Methods of Functional Analysis and Topology
Vol. 32 (2026), no. 1, pp. 18–24
doi.org/10.31392/MFAT-npu26_1.2026.03

MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

LINEAR MAPS PRESERVING PARTIAL ISOMETRIES AND OPERATOR PAIRS WHOSE PRODUCTS ARE PROJECTIONS

MOHAMED AMINE AOUICHAOUI

Abstract. Let $\mathcal{H}$ be a complex Hilbert space of dimension at least 3, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Based on results by Molnár, this paper revisits the problem addressed in [18], which characterizes surjective maps $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ that preserve the set of partial isometric operators in both directions. We focus exclusively on the linear case, rather than the more general additive case. Furthermore, we provide an alternative proof of the main result in [9] from a different point of view. Finally, we propose new directions for exploring maps that preserve higher-order partial isometric operators in both directions.

1. Introduction

Here $\mathcal{H}$ stands for a complex Hilbert space with dimension at least 3. Let $\mathcal{B}(\mathcal{H})$ and $\mathcal{PI}(\mathcal{H})$ denote the sets of all bounded linear operators and partial isometries on $\mathcal{H}$, respectively. For any operator $T \in \mathcal{B}(\mathcal{H})$, we use $\mathcal{N}(T)$, $\mathcal{R}(T)$, T^*, and $\|T\|$ to represent the kernel, range, adjoint operator, and operator norm of T, respectively. The unit circle $|z| = 1$ is denoted $\mathbb{T}$.

Linear preserver problems constitute a rich and active area of research within matrix and operator theory [11, 13, 16]. Researchers have developed a wide range of results and techniques in the study of linear preserver problems, particularly those concerning mappings that preserve important classes of matrices or operators. For example, Shi and Ji [18] characterized additive surjective maps that preserve partial isometric operators in both directions, while Ji and Gao studied those that preserve operator pairs whose products are projections [9]. More recently, the authors in [7] have further explored maps preserving certain orthogonality properties of operators on $\mathcal{B}(\mathcal{H})$. These developments motivate us to reconsider the previously addressed problems from a different point of view. In particular, based on results by Molnár, this paper revisits the problem addressed in [18]. We focus exclusively on the linear case, rather than the more general additive case. Using a classical result of Murray and von Neumann [14], this case appears as a special case of the general result in [18]. Specifically, before characterizing such maps, the authors in [18] demonstrated that the continuity assumption in Molnár's result [12, Theorem 1] could be omitted. Subsequently, they characterized additive surjective maps that preserve partial isometric operators in both directions. In our work, we demonstrate that for the linear case, the continuity of surjective linear maps that preserve partial isometries in both directions follows directly from a result by Murray and von Neumann [14, page 239]. It is noteworthy that the approach presented in this paper is distinct from that employed in the finite-dimensional case, as discussed in [10, Theorem 3.3], and also differs from the methodology used in [6, Theorem 3.3] which is restricted to separable complex Hilbert spaces. Furthermore, as a significant application, we offer a streamlined proof of the main result from [9]. The final section of the paper opens the door to the study of maps that preserve higher-order partial isometries in both directions.

2020 *Mathematics Subject Classification.* 47B49, 47B48.

Keywords. Linear preserver problems, partial isometry, projection, product of operators, decomposition of operators.

2. Linear maps preserving the set of partial isometries and Operator Pairs Whose Products Are Projections

We begin by recalling some fundamental concepts and definitions.

Notation 2.1. *In [12], Molnár introduced a partial order on $\mathcal{PI}(\mathcal{H})$: for $P, Q \in \mathcal{PI}(\mathcal{H})$, we write $P \leqslant Q$ if the following conditions hold:*
(1) $\mathcal{N}(P)^{\perp} \subseteq \mathcal{N}(Q)^{\perp}$,
(2) $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$,
(3) $P|_{\mathcal{N}(P)^{\perp}} = Q|_{\mathcal{N}(P)^{\perp}}$.

