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A CLASS OF VARIATIONAL INEQUALITY IN HYPERBOLIC FRAMEWORK

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Abstract. In this paper, we extends the classical theory of variational inequalities to the hyperbolic scalar setting using the structure of $\mathbb{D}$-Hilbert spaces. We introduce and analyze a new class of variational inequalities, termed general mildly $\mathbb{D}$-nonlinear variational inequalities, which generalize classical formulations by incorporating $\mathbb{D}$-nonlinear and product-type mappings. We characterize these problems in terms of their idempotent components and demonstrate that several known variational inequality problems, including Stampacchia-type and complementarity problems, emerge as special cases.

1. Introduction

In 1959, Antonio Signorini introduced what is now known as the Signorini problem, marking the first occurrence of a problem formulated within the framework of variational inequalities. A complete resolution to this problem was subsequently provided by Gaetano Fichera in 1963, with the foundational contributions to this field, see [7, 8], and [9].
Current research efforts remain focused on refining the theoretical underpinnings of variational inequalities, examining solution properties, studying the convergence behavior of iterative algorithms, and enhancing the robustness of variational inequality models (see [6, 17]). The theory of variational inequalities continues to be an active and evolving area of mathematical inquiry, with ongoing advancements in both theoretical perspectives and practical applications. For a comprehensive exposition on variational inequalities, we refer to [1, 4, 6, 10, 18, 19], and [20].
Recently, the scope of variational inequality problems has been significantly extended and generalized in various directions. A study of variational inequalities in hyperbolic framework was initiated in [2]. In this work, our principal objective is to establish extensions of certain results in the theory of variational inequalities within the framework of hyperbolic scalars, leading to the formulation of the general $\mathbb{D}$-nonlinear variational inequality.

2. Preliminaries

This section consolidates fundamental concepts concerning hyperbolic numbers, drawing upon key insights from [3, 15], and supplementary material from [16]. We provide an introduction to hyperbolic numbers and examine their essential properties relevant to the context of this study.
The set of hyperbolic numbers, denoted by $\mathbb{D}$, is defined as

$$\mathbb{D} := \left\{ \mathfrak{h} = x + y\mathbf{k} \mid x, y \in \mathbb{R}, \mathbf{k}^2 = 1, \mathbf{k} \notin \mathbb{R} \right\}.$$

The conjugate of a hyperbolic number $\mathfrak{h} = x + y\mathbf{k}$ is given by $\mathfrak{h}^\diamond := x - y\mathbf{k}$. The hyperbolic modulus of $\mathfrak{h}$ is defined as $|\mathfrak{h}|_{hyp} := \sqrt{x^2 - y^2}$.
Two fundamental hyperbolic numbers, $\mathbf{e} = \frac{1}{2}(1 + \mathbf{k})$ and $\mathbf{e}^\dagger = \frac{1}{2}(1 - \mathbf{k})$, exhibit distinct algebraic properties in hyperbolic analysis. They satisfy idempotency conditions, $(\mathbf{e})^2 = \mathbf{e}$

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and $(\mathbf{e}^\dagger)^2 = \mathbf{e}^\dagger$, and act as zero divisors with $\mathbf{e}\cdot\mathbf{e}^\dagger = 0$. Furthermore, they satisfy $\mathbf{e}+\mathbf{e}^\dagger = 1$ and $\mathbf{e}-\mathbf{e}^\dagger = \mathbf{k}$. These properties yield the idempotent representation of hyperbolic numbers:

$$\mathfrak{h} = x + \mathbf{k}y = (x+y)\mathbf{e} + (x-y)\mathbf{e}^\dagger = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger.$$

A hyperbolic number $\mathfrak{h} = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ is a zero divisor if and only if either $\mathfrak{h}_{\mathbf{e}} = 0$ or $\mathfrak{h}_{\mathbf{e}^\dagger} = 0$. The set of all zero divisors is denoted by $\mathbb{NC}$, and its union with zero is represented as $\mathbb{NC}_0$.
The subset of positive hyperbolic numbers, denoted by $\mathbb{D}^+$, consists of hyperbolic numbers whose idempotent components are non-negative:

$$\mathbb{D}^+ := \left\{\mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \mid \mathfrak{h}_{\mathbf{e}}, \mathfrak{h}_{\mathbf{e}^\dagger} \geq 0\right\}.$$

For hyperbolic numbers $\mathfrak{h}, \mathfrak{g} \in \mathbb{D}$, the notation $\mathfrak{h} \preceq \mathfrak{g}$ signifies that $\mathfrak{g} - \mathfrak{h} \in \mathbb{D}^+$, thereby inducing a partial order relation on $\mathbb{D}$. Expressing $\mathfrak{h}$ and $\mathfrak{g}$ in their idempotent forms,

$$\mathfrak{h} = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger, \quad \mathfrak{g} = \mathfrak{g}_{\mathbf{e}}\mathbf{e} + \mathfrak{g}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger,$$

