$ , we have  $\{T(t)x : x \in D(A)\} \subseteq D(A)$  and  $AT(t)x = T(t)Ax$ , for all  $x \in D(A)$  [7];
- (2)  $T(t)x - x = \int_0^t T(s)Ax ds = \int_0^t AT(s)x ds$ ,  $x \in D(A)$  [7];
- (3)  $T(t)x - x = A \int_0^t T(s)x ds$ ,  $x \in \mathcal{X}$  [7];
- (4) Moreover [8], if  $(T(t))_{t \geq 0}$  is exponentially equicontinuous, then, for all  $\lambda > \omega$ , the resolvent  $R(\lambda; A) = (\lambda I - A)^{-1}$ , is well defined, continuous on  $\mathcal{X}$  and satisfies the equality

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} T(t)x dt, \quad \forall x \in \mathcal{X}.$$

For each  $x \in D(A)$  (resp.  $x \in \mathcal{X}$ ), the integrals occurring in (2.) (resp. (3.)) are Riemann integrals of an  $\mathcal{X}$ -valued, continuous function on  $[0, t]$ .

### 3. MAIN RESULTS

We need to formulate a simple lemma, which we shall use to prove the principal theorem 3.2.

**Lemma 3.1.** *Let  $(\mathcal{X}, \mathcal{P})$  be a sequentially complete locally convex space. Let  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  be a  $C_0$ -semigroup of universally bounded operators on  $\mathcal{X}$  satisfying*

- (i)  $\exists M \geq 1, \exists \omega \in \mathbb{R}$  such that  $\|T(t)\|_{\mathcal{P}} \leq Me^{\omega t}, \forall t \geq 0$ ;
- (ii)  $\frac{1}{t} \|T(t)\|_{\mathcal{P}} \rightarrow 0$ , as  $t \rightarrow \infty$ .

*If  $A$  is the infinitesimal generator of  $(T(t))_{t \geq 0}$ , then  $\frac{1}{n} \|[\lambda R(\lambda, A)]^n\|_{\mathcal{P}} \rightarrow 0$ , as  $n \rightarrow \infty$ , for all  $\lambda > \max\{\omega, 0\}$ .*

*Proof.* Let  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  be a  $C_0$ -semigroups of universally bounded operators on  $\mathcal{X}$  and  $A$  its infinitesimal generator. Since  $\lim_{t \rightarrow \infty} \frac{1}{t} \|T(t)\|_{\mathcal{P}} = 0$ , for  $\epsilon > 0$  there exists  $t_0$  such that

$$\|T(t)\|_{\mathcal{P}} \leq \epsilon t, \text{ for all } t \geq t_0.$$

By (i), we have

$$\|T(t)\|_{\mathcal{P}} \leq Me^{\omega_0 t}, \text{ for all } t \in [0; t_0] \text{ and for } \omega_0 = \max\{\omega, 0\}.$$

Using (i) and Lemma 2.5, we have

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x dt.$$

From [3, Proposition 2.3], we have

$$R(\lambda, A)^n x = \frac{1}{(n-1)!} \int_0^\infty e^{-\lambda t} t^{n-1} T(t)x dt \text{ for all } \lambda > \max\{\omega, 0\}.$$

Therefore

$$\frac{\|[\lambda R(\lambda, A)]^n\|_{\mathcal{P}}}{n} \leq \frac{\lambda^n}{n!} Me^{\omega_0 t_0} \int_0^{t_0} e^{-\lambda t} t^{n-1} dt + \epsilon \frac{\lambda^n}{n!} \int_{t_0}^\infty e^{-\lambda t} t^n dt.$$

Since  $\frac{\lambda^{n+1}}{n!} \int_0^\infty e^{-\lambda t} t^n dt = 1$  for all  $\lambda > 0$  and  $n \in \mathbb{N}^*$ ,

$$\frac{\|[\lambda R(\lambda, A)]^n\|_{\mathcal{P}}}{n} \leq \frac{Me^{\omega_0 t_0}}{n} + \frac{\epsilon}{\lambda}.$$

Hence,

$$\limsup_{n \rightarrow +\infty} \frac{\|[\lambda R(\lambda, A)]^n\|_{\mathcal{P}}}{n} \leq \frac{\epsilon}{\lambda}.$$

Letting  $\epsilon \rightarrow 0^+$ , the proof is complete.  $\square$

**Theorem 3.2.** *Let  $(\mathcal{X}, \mathcal{P})$  be a barreled,  $B$ -complete locally convex space. Let  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  be a  $C_0$ -semigroup of universally bounded operators on  $\mathcal{X}$  satisfying*

- (i)  $\exists M \geq 1$  such that  $\|T(t)\|_{\mathcal{P}} \leq Me^{\omega t}$ ,  $\forall t \geq 0$ ;
- (ii)  $\frac{1}{t} \|T(t)\|_{\mathcal{P}} \rightarrow 0$ , as  $t \rightarrow \infty$ .