Definition 2.2. *A linear bijection on $\mathcal{H}$ that preserves the norm is called a unitary operator, while a conjugate-linear bijection on $\mathcal{H}$ that preserves the norm is called an anti-unitary operator.*

Definition 2.3. *A mapping $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to preserve:*
(1) *partial isometries, if for every $A \in \mathcal{PI}(\mathcal{H})$, the operator $\phi(A)$ is a partial isometry;*
(2) *partial isometries in both directions, if for every $A \in \mathcal{B}(\mathcal{H})$, the operator $\phi(A)$ is a partial isometry if and only if A is a partial isometry.*

Definition 2.4. *Two operators $A, B \in \mathcal{B}(\mathcal{H})$ are said to be orthogonal if $A^*B = AB^* = 0$. This condition implies that the ranges of A and B, as well as the orthogonal complements of their kernels, are mutually orthogonal.*

The following result by Molnár, which is useful in the context of the orthogonality of partial isometries, can be found in [13].

Proposition 2.5. *Suppose that $T, S \in \mathcal{B}(\mathcal{H})$ are partial isometries. The operator $T + \lambda S$ is a partial isometry for every $\lambda \in \mathbb{T}$ if and only if T and S are orthogonal to each other.*

In the following, we also recall a fundamental result by Molnár [12, Theorem 1].

Theorem 2.6. *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$. Suppose that $\phi : \mathcal{PI}(\mathcal{H}) \to \mathcal{PI}(\mathcal{H})$ is a bijective transformation which preserves the partial ordering and the orthogonality between partial isometries in both directions. If ϕ is continuous (in the operator norm) at a single element of $\mathcal{PI}(\mathcal{H})$ different from 0, then ϕ can be written in one of the following forms:*
(1) *there exist unitaries U, V on $\mathcal{H}$ such that*
$$\phi(R) = URV \quad (R \in \mathcal{PI}(\mathcal{H}))$$
(2) *there exist anti-unitaries U, V on $\mathcal{H}$ such that*
$$\phi(R) = URV \quad (R \in \mathcal{PI}(\mathcal{H}))$$
(3) *there exist unitaries U, V on $\mathcal{H}$ such that*
$$\phi(R) = UR^*V \quad (R \in \mathcal{PI}(\mathcal{H}))$$
(4) *there exist anti-unitaries U, V on $\mathcal{H}$ such that*
$$\phi(R) = UR^*V \quad (R \in \mathcal{PI}(\mathcal{H}))$$

It is important to recall a remark made by Molnár after the proof of [12, Theorem 1], in which he suggested that a significant advancement would be to determine whether the result holds without the continuity assumption, if that is indeed the case. In [18], the authors made this advancement and subsequently provided a characterization of additive surjective maps that preserve partial isometries in both directions. In the present work, we focus exclusively on the linear case and demonstrate that, in this context, the continuity of such maps follows directly from a classical result by Murray and von Neumann [14, page 239].

Theorem 2.7. *Let $\mathcal{H}$ be a complex Hilbert space with dimension at least 3, and let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a surjective linear map. The map ϕ preserves partial isometries in both directions if and only if it takes one of the following explicit forms:*

(1) *There exist unitary operators U and V on $\mathcal{H}$ such that*
$$\phi(R) = URV, \quad \forall R \in \mathcal{B}(\mathcal{H}).$$
(2) *There exist anti-unitary operators U and V on $\mathcal{H}$ such that*
$$\phi(R) = UR^*V, \quad \forall R \in \mathcal{B}(\mathcal{H}).$$

