Methods of Functional Analysis and Topology
Vol. 32 (2026), no. 1, pp. 25–34
doi.org/10.31392/MFAT-npu26_1.2026.04

EXISTENCE OF SOLUTIONS FOR LOWER SEMI-CONTINUOUS NON-CONVEX DIFFERENTIAL INCLUSIONS WITH ϕ−LAPLACIAN

NAJIB ASKOURAYE AND MYELKEBIR AITALIOUBRAHIM

Abstract. We show the existence of solutions satisfying Cauchy or terminal boundary conditions for first order differential inclusion $(\phi(x(t)))' \in F(t, x(t))$. We consider the second order problem $(\phi(x'(t)))' \in F(t, x(t))$ with many boundary conditions. The set-valued map F has non-convex values and the function ϕ satisfies a weak condition. The resolution method use the topological degree without the method of upper and lower solutions.

1. Introduction

In this paper, we shall prove the existence of solutions to the following differential inclusions:

$$\begin{cases} \big(\phi(x(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x(0) = r; \end{cases} \tag{1.1}$$

$$\begin{cases} \big(\phi(x(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x(T) = r; \end{cases} \tag{1.2}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x(0) = r,\ x(T) = r'; \end{cases} \tag{1.3}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x(0) = r,\ x'(0) = r'; \end{cases} \tag{1.4}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x(T) = r,\ x'(T) = r'; \end{cases} \tag{1.5}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x'(0) = r,\ x(T) = r'; \end{cases} \tag{1.6}$$

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x(0) = r,\ x'(T) = r', \end{cases} \tag{1.7}$$

where F is a multi-valued map, $\phi :]-a, a[\to \mathbb{R}$ is a function and $(r, r') \in \mathbb{R}^2$. We recall here that the ϕ-Laplacian operator is defined as a generalization of the following $p-$Laplacian operator $\phi_p(s) = |s|^{p-2}\, s$ with $p > 1$ and $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

Various studies have focused on the existence of solutions to differential inclusions under diverse conditions and boundary constraints. Ferracuti and Papalini [8], have examined the scalar differential inclusion

$$\left(D(u(t))\Phi\left(u'(t)\right)\right)' \in G\left(t, u(t), u'(t)\right) \quad \text{a.e. } t \in I = [0, T],$$

2020 *Mathematics Subject Classification.* 34A60, 34B10, 34B15.
Keywords. Differential inclusion, boundary value problem, topological degree, lower semi-continuous multifunction, measurability.

where $D(u)$ is a positive continuous function, $G(t,u,u')$ is a Carathéodory multifunction and Φ is an increasing homeomorphism that may either possess a bounded domain such as $]-a,a[$ or represent the p-Laplacian operator. By utilizing fixed-point methods, sometimes in combination with the method of lower and upper solutions, they have demonstrated the existence of solutions that satisfy various boundary conditions.

Jebelean and Serban [11], have investigated the existence results to systems of differential inclusions of the form $-\big(\phi(x'(t))\big)' \in \partial f(t,x(t))$, under Dirichlet, periodic and Neumann boundary conditions. Here, $\partial f(t,x)$ denotes the generalized Clarke gradient of $f(t,\cdot)$ at $x \in \mathbb{R}^n$.

Recently, Papageorgiou and other authors, see [12], have considered the nonlinear multivalued Duffing system

$$\begin{cases} -a(x'(t))' - r(t)|x'(t)|^{p-2}x'(t) \in F(t,x(t)) & \text{a.e. on } [0,T], \\ x(0) = x(T) = 0, \quad 1 < p < +\infty, \end{cases}$$

where $r : [0,T] \to \mathbb{R}^N$ is a function and $F : [0,T] \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a multifunction with convex or nonconvex values.

More recently, in [4], Aitalioubrahim and Ayyadi have solved the problem

$$\big(\phi(x(t))\big)' \in F(t,x(t)),$$

satisfying periodic or initial boundary conditions, where F is lower semi-continuous with respect to the second argument and $\phi :]a,b[\to \mathbb{R}$, $-\infty \le a < b \le +\infty$, is an increasing homeomorphism. Their work is based on the existence of upper and lower solutions and on the topological degree. By the same method and hypotheses, Aitalioubrahim and Tebbaa, see [3, 5], have proved the existence of solutions for the problem $\big(\phi(x'(t))\big)' \in F(t,x(t))$ satisfying periodic, Dirichlet, Cauchy or terminal boundary conditions. For other results regarding this type of problem see [1, 2].