*If  $A$  is the infinitesimal generator of  $(T(t))_{t \geq 0}$ , then the following conditions are equivalent:*

- (1)  $A$  has a closed range and its co-dimension is  $\aleph_0$ ;
- (2) The operator  $\lambda R(\lambda, A)$  is uniformly mean ergodic,  $\lambda > 0$ .

*Proof.* Let  $\mathbf{Y} = \overline{R(A)}$ .

(1)  $\implies$  (2). For  $\lambda > 0$ ,  $(\lambda I - A)R(\lambda, A) = I$  and for  $x \in D(A)$ ,  $R(\lambda, A)(\lambda I - A)x = x$  (see the proof of [3, Lemma 2.9]). Hence  $\lambda R(\lambda, A) = I + AR(\lambda, A)$  and  $(\lambda R(\lambda, A) - I)\mathcal{X} \subset R(A)$ . By (1),  $\mathbf{Y} = R(A)$ . Then, for  $y \in \mathbf{Y}$ , there exists  $x \in D(A)$  such that  $Ax = y$ , and

$$x = R(\lambda, A)(\lambda I - A)x = \lambda R(\lambda, A)x - R(\lambda, A)y,$$

thus

$$R(\lambda, A)y = (\lambda R(\lambda, A) - I)x.$$

Hence

$$y = (\lambda I - A)R(\lambda, A)y = (\lambda I - A)(\lambda R(\lambda, A) - I)x = (\lambda R(\lambda, A) - I)(\lambda I - A)x$$

and  $\mathbf{Y} = (\lambda R(\lambda, A) - I)\mathcal{X}$ . Hence, by applying the uniform ergodic theorem for a universally bounded operator of [14] to the operator  $\lambda R(\lambda, A)$  we obtain (2), since  $\|(\lambda R(\lambda, A))^n\|_{\mathcal{P}}/n \rightarrow 0$  by Lemma 3.1.

(2)  $\implies$  (1). By the uniform ergodic theorem for a universally bounded operator [14],  $(I - R(1, A))\mathcal{X}$  is closed. In the proof of (1)  $\implies$  (2) we have shown  $(I - \lambda R(\lambda, A))\mathcal{X} = R(A)$  and  $R(A)$  is closed.  $\square$

**Theorem 3.3.** *Let  $(\mathcal{X}, \mathcal{P})$  be a sequentially complete locally convex space,  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  be a  $C_0$ -semigroup of universally bounded operators on  $\mathcal{X}$  satisfying*

- (i)  $\exists M \geq 1$  such that  $\|T(t)\|_{\mathcal{P}} \leq Me^{\omega t}$ ,  $\forall t \geq 0$ ;
- (ii)  $\frac{1}{t} \|T(t)\|_{\mathcal{P}} \rightarrow 0$ , as  $t \rightarrow \infty$ .

*Let  $A$  be the infinitesimal generator of  $(T(t))_{t \geq 0}$ .*

- (1) *If there exists an universally bounded operator  $E$  on  $\mathcal{X}$  such that*

$$\left\| t^{-1} \int_0^t T(s)ds - E \right\|_{\mathcal{P}} \rightarrow 0, \text{ as } t \rightarrow \infty, \quad (3.1)$$

*or*

- (2) *If there exists a projection  $P$  on  $\{x : T(t)x = x, \forall t > 0\}$  such that*

$$\|\lambda R(\lambda, A) - P\|_{\mathcal{P}} \rightarrow 0, \text{ as } \lambda \rightarrow 0^+, \quad (3.2)$$

*then  $A$  has a closed range.*

*Proof.* Assume (1). Since  $\|\cdot\|_{\mathcal{P}} - \lim_{t \rightarrow \infty} \frac{T(t)}{t} = 0$ , it follows from (1) and [1, Remarks 4(ii) & 5(iii)] that  $E^2 = E$ , with  $R(E) = \{x : T(t)x = x, \forall t \geq 0\}$ . Since,  $AT(t)x = T(t)Ax$ , for all  $x \in D(A)$ , we get  $T(t)\mathbf{Y} \subset \mathbf{Y}$ . For  $x \in D(A)$ , we have  $(T(t) - I)x = \int_0^t T(r)Ax dr \in \mathbf{Y}$ . Since  $\overline{D(A)} = \mathcal{X}$ , we have that  $\bigcup_{t \geq 0} \overline{(I - T(t))\mathcal{X}} \subset \mathbf{Y}$ . On the other hand, for  $x \in D(A)$ ,  $Ax = \lim_{h \rightarrow 0} h^{-1}(T(h) - I)x$ , so that  $\mathbf{Y} \subset \overline{\bigcup_{t \geq 0} (I - T(t))\mathcal{X}}$  and equality holds. Hence by (1),  $\mathcal{X} = R(E) \oplus \mathbf{Y}$ . Let  $S(t) = T(t)|_{\mathbf{Y}}$  be the restriction of  $T(t)$  to  $\mathbf{Y}$  and  $A_1$  its infinitesimal generator with domain  $D(A_1) = D(A) \cap \mathbf{Y}$ . Then  $\mathbf{Y} = \overline{R(A_1)}$ , see [3, Lemma 13 and Remark 14]. Now, we may assume that  $\mathcal{X} = \mathbf{Y}$ , and