Proof. We observe that the sufficiency is evident. Thus, we proceed to prove the necessity. We begin by establishing the continuity of ϕ. To this end, let $R \in \mathcal{B}(\mathcal{H})$ be a Hermitian operator with $\|R\| \leq 1$. As noted in [14, page 239], R can be expressed as
$$R = \frac{1}{2}(U + U^*), \text{ for some unitary operator } U.$$
Since both $\phi(U)$ and $\phi(U^*)$ are partial isometries, we deduce the following estimate for the norm of $\phi(R)$:
$$\|\phi(R)\| \leq \frac{1}{2}(\|\phi(U)\| + \|\phi(U^*)\|) \leq 1.$$
For any operator $R \in \mathcal{B}(\mathcal{H})$, it can be expressed as $R = A + \mathrm{i}B$, where A and B are Hermitian operators. Specifically, $A = \frac{R+R^*}{2}$ and $B = \frac{R-R^*}{2i}$, representing the real and imaginary parts of R, respectively. Using this decomposition, we can easily conclude that $\|\phi(R)\| \leq 2$ whenever $\|R\| \leq 1$, thereby proving the continuity of ϕ.

Now, we observe that ϕ is a bijection on $\mathcal{B}(\mathcal{H})$. Indeed, if $\phi(T) = 0$, then T must be a partial isometry. For any scalar $\lambda \in \mathbb{C}$, we have $\phi(\lambda T) = 0$, implying that λT is also a partial isometry. Hence, $T = 0$, which shows that ϕ is injective. Since ϕ is assumed to be surjective, it follows that ϕ is a bijection on $\mathcal{B}(\mathcal{H})$, (and consequently also a bijection on $\mathcal{PI}(\mathcal{H})$).

Next, by assumption, $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a linear map that preserves partial isometries in both directions. Therefore, by Proposition 2.5, for all $T, S \in \mathcal{PI}(\mathcal{H})$, the following equivalences hold:
$$\begin{aligned} T \perp S &\Leftrightarrow T + \lambda S \in \mathcal{PI}(\mathcal{H}), \quad \text{for all } \lambda \in \mathbb{T}, \\ &\Leftrightarrow \phi(T) + \lambda\phi(S) \in \mathcal{PI}(\mathcal{H}), \quad \text{for all } \lambda \in \mathbb{T}, \\ &\Leftrightarrow \phi(T) \perp \phi(S). \end{aligned}$$
Thus, the map $\phi : \mathcal{PI}(\mathcal{H}) \to \mathcal{PI}(\mathcal{H})$ preserves the orthogonality between partial isometries in both directions.

The result now follows from Theorem 2.6 in the same manner as in [18]. For the sake of clarity and completeness, we provide the remaining details below.

Let T and S be partial isometries such that $T \leq S$. This implies that
$$(S-T)x = Sx, \quad \forall x \in \mathcal{N}(S)^\perp \ominus \mathcal{N}(T)^\perp, \quad \text{and} \quad (S-T)x = 0, \quad \forall x \in \mathcal{N}(T)^\perp \oplus \mathcal{N}(S).$$
Hence, $S - T$ is a partial isometry (with initial space $\mathcal{N}(S)^\perp \ominus \mathcal{N}(T)^\perp$). This also implies that $T \perp (S - T)$. Consequently, we can express S as $S = T + (S - T)$, with $T \perp (S - T)$. Since ϕ preserves the orthogonality of partial isometries, it follows that:
$$\phi(S) = \phi(T) + \phi(S - T) \quad \text{and} \quad \phi(T) \perp \phi(S - T).$$

This leads to the conclusion that $\phi(T) \leq \phi(S)$. Moreover, since ϕ^{-1} shares the same properties as ϕ, we conclude that ϕ preserves the partial ordering between partial isometries in both directions.

As a result, ϕ, when restricted to $\mathcal{PI}(\mathcal{H})$, is a continuous linear bijection that preserves both the partial ordering and orthogonality between partial isometries in both directions. Hence, ϕ can be represented in either the first or the fourth form of Theorem 2.6.