where $\mathfrak{h}_{\mathbf{e}}, \mathfrak{h}_{\mathbf{e}^\dagger}, \mathfrak{g}_{\mathbf{e}}, \mathfrak{g}_{\mathbf{e}^\dagger} \in \mathbb{R}$, it follows that $\mathfrak{h} \succeq \mathfrak{g}$ if and only if $\mathfrak{h}_{\mathbf{e}} \geq \mathfrak{g}_{\mathbf{e}}$ and $\mathfrak{h}_{\mathbf{e}^\dagger} \geq \mathfrak{g}_{\mathbf{e}^\dagger}$.
The theory of $\mathbb{BC}$-modules has been extensively explored in [11, 12], and [13]. When considering the specific case $\mathbb{BC} = \mathbb{D}$, the results established for $\mathbb{BC}$-modules naturally extend to $\mathbb{D}$-modules. This paper focuses on the study of $\mathbb{D}$-modules.
A module $\mathfrak{X}_{\mathbb{D}} = (\mathfrak{X}_{\mathbb{D}}, +, \cdot)$ over a ring R is called a topological R-module if there exists a topology $\tau_{\mathfrak{X}_{\mathbb{D}}}$ on $\mathfrak{X}_{\mathbb{D}}$ such that the corresponding operations $+ : \mathfrak{X}_{\mathbb{D}} \times \mathfrak{X}_{\mathbb{D}} \to \mathfrak{X}_{\mathbb{D}}$ and $\cdot : R \times \mathfrak{X}_{\mathbb{D}} \to \mathfrak{X}_{\mathbb{D}}$ are continuous. In particular, if we take $R = \mathbb{D}$, the ring of hyperbolic numbers, the module $\mathfrak{X}_{\mathbb{D}}$ is called a topological $\mathbb{D}$-module. The dual space of $\mathfrak{X}_{\mathbb{D}}$, denoted $\mathfrak{X}^*_{\mathbb{D}}$, consists of all continuous $\mathbb{D}$-linear functionals $\Psi : \mathbb{D} \to \mathbb{D}$. In particular if we take $\mathfrak{X}_{\mathbb{D}} = \mathbb{D}_{\mathbb{D}}$, the set of hyperbolic numbers over the ring $\mathbb{D}$, then it is called hyperbolic module or $\mathbb{D}$-module and we denote it by $\mathfrak{X}$.
Let $\mathfrak{X}$ be a $\mathbb{D}$-module. It admits an idempotent decomposition (see [14, 21]):

$$\mathfrak{X} = \mathbf{e}\mathfrak{X}_{\mathbf{e}} + \mathbf{e}^\dagger\mathfrak{X}_{\mathbf{e}^\dagger},$$

where $\mathfrak{X}_{\mathbf{e}} = \mathbf{e}\mathfrak{X}$ and $\mathfrak{X}_{\mathbf{e}^\dagger} = \mathbf{e}^\dagger\mathfrak{X}$ are real-linear spaces and also function as $\mathbb{D}$-modules. Any element $\mathfrak{h} \in \mathfrak{X}$ can be uniquely expressed as $\mathfrak{h} = \mathbf{e}\mathfrak{h}_{\mathbf{e}} + \mathbf{e}^\dagger\mathfrak{h}_{\mathbf{e}^\dagger}$. If $\mathfrak{X}_{\mathbf{e}}$ and $\mathfrak{X}_{\mathbf{e}^\dagger}$ are normed spaces with respective norms $||\cdot||_{\mathbf{e}}$ and $||\cdot||_{\mathbf{e}^\dagger}$, then the function

$$||\mathfrak{h}|| = \sqrt{\frac{||\mathfrak{h}_{\mathbf{e}}||^2_{\mathbf{e}} + ||\mathfrak{h}_{\mathbf{e}^\dagger}||^2_{\mathbf{e}^\dagger}}{2}}$$

defines a real-valued norm on $\mathfrak{X}$ satisfying the bound $||\lambda\mathfrak{h}|| \preceq \sqrt{2}|\lambda|||\mathfrak{h}||$ for any $\lambda \in \mathbb{D}$ and $\mathfrak{h} \in \mathfrak{X}$.

Additionally, the space $\mathfrak{X}$ can be equipped with a hyperbolic-valued norm defined as

$$||\mathfrak{h}||_{\mathbb{D}} = ||\mathbf{e}\mathfrak{h}_{\mathbf{e}} + \mathbf{e}^\dagger\mathfrak{h}_{\mathbf{e}^\dagger}||_{\mathbb{D}} = ||\mathfrak{h}_{\mathbf{e}}||_{\mathbf{e}}\mathbf{e} + ||\mathfrak{h}_{\mathbf{e}^\dagger}||_{\mathbf{e}^\dagger}\mathbf{e}^\dagger,$$

which satisfies the property $||\lambda\mathfrak{h}||_{\mathbb{D}} = |\lambda|_{hyp}||\mathfrak{h}||_{\mathbb{D}}$. Moreover, the real-valued norm induced by the hyperbolic norm is given by

$$|||\mathfrak{h}||_{\mathbb{D}}| = ||\mathfrak{h}||.$$

For more details, see [3, Section 4.2].

A $\mathbb{D}$-inner product on the $\mathbb{D}$-module $\mathfrak{X}$ can be naturally defined in terms of the inner products of its idempotent components. Assuming that $\mathfrak{X}_{\mathbf{e}}$ and $\mathfrak{X}_{\mathbf{e}^\dagger}$ are inner product spaces equipped with inner products $(\cdot,\cdot)_{\mathbf{e}}$ and $(\cdot,\cdot)_{\mathbf{e}^\dagger}$, respectively, the inner product on $\mathfrak{X}$ is given by

$$\begin{aligned}(\mathfrak{h}, \mathfrak{g})_{\mathbb{D}} &= (\mathbf{e}\mathfrak{h}_{\mathbf{e}} + \mathbf{e}^\dagger\mathfrak{h}_{\mathbf{e}^\dagger}, \mathbf{e}\mathfrak{g}_{\mathbf{e}} + \mathbf{e}^\dagger\mathfrak{g}_{\mathbf{e}^\dagger})_{\mathbb{D}} \\ &= \mathbf{e}(\mathfrak{h}_{\mathbf{e}}, \mathfrak{g}_{\mathbf{e}})_{\mathbf{e}} + \mathbf{e}^\dagger(\mathfrak{h}_{\mathbf{e}^\dagger}, \mathfrak{g}_{\mathbf{e}^\dagger})_{\mathbf{e}^\dagger}.\end{aligned}$$

A $\mathbb{D}$-inner product in $\mathbb{D}$ itself can be introduced as

$$(\mathfrak{h}, \mathfrak{g})_{\mathbb{D}} = \mathfrak{h} \cdot \mathfrak{g}^{\diamond}. \tag{2.1}$$

The inner product in (2.1) coincides with the one derived from the idempotent decomposition. The induced hyperbolic norm is given by

$$||\mathfrak{h}||_{\mathbb{D}} = (\mathfrak{h}, \mathfrak{h})_{\mathbb{D}}^{1/2}.$$

A corresponding real-valued norm can be defined as

$$||\mathfrak{h}||^2 = \frac{1}{2}\left((\mathfrak{h}_{\mathbf{e}}, \mathfrak{h}_{\mathbf{e}})_{\mathbf{e}} + (\mathfrak{h}_{\mathbf{e}^\dagger}, \mathfrak{h}_{\mathbf{e}^\dagger})_{\mathbf{e}^\dagger}\right).$$

For more details, see [3, Section 4.3] and [14].