$\left\| t^{-1} \int_0^t S(s) ds \right\|_{\mathcal{P}} \rightarrow 0, t \rightarrow \infty$ . Thus, for fixed  $t$  large enough,  $\left\| t^{-1} \int_0^t S(r) dr \right\|_{\mathcal{P}} < 1$ , then  $I - t^{-1} \int_0^t S(r) dr$  is invertible, and so is  $\int_0^t (S(r) - I) dr$ . Let  $y \in \mathbf{Y}$ , then

$$\begin{aligned} \int_0^t (S(s) - I) y ds &= \int_0^t (T(s) - I) y ds \\ &= \int_0^t \left( A \int_0^s T(r) y dr \right) ds = A \int_0^t \left( \int_0^s T(r) y dr \right) ds. \end{aligned}$$

Hence  $\int_0^t (S(s) - I) y ds \in R(A)$ . Since  $\mathbf{Y} = \left( \int_0^t (S(s) - I) ds \right) \mathbf{Y}$ . It follows that  $\mathbf{Y} \subseteq R(A)$ . Therefore,  $\mathbf{Y} = R(A)$  and  $R(A)$  is closed.

(2) implies  $A$  has a closed range has been already proved in Theorem 3.2.  $\square$

**Theorem 3.4.** *Let  $(\mathcal{X}, \mathcal{P})$  be a barrelled  $B$ -complete locally convex space. Let  $(T(t))_{t \geq 0} \subset \mathcal{B}_{\mathcal{P}}(\mathcal{X})$  be a  $C_0$ -semigroup of universally bounded operators on  $\mathcal{X}$  satisfying*

- (i)  $\exists M \geq 1$  such that  $\|T(t)\|_{\mathcal{P}} \leq M e^{\omega t}, \forall t \geq 0$ ;
- (ii)  $\frac{1}{t} \|T(t)\|_{\mathcal{P}} \rightarrow 0$ , as  $t \rightarrow \infty$ .

If  $A$  is the infinitesimal generator of  $(T(t))_{t \geq 0}$ , then the following conditions are equivalent:

- (1) There exists an universally bounded operator  $E$  such that

$$\left\| t^{-1} \int_0^t T(s) ds - E \right\|_{\mathcal{P}} \rightarrow 0, \text{ as } t \rightarrow \infty; \quad (3.3)$$

- (2)  $A$  has a closed range;
- (3) The operator  $\lambda R(\lambda, A)$  is uniformly mean ergodic,  $\lambda > 0$ ;
- (4) There exists a projection  $E$  on  $\{x : T(t)x = x, \forall t > 0\}$  such that

$$\|\lambda R(\lambda, A) - E\|_{\mathcal{P}} \rightarrow 0, \text{ as } \lambda \rightarrow 0^+. \quad (3.4)$$

*Proof.* It suffices to show that (1)  $\implies$  (4) and (2)  $\implies$  (1).

For (1)  $\implies$  (4). Since  $\mathcal{X}$  is barrelled, then  $(C(r))_{r \geq 0}$  is equicontinuous [2, Remark 5.6. (i)]. By hypothesis,  $(C(r))_{r \geq 0}$  converges in  $\mathcal{B}_{\mathcal{P}}(\mathcal{X})$  to some  $P \in \mathcal{B}_{\mathcal{P}}(\mathcal{X})$ . By (i), we have

$$p(C(r)x) \leq \frac{M}{r\omega} e^{\omega r} p(x), \forall x \in \mathcal{X}, r \geq 0.$$

Then

$$p\left(\int_0^r T(s)x ds\right) \leq \frac{M}{\omega} e^{\omega r} p(x), \forall x \in \mathcal{X}, r \geq 0.$$

Thus, by [2, Lemma 5.2], we get

$$\lambda R(\lambda, A)x = \lambda^2 \int_0^{\infty} s e^{-\lambda s} C(s)x ds, \text{ for } x \in \mathcal{X}.$$