Without loss of generality, let us assume that ϕ can be written in the form described in (1) of Theorem 2.6. Actually, there exist unitary operators U and V on $\mathcal{H}$ such that for all $R \in \mathcal{PI}(\mathcal{H})$,

$$\phi(R) = URV.$$

It is a well-established result that any bounded linear operator $T \in \mathcal{B}(\mathcal{H})$ can be expressed as a finite linear combination of partial isometries. Indeed, since $T = T_1 + iT_2$, where T_1 and T_2 are Hermitian, it will be sufficient to show this for Hermitian T's. In this case, the result follows from the spectral theorem in finite-dimensional spaces [8, Theorem 2.5.6] and from [15, Theorem 3] in both separable and non-separable infinite-dimensional spaces.

By the linearity of ϕ, one obtains

$$\phi(T) = UTV, \quad \forall\, T \in \mathcal{B}(\mathcal{H}),$$

which confirms that ϕ acts according to the form (1) described in Theorem 2.7. The other case can be derived similarly, leading to the corresponding conclusion. Thus, we have completed the proof. □

Let ϕ be a map on $\mathcal{B}(\mathcal{H})$. We shall say that ϕ preserves operator pairs whose products are non-zero projections if for any $A, B \in \mathcal{B}(\mathcal{H})$, $\phi(A)\phi(B)$ is a non-zero projection whenever AB is a non-zero projection. Moreover, if for any $A, B \in \mathcal{B}(\mathcal{H})$, $\phi(A)\phi(B)$ is a non-zero projection if and only if AB is a non-zero projection, we say that ϕ preserves operator pairs whose products are non-zero projections in both directions.

We show below how [9, Theorem 2.1] follows under the additional assumption that ϕ is linear, by means of Theorem 2.7. Our idea of proof is almost entirely different from the original ones given by Ji and Gao, and may facilitate new methods for treating these kinds of problems.

Theorem 2.8. *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geqslant 3$, and let ϕ be a linear surjective map on $\mathcal{B}(\mathcal{H})$. Then ϕ preserves operator pairs whose products are non-zero projections in both directions if and only if there exist a unitary or anti-unitary operator U on $\mathcal{H}$ and a constant λ with $\lambda^2 = 1$ such that $\phi(A) = \lambda U^* A U$, for all $A \in \mathcal{B}(\mathcal{H})$.*

Proof. We first observe that the sufficiency of the result is evident. Therefore, we proceed to prove the necessity of the theorem.

Let us begin by proving that $\phi(R^*) = \phi(R)^*$ for all $R \in \mathcal{B}(\mathcal{H})$. Since ϕ is linear and $R = R_1 + iR_2$, where R_1 and R_2 are Hermitian operators, it suffices to show this for Hermitian R's. Let R be a Hermitian operator in $\mathcal{B}(\mathcal{H})$. According to [15, Theorem 3], R can be written as a finite sum

$$R = \sum_k \lambda_k P_k,$$

where P_k are projections and λ_k are real scalars. Therefore, we have

$$\phi(R) = \sum_k \lambda_k \phi(P_k).$$

Since ϕ preserves operator pairs whose products are non-zero projections in both directions, it follows that $\phi(I) = \pm I$, and hence each $\phi(P_k)$ is Hermitian. This implies that

$$\phi(R)^* = \phi(R),$$

for every Hermitian operator R.

Now, since ϕ preserves operator pairs whose products are non-zero projections in both directions and satisfies $\phi(R^*) = \phi(R)^*$ for all $R \in \mathcal{B}(\mathcal{H})$, the following equivalences hold:

$$\begin{aligned} T \in \mathcal{PI}(\mathcal{H}) \setminus \{0\} &\iff TT^* \text{ is a non-zero projection} \\ &\iff \phi(T)\phi(T^*) \text{ is a non-zero projection} \\ &\iff \phi(T)\phi(T)^* \text{ is a non-zero projection} \\ &\iff \phi(T) \in \mathcal{PI}(\mathcal{H}) \setminus \{0\}. \end{aligned}$$