Using the topological degree, we study the existence results for Problems (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) without the method of upper and lower solutions. The right hand side is lower semi-continuous with respect to the second argument. ϕ is not necessary increasing for (1.1), (1.2), (1.4), (1.5), (1.6) and (1.7). These results extend the ones in [3, 4, 5].

2. Preliminaries

In this section, we introduce definitions, notations and preliminary which are used throughout this paper. Let E be a Banach space equipped with the norm $\|.\|$. By $\mathcal{C}([0,T],E)$, we denote the Banach space of all continuous functions from $[0,T]$ into E equipped with the norm

$$\|x\|_\infty := \sup\big\{\|x(t)\|; t \in [0,T]\big\}.$$

$\mathcal{C}^1([0,T],E)$ denotes the Banach space of continuously differentiable functions on $[0,T]$ and $L^1([0,T],\mathbb{R})$ refers to the Banach space of Lebesgue integrable functions from $[0,T]$ into $\mathbb{R}$. We say that a subset U of $[0,T] \times \mathbb{R}$ is $\mathcal{L} \otimes \mathcal{B}$ -measurable if U belongs to the σ-algebra generated by all sets of the form $I \times X$ where I is Lebesgue measurable in $[0,T]$ and X is Borel measurable in $\mathbb{R}$. A multifunction is said to be measurable if its graph is measurable.

Definition 2.1. A subset U of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v \in U$ and $I \subset [0,T]$ measurable, the function $u\chi_I + v\chi_{[0,T]\setminus I} \in U$, where χ denotes the characteristic function.

Definition 2.2. Let E be a separable Banach space, X a nonempty closed subset of E and $G : X \to 2^E$ a multi-valued map with nonempty closed values. We say that G is lower semi-continuous if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set C in E.

Now, let $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ be a multi-valued map with nonempty compact values. Assign to F the multi-valued operator,

$$\mathcal{F} : \mathcal{C}([0,T],\mathbb{R}) \to 2^{L^1([0,T],\mathbb{R})}$$

defined by

$$\mathcal{F}(x) = \left\{ y \in L^1([0,T],\mathbb{R}) : y(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T] \right\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F. We say that F is of the lower semi-continuous type if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values. In this context, we need the following lemma.

Lemma 2.3. [7] *Let E be a separable metric space and $\Gamma : E \to 2^{L^1([0,T],\mathbb{R})}$ a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then Γ has a continuous selection, i.e. there exists a continuous function $g : E \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \Gamma(y)$ for every $y \in E$.*

In the sequel, we need the concept of compact and completely continuous function.

Definition 2.4. Let X and Y be topological spaces. A function $f : X \to Y$ is said compact if $f(X)$ is relatively compact. If X is a metric space, $f : X \to Y$ is called completely continuous if for any bounded subset $B \subset X$, $f(B)$ is relatively compact.

We will use in this work the following assumptions.

- (H1) $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued map with nonempty compact values satisfying
  - (i) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}-$measurable,
  - (ii) $x \mapsto F(t,x)$ is lower semi-continuous for almost all $t \in [0,T]$.
- (H2) There exists a function $m \in L^1([0,T],\mathbb{R}^+)$ such that for almost all $t \in [0,T]$ and all $x \in \mathbb{R}$
$$\big\|F(t,x)\big\| \le m(t).$$
- (H3) $\phi :]-a,a[\to \mathbb{R}$ is an increasing homeomorphism, where $0 < a < +\infty$.
- (H3)′ $\phi :]-a,a[\to \mathbb{R}$ is an homeomorphism, where $0 < a < +\infty$.

We denote by $W_a^{2,1}([0,T],\mathbb{R})$ the class of all functions $x \in C^1([0,T],\mathbb{R})$ such that $\|x'\|_\infty < a$, $\phi(x')$ is absolutely continuous. $W^{1,1}([0,T],\mathbb{R})$ denotes the Sobolev class of absolutely continuous functions $x : [0,T] \to \mathbb{R}$. In the sequel, we will use the following important lemmas.