Definition 2.1 ([3], Page 49)**.** A sequence $\{\mathfrak{h}_n\}$ in a $\mathbb{D}$-module $\mathfrak{X}$ converges to $\mathfrak{h}_0 \in \mathfrak{X}$ with respect to the hyperbolic norm $||\cdot||_{\mathbb{D}}$ if, for every $\varepsilon \succ 0$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$, we have $||\mathfrak{h}_n - \mathfrak{h}_0||_{\mathbb{D}} \preceq \varepsilon$. A $\mathbb{D}$-Hilbert space is a $\mathbb{D}$-module $\mathfrak{X}$ that is complete under the hyperbolic norm induced by the inner product. Consequently, $(\mathfrak{X}, (\cdot,\cdot)_{\mathbb{D}})$ is a $\mathbb{D}$-Hilbert space if and only if $(\mathfrak{X}_{\mathbf{e}}, (\cdot,\cdot)_{\mathbf{e}})$ and $(\mathfrak{X}_{\mathbf{e}^\dagger}, (\cdot,\cdot)_{\mathbf{e}^\dagger})$ are real Hilbert spaces.

Remark 2.2. Every real metric is inherently a $\mathbb{D}$-metric. A set $\mathfrak{X}$ is a $\mathbb{D}$-metric space if and only if it is a metric space where the metric can be expressed as

$$d(\mathfrak{h}, \mathfrak{g}) = ||d_{\mathbb{D}}(\mathfrak{h}, \mathfrak{g})||,$$

where the algebra norm in $\mathbb{D}$ is given by

$$||\mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger|| = \max\{|\mathfrak{h}_{\mathbf{e}}|, |\mathfrak{h}_{\mathbf{e}^\dagger}|\}.$$

This norm satisfies the properties:
(i) $|\mathfrak{h}|_{hyp} \leq ||\mathfrak{g}||$, and
(ii) $||\mathfrak{h}|| \leq ||\mathfrak{g}||$ whenever $|\mathfrak{h}|_{hyp} \leq |\mathfrak{g}|_{hyp}$.
Under this framework, a hyperbolic-valued function Ψ is $\mathbb{D}$-Lipschitz, $\mathbb{D}$-bounded, and $\mathbb{D}$-coercive if and only if its real-valued components satisfy these properties in the classical sense.

A subset $\mathcal{B}$ of $\mathbb{D}$ is called a product-type set if it can be expressed as $\mathcal{B} = \mathcal{B}_{\mathbf{e}}\mathbf{e} + \mathcal{B}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$, where $\mathcal{B}_{\mathbf{e}} := \pi_{\mathbf{e}}(\mathcal{B})$ and $\mathcal{B}_{\mathbf{e}^\dagger} := \pi_{\mathbf{e}^\dagger}(\mathcal{B})$ are the projections of $\mathcal{B}$ onto the coordinate axes determined by the basis $\{\mathbf{e}, \mathbf{e}^\dagger\}$ in $\mathbb{R}^2$. A function $\Psi : \mathcal{B} = \mathcal{B}_{\mathbf{e}}\mathbf{e} + \mathcal{B}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \subset \mathbb{D} \to \mathbb{D}$ is said to be of product-type if there exist functions $\Psi_{\mathbf{e}} : \mathcal{B}_{\mathbf{e}} \to \mathbb{R}$ and $\Psi_{\mathbf{e}^\dagger} : \mathcal{B}_{\mathbf{e}^\dagger} \to \mathbb{R}$ such that for every $\mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathcal{B}$, the function satisfies $\Psi(\mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger) = \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\mathbf{e} + \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\mathbf{e}^\dagger$. For more details on product-type set, product-type function, and their properties, one can refer to [2] and [5].

3. General Mildly $\mathbb{D}$-nonlinear Variational Inequality.

In this section, we extend the results of [20] to the hyperbolic scalar setting. This work explores a specific class of variational inequalities, known as general mildly $\mathbb{D}$-nonlinear variational inequalities, which generalize classical variational inequality formulations within a hyperbolic framework.

Problem 3.1. Let $\mathcal{H} = \mathcal{H}_{\mathbf{e}}\mathbf{e} + \mathcal{H}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ be a $\mathbb{D}$-Hilbert space with dual space $\mathcal{H}'$, where the $\mathbb{D}$-inner product and $\mathbb{D}$-norm are denoted by $(\cdot,\cdot)_{\mathbb{D}}$ and $||\cdot||_{\mathbb{D}}$, respectively. Let $\mathcal{B}$ be a product-type $\mathbb{D}$-closed and $\mathbb{D}$-convex subset of $\mathcal{H}$. We denote by $\langle\cdot,\cdot\rangle_{\mathbb{D}}$ the pairing between $\mathcal{H}'$ and $\mathcal{H}$.

Given continuous product-type mappings $\Psi, \psi : \mathcal{H} \to \mathcal{H}'$, we consider the problem of finding $\mathfrak{h} = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathcal{H}$ such that $\psi(\mathfrak{h}) = \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\mathbf{e} + \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\mathbf{e}^\dagger \in \mathcal{B}$, and

$$\langle\Psi(\mathfrak{h}), \psi(\mathfrak{g}) - \psi(\mathfrak{h})\rangle_{\mathbb{D}} \succeq \langle\Phi(\mathfrak{h}), \psi(\mathfrak{g}) - \psi(\mathfrak{h})\rangle_{\mathbb{D}}, \quad \forall\psi(\mathfrak{g}) \in \mathcal{B}, \tag{3.2}$$

which expands as

$$\begin{aligned}&\langle \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}), \psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}) - \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\rangle_{\mathbf{e}}\, \mathbf{e} + \langle \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}), \psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}) - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\rangle_{\mathbf{e}^\dagger}\, \mathbf{e}^\dagger \\ &\succeq \langle \Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}), \psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}) - \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\rangle_{\mathbf{e}}\, \mathbf{e} + \langle \Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}), \psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}) - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\rangle_{\mathbf{e}^\dagger}\, \mathbf{e}^\dagger,\end{aligned}$$

for all $\psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}) \in \mathcal{B}_{\mathbf{e}}, \psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}) \in \mathcal{B}_{\mathbf{e}^\dagger}$.