Using the identity  $\lambda^2 \int_0^{\infty} s e^{-\lambda s} ds = 1$ , for all  $\lambda > 0$ , we obtain

$$\begin{aligned} p(\lambda R(\lambda, A)x - Px) &\leq \int_0^{\infty} \lambda^2 s e^{-\lambda s} p(C(s)x - Px) ds \\ &\leq \int_0^{\infty} t e^{-t} p\left(C\left(\frac{t}{\lambda}\right)x - Px\right) dt \\ &\leq \int_0^{\infty} t e^{-t} \left\| C\left(\frac{t}{\lambda}\right) - P \right\|_{\mathcal{P}} dt p(x), \forall x \in \mathcal{X}, \lambda > 0. \end{aligned}$$

By the uniform ergodicity of  $(T(t))_{t \geq 0}$ ,  $\left\| C\left(\frac{t}{\lambda}\right) - P \right\|_{\mathcal{P}} \rightarrow 0, \lambda \rightarrow 0^+$  pointwise for  $t \geq 0$ .

(2)  $\implies$  (1). Let  $\mathbf{Y} = R(A)$  and let  $A_1$  the generator of the restriction of  $(T(t))_{t \geq 0}$  to  $\mathbf{Y}$ .

Then it is the restriction of  $A$  to  $\mathbf{Y} \cap D(A)$ . It is shown in the proof of Theorem 3.2 that condition (2) implies  $\mathbf{Y} = (I - \lambda R(\lambda, A))\mathcal{X}$  and, by [14],  $I - \lambda R(\lambda, A)$  is invertible on  $\mathbf{Y}$ .

Let  $y \in D(A_1)$  such that  $A_1 y = 0$ , then

$$y = R(\lambda, A)(\lambda I - A)y = \lambda R(\lambda, A)y - R(\lambda, A)A_1 y = \lambda R(\lambda, A)y,$$

thus

$$(I - \lambda R(\lambda, A))y = 0,$$

which implies that  $y = 0$ . Thus,  $A_1$  is one-to-one. On the other hand, by applying [3, Lemma 2.9] to  $(T(t)|_{\mathbf{Y}})_{t \geq 0}$ , we have

$$\text{Im } A_1 = (I - \lambda R(\lambda, A_1))\mathbf{Y} = (I - \lambda R(\lambda, A))\mathbf{Y} = \mathbf{Y}.$$

Therefore,  $A_1 : D(A_1) \rightarrow \mathbf{Y}$  is bijective and  $A_1^{-1}$  is defined on all  $\mathbf{Y}$ . Since  $A_1$  is closed, also  $A_1^{-1}$  is closed. Since  $\mathcal{X}$  is barrelled, by the Closed Graph Theorem, it follows that  $A_1^{-1}$  is continuous.

Let  $y \in \mathbf{Y} = R(A_1)$  there is an  $x \in D(A) \cap \mathbf{Y}$  such that  $Ax = y$  and for all  $p \in \mathcal{P}$ ,  $p(x) \leq \|A_1^{-1}\|_{\mathcal{P}} p(y)$ . For every  $t > 0$ , we have

$$(T(t) - I)x = \int_0^t T(r)Ax dr = \int_0^t T(r)y dr.$$

Thus  $C(t)y = \frac{(T(t)-I)x}{t}$ , and so for  $y \in \mathbf{Y}$  we get

$$p\left(t^{-1} \int_0^t T(r)y dr\right) \leq \|A_1^{-1}\|_{\mathcal{P}} (\|T(t)\|_{\mathcal{P}} + 1) t^{-1} p(y),$$

for all  $p \in \mathcal{P}$ . Hence, on  $\mathbf{Y}$ , we have  $\|C(t)\|_{\mathcal{P}} \rightarrow 0$ . But  $\mathcal{X} = \mathbf{Y} \oplus \{x : R(\lambda, A)x = x\}$  by Theorem 3.2 and the uniform ergodic theorem, hence  $\mathcal{X} = \mathbf{Y} \oplus \{x : T(t)x = x, \forall t > 0\}$ . Therefore (3.3) is satisfied.  $\square$

**Conflict of interest** The authors declare that there is no conflict of interest.

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*Abdellah Akrym*: [akrym.maths@gmail.com](mailto:akrym.maths@gmail.com)

Department of Mathematics, Faculty of Sciences, Chouaib Doukkali University, El Jadida, Morocco.

*Abdeslam EL Bakkali*: [abdeslamelbakkalii@gmail.com](mailto:abdeslamelbakkalii@gmail.com)

Department of Mathematics, Faculty of Sciences, Chouaib Doukkali University, El Jadida, Morocco.

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