Lemma 2.5. [9] *If assumptions (H1) and (H2) are satisfied, then F is of the lower semi-continuous type.*

Lemma 2.6. [5] *If assumption (H3) is satisfied, for any $c \in]-aT,aT[$ and $h \in \mathcal{C}([0,T],\mathbb{R})$, there exists an unique $\alpha = G_\phi(c,h) \in \mathbb{R}$ such that*

$$\int_0^T \phi^{-1}(h(t) - \alpha)dt = c.$$

Moreover, for all $c \in]-aT,aT[$, the function $\varphi : h \mapsto G_\phi(c,h)$ is continuous and sends bounded sets into bounded sets.

3. Existence results of BVPs (1.1)-(1.7)

The main results of this section concern the boundary value problems (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7), (BVP for short). Before we state and prove these results, we give the definition of a solution of BVP (1.1)-(1.7) and that of the topological degree. For more details on topological degree, we refer the reader to [6, 10].

Definition 3.1. A function $x : [0,T] \to \mathbb{R}$ is said to be solution of (1.1) (resp. (1.2)) if $x \in W^{1,1}([0,T],\mathbb{R})$ and x satisfies the conditions of (1.1) (resp. (1.2)). It is said to be solution of (1.3) (resp. (1.4), (1.5), (1.6), (1.7)) if $x \in W_a^{2,1}([0,T],\mathbb{R})$ and x satisfies the conditions of (1.3) (resp. (1.4), (1.5), (1.6), (1.7)).

Definition 3.2. Let E be a real Banach space and id be the identity on E. A degree is an application, which associates to any open bounded set $U \subset E$ and any continuous compact function $f : \overline{U} \to E$, an integer $\deg(id - f, U)$ with the following properties:

(i) (Existence): If $0 \notin (id - f)(\partial U)$, where ∂U is the boundary of U and $\deg(id - f, U) \neq 0$, then there exists $x \in U$ such that $x = f(x)$.

(ii) (Normalisation): If $0 \notin \partial U$, then $\deg(id, U) = 1$ if and only if $0 \in U$.

(iii) (Additivity): If $0 \notin (id - f)(\overline{U} \setminus U_1 \cup U_2)$, where U_1 and U_2 are disjoint open subsets of U and $\overline{U} \setminus U_1 \cup U_2$ is the relative complement of $U_1 \cup U_2$ in $\overline{U}$, then

$$\deg(id - f, U) = \deg(id - f, U_1) + \deg(id - f, U_2).$$

(iv) (Homotopy): If $H : [0,1] \times \overline{U} \to E$ is a continuous compact function such that $0 \notin (id - H(\lambda,\cdot))(\partial U)$, for every $\lambda \in [0,1]$, then

$$\deg(id - H(\lambda,\cdot), U) = \deg(id - H(0,\cdot), U), \quad \forall \lambda \in [0,1].$$

(v) (Excision): If $V \subset U$ is open and $0 \notin (id - f)(\overline{U} \setminus V)$, then

$$\deg(id - f, U) = \deg(id - f, V).$$

In the next, we will prove the following theorem.

Theorem 3.3. *Assume that (H1) and (H2) are hold.*

(i) *If (H3)′ is satisfied and* $r \in]-a, a[$*, then Problem (1.1) (resp. (1.2) and (1.6)) has at least one solution* x *on* $[0,T]$.

(ii) *If (H3) is satisfied and* $|r - r'| < aT$*, then Problem (1.3) has at least one solution* x *on* $[0,T]$.

(iii) *If (H3)′ is satisfied and* $r' \in]-a, a[$*, then Problem (1.4) (resp. (1.5) and (1.7)) has at least one solution* x *on* $[0,T]$.