This holds if and only if there exist solutions $\mathfrak{h}_{\mathbf{e}} \in \mathcal{H}_{\mathbf{e}}$ and $\mathfrak{h}_{\mathbf{e}^\dagger} \in \mathcal{H}_{\mathbf{e}^\dagger}$ such that $\psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) \in \mathcal{B}_{\mathbf{e}}$ and $\psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) \in \mathcal{B}_{\mathbf{e}^\dagger}$, and

$$\langle \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}), \psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}) - \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\rangle_{\mathbf{e}} \geq \langle \Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}), \psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}) - \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\rangle_{\mathbf{e}}, \quad \forall\ \psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}) \in \mathcal{B}_{\mathbf{e}},$$

and

$$\langle \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}), \psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}) - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\rangle_{\mathbf{e}^\dagger} \geq \langle \Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}), \psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}) - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\rangle_{\mathbf{e}^\dagger}, \ \forall\ \psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}) \in \mathcal{B}_{\mathbf{e}^\dagger},$$

where $\Phi(\mathfrak{h}) = \Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\mathbf{e}^\dagger$ is a product-type $\mathbb{D}$-nonlinear continuous mapping satisfying $\Phi(\mathfrak{h}) \in \mathcal{H}'$. The inequality (3.2) is referred to as the general mildly $\mathbb{D}$-nonlinear variational inequality.

4. Special Cases

We now discuss particular cases of Problem 3.1, highlighting its generality and significance in the theory of variational inequalities.

Case I. If the mapping ψ is the identity, i.e., $\psi(\mathfrak{h}) = \mathfrak{h} \in \mathcal{B}$, then Problem 3.1 reduces to finding $\mathfrak{h} \in \mathcal{B}$ such that

$$\langle \Psi(\mathfrak{h}), \mathfrak{g} - \mathfrak{h}\rangle_{\mathbb{D}} \succeq \langle \Phi(\mathfrak{h}), \mathfrak{g} - \mathfrak{h}\rangle_{\mathbb{D}}, \quad \forall\ \mathfrak{g} \in \mathcal{B}. \tag{4.3}$$

This inequality corresponds to the class of mildly (or strongly) $\mathbb{D}$-nonlinear variational inequalities.

Case II. If the $\mathbb{D}$-nonlinear operator $\Phi(\mathfrak{h})$ is identically zero, i.e., $\Phi(\mathfrak{h}) \equiv 0$, or if $\Phi(\mathfrak{h})$ is independent of $\mathfrak{h}$ (i.e., $\Phi(\mathfrak{h}) \equiv \phi$ for some function ϕ), then (3.2) simplifies to the problem of finding $\mathfrak{h} \in \mathcal{B}$ such that $\psi(\mathfrak{h}) \in \mathcal{B}$ and

$$\langle \Psi(\mathfrak{h}), \psi(\mathfrak{g}) - \psi(\mathfrak{h})\rangle_{\mathbb{D}} \succeq 0, \quad \forall\ \psi(\mathfrak{g}) \in \mathcal{B}. \tag{4.4}$$

This formulation represents a specific subclass of variational inequalities.

Case III. If $\Phi(\mathfrak{h}) \equiv 0$ and ψ is the identity mapping, then Problem 3.1 reduces to the classical variational inequality problem of finding $\mathfrak{h} \in \mathcal{B}$ such that

$$\langle \Psi(\mathfrak{h}), \mathfrak{g} - \mathfrak{h}\rangle_{\mathbb{D}} \succeq 0, \quad \forall\ \mathfrak{g} \in \mathcal{B}. \tag{4.5}$$

This problem corresponds to the classical formulation of Stampacchia-type variational inequalities.

Case IV. If $\mathcal{B}^* = \{\mathfrak{h} \in \mathcal{H}', \langle \mathfrak{h}, \mathfrak{g}\rangle_{\mathbb{D}} \succeq 0, \forall\ \mathfrak{g} \in \mathcal{B}\}$ denotes the polar cone of the $\mathbb{D}$-convex set $\mathcal{B}$ in $\mathcal{H}$, then Problem 3.1 is equivalent to finding $\mathfrak{h} \in \mathcal{H}$ such that

$$\psi(\mathfrak{h}) \in \mathcal{B}, \quad (\Psi(\mathfrak{h}) - \Phi(\mathfrak{h})) \in \mathcal{B}^*, \quad \text{and} \quad \langle \Psi(\mathfrak{h}) - \Phi(\mathfrak{h}), \psi(\mathfrak{h})\rangle_{\mathbb{D}} = 0. \tag{4.6}$$

This formulation leads to the general mildly $\mathbb{D}$-nonlinear complementarity problem.

It is evident that Problems (4.3)–(4.6) are all special instances of the general Problem 3.1. This underscores the unifying nature of Problem 3.1, which serves as the primary motivation for this work.

5. Iterative Algorithm

An iterative algorithm for the general mildly nonlinear variational inequality was studied in [18, 19] and [20]. In this section, we develop an iterative algorithm for solving Problem 3.1 within a hyperbolic framework.

Definition 5.1. [2] Let $\mathcal{H}$ be a $\mathbb{D}$-Hilbert space, then for each $\mathfrak{h} \in \mathcal{H}$, there exists a unique $\mathfrak{g} \in \mathcal{B}$ such that $||\mathfrak{h} - \mathfrak{g}||_{\mathbb{D}} = \inf_{\mathfrak{i} \in \mathcal{B}} ||\mathfrak{h} - \mathfrak{i}||_{\mathbb{D}}$. This $\mathfrak{g}$ is termed the projection of $\mathfrak{h}$ onto $\mathcal{B}$ and is denoted by $\mathbf{P}_{\mathcal{B}}(\mathfrak{h})$.