Moreover, since $\phi(0) = 0$ and ϕ is injective (see Corollary 2.3. in [9]), it follows from the above equivalence that ϕ preserves partial isometries in both directions. Consequently, ϕ must take one of the two forms described in Theorem 2.7. Finally, noting that $\phi(I) = \pm I$, the proof is complete. ☐

3. Linear Maps Preserving Higher-Order Partial Isometries

The class of m-isometries was introduced by Agler in [1], and further developments and properties of this class were explored by Agler and Stankus in their trilogy [2, 3, 4]. The case when $m = 1$ has garnered significant attention from operator theorists due to the centrality of isometry theory and its broad applications. Among the most significant classes of linear operators on Hilbert spaces are partial isometries. This class provides a generalization of that of isometries. In [17], Saddi and Sid Ahmed introduced the concept of m-partial isometries as a natural generalization of partial isometries, which are recovered in the case $m = 1$. Recently, the author of this paper presented an additional extension in [5]. These developments constitute substantial generalizations of m-isometries, advancing the theoretical framework and deepening our understanding of this class of operators.

For a positive integer m, we shall say that an operator $T \in \mathcal{B}(\mathcal{H})$ is:

(1) an *m-isometry* if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0.$$

(2) an *m-partial isometry* if

$$T \sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k} T^{m-k} = 0.$$

An interesting example of an m-partial isometry, which is not an m-isometry, arises naturally in the context of weighted shifts.

Example 3.1. *Let $(e_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$. For a bounded sequence of complex numbers $(\omega_n)_{n\geq 1}$, consider the associated weighted shift operator T on $\mathcal{H}$ defined by*

$$Te_n = \omega_n e_{n+1}, \quad \text{for all } n \geq 1.$$

We observe that

$$T^k e_n = \left(\prod_{n \leq j \leq k+n-1} \omega_j \right) e_{n+k}, \quad \forall\, k \geq 1.$$

Consequently,

$$T^{*k}e_n = \begin{cases} 0, & \text{if } n \leq k, \\ \left(\prod_{n-k\leq j\leq n-1} \overline{\omega_j}\right) e_{n-k}, & \text{if } n > k. \end{cases}$$

Therefore,

$$T^{*k}T^k e_n = \left(\prod_{n\leq j\leq k+n-1} |\omega_j|^2\right) e_n.$$

Thus, T is an m-partial isometry if and only if, for any integer $n \geq 1$, the following condition holds:

$$\omega_n \left((-1)^m + \sum_{1\leq k\leq m} (-1)^{m-k}\binom{m}{k}\left(\prod_{n\leq j\leq k+n-1} |\omega_j|^2\right)\right) = 0.$$

Note that in order for T to be a m-partial isometry that is not an m-isometry, it is necessary to allow weights to be zero for some indices. □

A natural question that remains open is the following.

Question. Let m be a positive integer, and let $\mathcal{H}$ be a complex Hilbert space of dimension at least 3. Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bijective continuous linear map. Assume that ϕ preserves m-partial isometries in both directions. Must there exist either two unitary operators U and V, or two anti-unitary operators U and V, on $\mathcal{H}$ such that

$$\phi(R) = URV \qquad \text{for all } R \in \mathcal{B}(\mathcal{H})?$$

It would be a nice improvement of Theorem 2.7 to show that the answer to this question is affirmative.

Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of interest

The author declares that there are no conflicts of interest.

Acknowledgements

The author would like to sincerely thank the reviewer for their careful reading of the manuscript and for providing valuable suggestions and comments, which helped to improve the clarity and quality of this work.

References

[1] J. Agler, *A disconjugacy theorem for Toeplitz operators*, Am. J. Math. **112** (1990), no. 1, 1–14 (English), doi:10.2307/2374849.

[2] J. Agler and M. Stankus, *m-isometric transformations of Hilbert space. I*, Integral Equations Oper. Theory **21** (1995), no. 4, 383–429 (English), doi:10.1007/BF01222016.