Corollary 5.2. [2, Corollary 4.4] *Let $\mathcal{H}$ be a $\mathbb{D}$-Hilbert space, and $\mathfrak{B} \subset \mathcal{H}$ be a product-type $\mathbb{D}$-closed $\mathbb{D}$-convex set. Then, the product-type operator $\mathbf{P}_{\mathfrak{B}}$ is non-expansive, i.e.,*

$$||\mathbf{P}_{\mathfrak{B}}(\mathfrak{h}) - \mathbf{P}_{\mathfrak{B}}(\mathfrak{h}^{'})||_{\mathbb{D}} \preceq ||\mathfrak{h} - \mathfrak{h}^{'}||_{\mathbb{D}}$$

for $\mathfrak{h} = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger, \mathfrak{h}^{'} = \mathfrak{h}^{'}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}^{'}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathcal{H}$.

Lemma 5.3. *Let $\mathcal{B} = \mathcal{B}_{\mathbf{e}}\mathbf{e} + \mathcal{B}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ be a $\mathbb{D}$-convex subset of $\mathcal{H}$. Then, $\mathfrak{h} = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathcal{H}$ is a solution to Problem (3.1) if and only if it satisfies the following relation:*

$$\psi(\mathfrak{h}) = \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}) - \rho\Lambda\Psi(\mathfrak{h}) - \Phi(\mathfrak{h})], \tag{5.7}$$

where $\rho \succ 0$ is a positive constant, and $\mathbf{P}_{\mathcal{B}}$ denotes the projection of $\mathcal{H}$ onto $\mathcal{B}$. Here, $\Lambda = \Lambda_{\mathbf{e}}\mathbf{e} + \Lambda_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ is the canonical isomorphism from $\mathcal{H}^{'}$ onto $\mathcal{H}$, satisfying the duality relation

$$\langle \phi, \mathfrak{h} \rangle_{\mathbb{D}} = (\Lambda\phi, \mathfrak{g})_{\mathbb{D}}, \quad \forall\ \mathfrak{g} \in \mathcal{H},\ \phi \in \mathcal{H}^{'}.$$

Proof. Since $\mathcal{B} = \mathcal{B}_{\mathbf{e}}\mathbf{e} + \mathcal{B}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ is a $\mathbb{D}$-convex set in $\mathcal{H}$ and $\mathfrak{h} = \mathfrak{h}_{\mathbf{e}}\mathbf{e} + \mathfrak{h}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathcal{H}$ is a solution of Problem 3.1. We must establish that $\mathfrak{h}$ satisfies the relation (5.7).

Write (5.7) in idempotent decomposition form, we obtain

$$\begin{aligned} \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\mathbf{e} + \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\mathbf{e}^\dagger \quad = \quad & \mathbf{P}_{\mathcal{B}_{\mathbf{e}}}\left[\psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) - \rho_{\mathbf{e}}\Lambda_{\mathbf{e}}\Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) - \Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})\right]\mathbf{e} \\ & + \mathbf{P}_{\mathcal{B}_{\mathbf{e}^\dagger}}\left[\psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) - \rho_{\mathbf{e}^\dagger}\Lambda_{\mathbf{e}^\dagger}\Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) - \Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})\right]\mathbf{e}^\dagger. \end{aligned}$$

From Lemma 3.1 in [19], each component satisfies the individual relation:

$$\psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) = \mathbf{P}_{\mathcal{B}_{\mathbf{e}}}[\psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) - \rho_{\mathbf{e}}\Lambda_{\mathbf{e}}\Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) - \Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})],$$

and

$$\psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) = \mathbf{P}_{\mathcal{B}_{\mathbf{e}^\dagger}}[\psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) - \rho_{\mathbf{e}^\dagger}\Lambda_{\mathbf{e}^\dagger}\Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) - \Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})].$$

Thus, it follows that $\mathfrak{h}$ satisfies the overall relation (5.7).

Conversely, the reverse implication follows directly from Lemma 3.1 in [19], completing the proof. □

As a direct consequence of Lemma 5.3, we deduce that Problem 3.1 can be reformulated as the fixed-point problem:

$$\mathfrak{h} = F(\mathfrak{h}),$$

where the product-type function $F : \mathcal{H} \to \mathcal{H}^{'}$ is given by

$$F(\mathfrak{h}) = \mathfrak{h} - \psi(\mathfrak{h}) + \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}) - \rho\Lambda(\Psi(\mathfrak{h}) - \Phi(\mathfrak{h}))]. \tag{5.8}$$

This reformulation is particularly useful in the study of approximation and numerical methods for variational inequalities. Specifically, it facilitates the construction of iterative schemes for obtaining approximate solutions to Problem 3.1.

We now introduce an $\mathbb{D}$-iterative approach to approximate the solution of Problem 3.1.

5.1. **General $\mathbb{D}$-Iterative Scheme. Algorithm 5.1:**

- Initialize with $\mathfrak{h}_0 \in \mathcal{H}$.
- Compute the sequence $\mathfrak{h}_n$ iteratively using the scheme:

$$\mathfrak{h}_{n+1} = \mathfrak{h}_n - \psi(\mathfrak{h}_n) + \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}_n) - \rho\Lambda(\Psi(\mathfrak{h}_n) - \Phi(\mathfrak{h}_n))], \quad n = 0, 1, 2, \ldots$$

where $\rho \succ 0$ is a given constant.

5.2. **Special Cases.**

(i) If $\psi(\mathfrak{h}) = \mathfrak{h} \in \mathcal{B}$, then Algorithm 5.1 simplifies to:

Algorithm 5.2:

- Given $\mathfrak{h}_0 \in \mathcal{H}$, compute iteratively:

$$\mathfrak{h}_{n+1} = \mathbf{P}_{\mathcal{B}}[\mathfrak{h}_n - \rho\Lambda(\Psi(\mathfrak{h}_n) - \Phi(\mathfrak{h}_n))], \quad n = 0, 1, 2, \ldots$$

(ii) If $\Phi(\mathfrak{h}) \equiv 0$, then Algorithm 5.1 takes the form:

Algorithm 5.3:

- Given $\mathfrak{h}_0 \in \mathcal{H}$, compute iteratively:

$$\mathfrak{h}_{n+1} = \mathfrak{h}_n - \psi(\mathfrak{h}_n) + \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}_n) - \rho\Lambda\Psi\mathfrak{h}_n], \quad n = 0, 1, 2, \ldots$$

(iii) If $\psi(\mathfrak{h}) = \mathfrak{h} \in \mathcal{B}$ and $\Phi(\mathfrak{h}) \equiv 0$, then Algorithm 5.1 reduces to:

Algorithm 5.4:

- Given $\mathfrak{h}_0 \in \mathcal{H}$, compute iteratively:

$$\mathfrak{h}_{n+1} = \mathbf{P}_{\mathcal{B}}[\mathfrak{h}_n - \rho\Lambda\Psi\mathfrak{h}_n], \quad n = 0, 1, 2, \ldots$$

To analyze the convergence of Algorithm 3.1, we introduce the following definitions

Definition 5.4. A function $\Psi : \mathcal{H} \to \mathcal{H}^{'}$ is said to be

(a) **Strongly $\mathbb{D}$-monotone** if there exists a constant $\alpha \succ 0$ in $\mathbb{D}$ such that

$$\langle \Psi(\mathfrak{h}) - \Psi(\mathfrak{g}), \mathfrak{h} - \mathfrak{g} \rangle_{\mathbb{D}} \succeq \alpha ||\mathfrak{h} - \mathfrak{g}||_{\mathbb{D}}^2, \quad \forall\, \mathfrak{h}, \mathfrak{g} \in \mathcal{H}.$$

(b) **$\mathbb{D}$-Lipschitz continuous** if there exists a constant $\beta \succ 0$ in $\mathbb{D}$ such that

$$||\Psi(\mathfrak{h}) - \Psi(\mathfrak{g})||_{\mathbb{D}} \preceq \beta ||\mathfrak{h} - \mathfrak{g}||_{\mathbb{D}}, \quad \forall \mathfrak{h}, \mathfrak{g} \in \mathcal{H}.$$

Defining a relationship where $\alpha \preceq \beta$. These properties will be instrumental in ensuring the convergence of the general iterative scheme to the solution of Problem 3.1.

Remark 5.5. If we take a function $\Psi = \Psi_{\mathbf{e}}\mathbf{e} + \Psi_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ a product-type and $\mathcal{H} = \mathcal{H}_{\mathbf{e}}\mathbf{e} + \mathcal{H}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$, then a function $\Psi : \mathcal{H} \to \mathcal{H}^{'}$ is said to be

(a) **Strongly $\mathbb{D}$-monotone** if there exists a constant $\alpha = \alpha_{\mathbf{e}} + \alpha_{\mathbf{e}^\dagger} \succ 0$ in $\mathbb{D}$ such that

$$\begin{aligned}\langle \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) - \Psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}}), \mathfrak{h}_{\mathbf{e}} - \mathfrak{g}_{\mathbf{e}} \rangle_{\mathbf{e}}\, \mathbf{e} \quad &+ \quad \langle \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) - \Psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger}), \mathfrak{h}_{\mathbf{e}^\dagger} - \mathfrak{g}_{\mathbf{e}^\dagger} \rangle_{\mathbf{e}^\dagger}\, \mathbf{e}^\dagger \\ &\succeq \alpha_{\mathbf{e}} ||\mathfrak{h}_{\mathbf{e}} - \mathfrak{g}_{\mathbf{e}}||_{\mathbf{e}}^2 \mathbf{e} + \alpha_{\mathbf{e}^\dagger} ||\mathfrak{h}_{\mathbf{e}^\dagger} - \mathfrak{g}_{\mathbf{e}^\dagger}||_{\mathbf{e}^\dagger}^2 \mathbf{e}^\dagger,\end{aligned}$$

$\forall\ \mathfrak{h}_{\mathbf{e}}, \mathfrak{g}_{\mathbf{e}} \in \mathcal{H}_{\mathbf{e}},\ \mathfrak{h}_{\mathbf{e}^\dagger}, \mathfrak{g}_{\mathbf{e}^\dagger} \in \mathcal{H}_{\mathbf{e}^\dagger}$.

(b) **$\mathbb{D}$-Lipschitz continuous** if there exists a constant $\beta = \beta_{\mathbf{e}} + \beta_{\mathbf{e}^\dagger} \succ 0$ in $\mathbb{D}$ such that

$$||\Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}) - \Psi_{\mathbf{e}}(\mathfrak{g}_{\mathbf{e}})||_{\mathbf{e}}\mathbf{e} + ||\Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}) - \Psi_{\mathbf{e}^\dagger}(\mathfrak{g}_{\mathbf{e}^\dagger})||_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \preceq \beta_{\mathbf{e}} ||\mathfrak{h}_{\mathbf{e}} - \mathfrak{g}_{\mathbf{e}}||_{\mathbf{e}}\mathbf{e} + \beta_{\mathbf{e}^\dagger} ||\mathfrak{h}_{\mathbf{e}^\dagger} - \mathfrak{g}_{\mathbf{e}^\dagger}||_{\mathbf{e}^\dagger}\mathbf{e}^\dagger,$$

$\forall\ \mathfrak{h}_{\mathbf{e}}, \mathfrak{g}_{\mathbf{e}} \in \mathcal{H}_{\mathbf{e}},\ \mathfrak{h}_{\mathbf{e}^\dagger}, \mathfrak{g}_{\mathbf{e}^\dagger} \in \mathcal{H}_{\mathbf{e}^\dagger}$.