[3] J. Agler and M. Stankus, *m-isometric transformations of Hilbert space. II*, Integral Equations Oper. Theory **23** (1995), no. 1, 1–48 (English), doi:10.1007/BF01261201.

[4] J. Agler and M. Stankus, *m-isometric transformations of Hilbert space. III*, Integral Equations Oper. Theory **24** (1996), no. 4, 379–421 (English), doi:10.1007/BF01191619.

[5] M. A. Aouichaoui, *$(m, \mathcal{N}_A)$-isometries and their harmonious applications to Hilbert-Schmidt operators*, Linear Algebra Appl. **697** (2024), 309–331 (English), doi:10.1016/j.laa.2024.01.008.

[6] A. Chahbi and S. Kabbaj, *On linear maps that preserve G-partial-isometries in Hilbert space*, Proyecciones **33** (2014), no. 4, 405–413 (English), doi:10.4067/S0716-09172014000400004.

[7] J. Han, W. Shi, and G. Ji, *Maps preserving certain orthogonality of operators on $\mathcal{B}(\mathcal{H})$*, Ann. Funct. Anal. **16** (2025), no. 2, 13 (English), Id/No 16, doi:10.1007/s43034-025-00406-8.

[8] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge university press, 2012.

[9] G. Ji and Y. Gao, *Maps preserving operator pairs whose products are projections*, Linear Algebra Appl. **433** (2010), no. 7, 1348–1364 (English), doi:10.1016/j.laa.2010.05.014.

[10] C. K. Li, M. C. Tsai, Y. S. Wang, and N. C. Wong, *Nonsurjective maps between rectangular matrix spaces preserving disjointness, triple products, or norms*, J. Oper. Theory **83** (2020), no. 1, 27–53 (English), doi:10.7900/jot.2018may14.2238.

[11] C. K. Li and N. K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. **162-164** (1992), 217–235 (English), doi:10.1016/0024-3795(92)90377-M.

[12] L. Molnár, *On certain automorphisms of sets of partial isometries.*, Arch. Math. **78** (2002), no. 1, 43–50 (English), doi:10.1007/s00013-002-8215-3.

[13] L. Molnár, *Selected preserver problems on algebraic structures of linear operators and on function spaces*, Lect. Notes Math., vol. 1895, Berlin: Springer, 2007 (English), doi:10.1007/3-540-39944-5.

[14] F. J. Murray and J. von Neumann, *On rings of operators. II.*, Trans. Am. Math. Soc. **41** (1937), 208–248 (English), doi:10.1090/S0002-9947-1937-1501899-4.

[15] C. Pearcy and D. Topping, *Sums of small numbers of idempotents*, Mich. Math. J. **14** (1967), 453–465 (English), doi:10.1307/mmj/1028999848.

[16] S. Pierce, M. H. Lim, R. Loewy, C. K. Li, N. K. Tsing, B. McDonald, and L. Beasley, *A survey of linear preserver problems*, Linear Multilinear Algebra **33** (1992), no. 1-2, 1–129 (English).

[17] A. Saddi and O. A. M. S. Ahmed, *m-partial isometries on Hilbert spaces*, Int. J. Funct. Anal. Oper. Theory Appl. **2** (2010), no. 1, 67–83 (English), pphmj.com/abstract/5091.htm.

[18] W. Shi and G. Ji, *Additive maps preserving partial isometries on $B(H)$*, J. Shaanxi Norm. Univ., Nat. Sci. Ed. **43** (2015), no. 5, 14–17 (Chinese), doi:10.15983/j.cnki.jsnu.2015.05.154.

Mohamed Amine Aouichaoui: amine.aouichaoui@fsm.rnu.tn, amine.aouichaoui@fsm.u-monastir.tn
Department of Mathematics, University of Monastir, Faculty of Sciences of Monastir, Monastir, 5019, Tunisia

Received 18/05/2025; Revised 04/11/2025