Theorem 5.6. *Let $\Psi, \psi : \mathcal{H} \to \mathcal{H}'$ be product-type functions such that Ψ is strongly $\mathbb{D}$-monotone and ψ is $\mathbb{D}$-Lipschitz continuous. Suppose that another product-type function Φ is also $\mathbb{D}$-Lipschitz continuous. Then, the sequence $\{\mathfrak{h}_n\} = \{\mathfrak{h}_{n,\mathbf{e}}\}\mathbf{e} + \{\mathfrak{h}_{n,\mathbf{e}^\dagger}\}\mathbf{e}^\dagger$ generated by Algorithm 5.1 strongly converges to $\mathfrak{h}$ in $\mathcal{H}$, provided the parameters satisfy the condition*

$$\left|\rho - \frac{\alpha + \gamma(k-1)}{\beta^2 - \gamma^2}\right|_{hyp} \prec \frac{\sqrt{(\alpha + \gamma(k-1))^2 - (\beta^2 - \gamma^2)k(2-k)}}{\beta^2 - \gamma^2}, \quad k \prec 1,$$

$$\alpha \succ \gamma(1-k) + \sqrt{(\beta^2 - \gamma^2)k(2-k)}, \quad and \gamma(1-k) \prec \alpha,$$

where $\mathfrak{h}_{n+1}$ and $\mathfrak{h}$ are solutions satisfying Algorithm 5.1 and Problem 3.1, respectively.

Proof. From Lemma 5.3, the solution $\mathfrak{h}$ of Problem 3.1 can be expressed using relation (5.7). Using (5.7) and (5.8), we set:

$$\begin{aligned}
||\mathfrak{h}_{n+1} - \mathfrak{h}||_{\mathbb{D}} &= ||\mathfrak{h}_n - \mathfrak{h} - (\psi(\mathfrak{h},) - \psi(\mathfrak{h})) + \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}_n) - \rho\Lambda(\Psi\mathfrak{h}_n - \Phi(\mathfrak{h}))] \\
&\quad - \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}) - \rho\Lambda(\Psi\mathfrak{h} - \Phi(\mathfrak{h}))]||_{\mathbb{D}} \\
&\preceq ||\mathfrak{h}_n - \mathfrak{h} - (\psi(\mathfrak{h}_n) - \psi(\mathfrak{h}))]|| \\
&\quad + ||\mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}_n) - \rho\Lambda(\Psi\mathfrak{h}_n - \Phi(\mathfrak{h}_n))] \\
&\quad - \mathbf{P}_{\mathcal{B}}[\psi(\mathfrak{h}) - \rho\Lambda(\Psi\mathfrak{h} - \Phi(\mathfrak{h}))]||_{\mathbb{D}} \\
&\preceq 2||\mathfrak{h}_n - \mathfrak{h} - (\psi(\mathfrak{h}_n) - \psi(u))||_{\mathbb{D}} \\
&\quad + ||\mathfrak{h}_n - \mathfrak{h} - \rho\Lambda(\Psi(\mathfrak{h}_n) - \Psi(\mathfrak{h})) + \rho\Lambda(\Phi(\mathfrak{h}_n) - \Phi(\mathfrak{h}))||_{\mathbb{D}},
\end{aligned}$$

where we use the fact from Corollary 5.2 that $\mathbf{P}_{\mathcal{B}}$ is a non-expansive mapping.
We can also write

$$\begin{gathered}
||\mathfrak{h}_{n+1} - \mathfrak{h}||_{\mathbb{D}} \preceq 2||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}} - \psi_{\mathbf{e}}(\mathfrak{h}_{n,\mathbf{e}}) - \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})||_{\mathbf{e}}\mathbf{e} \\
+ 2||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger} - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{n,\mathbf{e}^\dagger}) - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}})||_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \\
+ ||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}} - \rho_{\mathbf{e}}\Lambda_{\mathbf{e}}(\Psi_{\mathbf{e}}(\mathfrak{h}_{n,\mathbf{e}}) - \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})) + \rho_{\mathbf{e}}\Lambda_{\mathbf{e}}(\Phi_{\mathbf{e}}(\mathfrak{h}_{n,\mathbf{e}}) - \Phi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}))||_{\mathbf{e}}\mathbf{e} \\
+ ||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger} - \rho_{\mathbf{e}^\dagger}\Lambda_{\mathbf{e}^\dagger}(\Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{n,\mathbf{e}^\dagger}) - \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})) \\
+ \rho_{\mathbf{e}^\dagger}\Lambda_{\mathbf{e}^\dagger}(\Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{n,\mathbf{e}^\dagger}) - \Phi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}))||_{\mathbf{e}^\dagger}\mathbf{e}^\dagger,
\end{gathered}$$

By employing the strong $\mathbb{D}$-monotonicity and $\mathbb{D}$-Lipschitz continuity of Ψ and ψ as well as the technique of Noor [19], we obtain:

$$||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}} - \psi_{\mathbf{e}}(\mathfrak{h}_{n,\mathbf{e}}) - \psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}})||_{\mathbf{e}}^2 \preceq (1 - 2\delta_{\mathbf{e}} + \sigma_{\mathbf{e}}^2)||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}}||_{\mathbf{e}}^2, \tag{5.9}$$

$$||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger} - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{n,\mathbf{e}^\dagger}) - \psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger})||_{\mathbf{e}^\dagger}^2 \preceq (1 - 2\delta_{\mathbf{e}^\dagger} + \sigma_{\mathbf{e}^\dagger}^2)||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger}||_{\mathbf{e}^\dagger}^2, \tag{5.10}$$

$$||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}} - \rho_{\mathbf{e}}\Lambda_{\mathbf{e}}(\Psi_{\mathbf{e}}(\mathfrak{h}_{n,\mathbf{e}}) - \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}))||_{\mathbf{e}}^2 \preceq (1 - 2\rho_{\mathbf{e}}\alpha_{\mathbf{e}} + \rho_{\mathbf{e}}^2\beta_{\mathbf{e}}^2)||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}}||_{\mathbf{e}}^2 \tag{5.11}$$

and

$$||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger} - \rho_{\mathbf{e}^\dagger}\Lambda_{\mathbf{e}^\dagger}(\Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{n,\mathbf{e}^\dagger}) - \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}))||_{\mathbf{e}^\dagger}^2 \preceq (1 - 2\rho_{\mathbf{e}^\dagger}\alpha_{\mathbf{e}^\dagger} + \rho_{\mathbf{e}^\dagger}^2\beta_{\mathbf{e}^\dagger}^2)||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger}||_{\mathbf{e}^\dagger}^2. \tag{5.12}$$

Combining (5.9), (5.9), (5.10) (5.11), and (5.12), and using the $\mathbb{D}$-Lipschitz continuity of Φ, we see that

$$\begin{gathered}
||\mathfrak{h}_{n+1} - \mathfrak{h}||_{\mathbb{D}} \preceq \left[\left\{\left(2\sqrt{1 - 2\delta_{\mathbf{e}} + \sigma_{\mathbf{e}}^2}\right) + \rho_{\mathbf{e}}\gamma_{\mathbf{e}} + \sqrt{1 - 2\alpha_{\mathbf{e}}\rho_{\mathbf{e}} + \rho_{\mathbf{e}}^2\beta_{\mathbf{e}}^2}\right\}||\mathfrak{h}_{n,\mathbf{e}} - \mathfrak{h}_{\mathbf{e}}||_{\mathbf{e}}\right]\mathbf{e} \\
\left[\left\{\left(2\sqrt{1 - 2\delta_{\mathbf{e}^\dagger} + \sigma_{\mathbf{e}^\dagger}^2}\right) + \rho_{\mathbf{e}^\dagger}\gamma_{\mathbf{e}^\dagger} + \sqrt{1 - 2\alpha_{\mathbf{e}^\dagger}\rho_{\mathbf{e}^\dagger} + \rho_{\mathbf{e}^\dagger}^2\beta_{\mathbf{e}^\dagger}^2}\right\}||\mathfrak{h}_{n,\mathbf{e}^\dagger} - \mathfrak{h}_{\mathbf{e}^\dagger}||_{\mathbf{e}^\dagger}\right]\mathbf{e}^\dagger \\
= \left\{(2\sqrt{1 - 2\delta + \sigma^2}) + \rho\gamma + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\right\}||\mathfrak{h}_n - \mathfrak{h}||_{\mathbb{D}} \\
= \{k + \rho\gamma + t(\rho)\}||\mathfrak{h}_n - \mathfrak{h}||_{\mathbb{D}} = \Theta||\mathfrak{h}_n - \mathfrak{h}||_{\mathbb{D}},
\end{gathered}$$

where we set:

$$\begin{aligned} k &= 2\sqrt{1-2\delta+\sigma^2}, \\ t(\rho) &= \sqrt{1-2\alpha\rho+\rho^2\beta^2}, \\ \theta &= k+\rho\gamma+t(\rho). \end{aligned}$$

The function $t(\rho)$ attains its minimum at $\overline{\rho} = \alpha/\beta^2$, yielding $t(\overline{\rho}) = \sqrt{1-\alpha^2/\beta^2}$. To ensure convergence, we require $\theta \prec 1$, which is satisfied under the conditions:

$$\begin{aligned} k &\prec 1, \\ \alpha &\succ \gamma(1-k)+\sqrt{(\beta^2-\gamma^2)k(2-k)}. \end{aligned}$$

This guarantees that $\theta \prec 1$ for all ρ satisfying

$$\left|\rho - \frac{\alpha+\gamma(k-1)}{\beta^2-\gamma^2}\right|_{hyp} \prec \frac{\sqrt{(\alpha+\gamma(k-1))^2-(\beta^2-\gamma^2)k(2-k)}}{\beta^2-\gamma^2},$$

with $k \prec 1$, $\alpha \succ \gamma(1-k)+\sqrt{(\beta^2-\gamma^2)k(2-k)}$, and $\gamma(1-k) \prec \alpha$. Since $\theta \prec 1$, the fixed point problem (5.7) has a unique solution $\mathfrak{h}$, implying the sequence $\mathfrak{h}_{n+1}$ generated by (5.8) strongly converges to the unique solution of (3.1). □

Now we introduce module variational inequalities:

Theorem 5.7. *Let $\mathfrak{X}_{\mathbb{D}}$ be a topological $\mathbb{D}$-module and $\mathcal{B} = \mathcal{B}_{\mathbf{e}}\mathbf{e} + \mathcal{B}_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ be a product type $\mathbb{D}$-compact, $\mathbb{D}$-convex subset of $\mathfrak{X}_{\mathbb{D}}$ and suppose $\Psi : \mathcal{B} \longrightarrow \mathfrak{X}^*_{\mathbb{D}}$ be product-type continuous. Then there exist a solution $\mathfrak{h} \in \mathcal{B}$ such that*

$$\langle \Psi(\mathfrak{g}), \mathfrak{g}-\mathfrak{h}\rangle_{\mathbb{D}} \succeq 0, \forall \mathfrak{g} \in \mathcal{B}$$

if and only if , there exist a solution $\mathfrak{h}_{\mathbf{e}} \in \mathcal{B}_{\mathbf{e}}$ and $\mathfrak{h}_{\mathbf{e}^\dagger} \in \mathcal{B}_{\mathbf{e}^\dagger}$ such that

$$\langle \Psi_{\mathbf{e}}(\mathfrak{h}_{\mathbf{e}}), \mathfrak{g}_{\mathbf{e}}-\mathfrak{h}_{\mathbf{e}}\rangle_{\mathbf{e}} \geq 0, \forall \mathfrak{g}_{\mathbf{e}} \in \mathcal{B}_{\mathbf{e}}$$

and

$$\langle \Psi_{\mathbf{e}^\dagger}(\mathfrak{h}_{\mathbf{e}^\dagger}), \mathfrak{g}_{\mathbf{e}^\dagger}-\mathfrak{h}_{\mathbf{e}^\dagger}\rangle_{\mathbf{e}^\dagger} \geq 0, \forall \mathfrak{g}_{\mathbf{e}^\dagger} \in \mathcal{B}_{\mathbf{e}^\dagger},$$

respectively.

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