Proof. By Lemma 2.5, F is of the lower semi-continuous type. Then, by Lemma 2.3, there exists a continuous function $g : \mathcal{C}([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in \mathcal{C}([0,T],\mathbb{R})$, where $\mathcal{F}$ is the Niemytzki operator associated with F. Now, consider the problems

$$\begin{cases} (\phi(y(t)))' = g(y)(t) \ \text{ a.e. on } [0,T]; \\ y(0) = r; \end{cases} \tag{3.8}$$

$$\begin{cases} (\phi(y(t)))' = g(y)(t) \ \text{ a.e. on } [0,T]; \\ y(T) = r; \end{cases} \tag{3.9}$$

$$\begin{cases} (\phi(u'(t)))' \ = \ g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(0) = r, \ u(T) = r'; \end{cases} \tag{3.10}$$

$$\begin{cases} (\phi(u'(t)))' \ = \ g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(0) = r, \ u'(0) = r'; \end{cases} \tag{3.11}$$

$$\begin{cases} (\phi(u'(t)))' \ = \ g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(T) = r, \ u'(T) = r'; \end{cases} \tag{3.12}$$

$$\begin{cases} (\phi(u'(t)))' \ = \ g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u'(0) = r, \ u(T) = r'; \end{cases} \tag{3.13}$$

and

$$\left\{ \begin{array}{lcl} (\phi(u'(t)))' & = & g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(0) = r, \ u'(T) = r'; \end{array} \right. \tag{3.14}$$

Remark that any solution of Problem (3.8) (resp. (3.9), (3.10), (3.11), (3.12), (3.13), (3.14)) is a solution of Problem (1.1) (resp. (1.2), (1.3), (1.4), (1.5), (1.6), (1.7)). For all $\lambda \in [0,1]$, consider the following modified problems

$$\left\{ \begin{array}{l} (\phi(y(t)))' = \lambda g(y)(t) \ \text{ a.e. on } [0,T]; \\ y(0) = r; \end{array} \right. \tag{3.15}$$

$$\left\{ \begin{array}{l} (\phi(y(t)))' = \lambda g(y)(t) \ \text{ a.e. on } [0,T]; \\ y(T) = r; \end{array} \right. \tag{3.16}$$

$$\left\{ \begin{array}{lcl} (\phi(u'(t)))' & = & \lambda g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(0) = r, \ u(T) = r'; \end{array} \right. \tag{3.17}$$

$$\left\{ \begin{array}{lcl} (\phi(u'(t)))' & = & \lambda g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(0) = r, \ u'(0) = r'; \end{array} \right. \tag{3.18}$$

$$\left\{ \begin{array}{lcl} (\phi(u'(t)))' & = & \lambda g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(T) = r, \ u'(T) = r'; \end{array} \right. \tag{3.19}$$

$$\left\{ \begin{array}{lcl} (\phi(u'(t)))' & = & \lambda g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u'(0) = r, \ u(T) = r'; \end{array} \right. \tag{3.20}$$

and

$$\left\{ \begin{array}{lcl} (\phi(u'(t)))' & = & \lambda g(u)(t), \quad \text{for a.e. } t \in [0,T]; \\ u(0) = r, \ u'(T) = r'; \end{array} \right. \tag{3.21}$$

Let us consider the operators

$$N_g : \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$$

and

$$\mathcal{H}_i : [0,1] \times \mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R}), \ i \in \{1,2,3,4,5,6,7\},$$

defined, for all $t \in [0,T]$, by

$$N_g(u)(t) = \int_0^t g(u)(s)ds,$$

for Problem (3.15),

$$\mathcal{H}_1(\lambda,u)(t) = \phi^{-1}\big(\phi(r) + \lambda N_g(u)(t)\big),$$

for Problem (3.16),

$$\mathcal{H}_2(\lambda,u)(t) = \phi^{-1}\big(\phi(r) + \lambda N_g(u)(t) - \lambda N_g(u)(T)\big),$$

for Problem (3.17),

$$\mathcal{H}_3(\lambda,u)(t) = r + \int_0^t \phi^{-1}\big(\lambda N_g(u)(s) - \alpha_{\lambda,u}\big)ds,$$

for Problem (3.18),

$$\mathcal{H}_4(\lambda,u)(t) = r + \int_0^t \phi^{-1}\big(\phi(r') + \lambda N_g(u)(s)\big)ds,$$

for Problem (3.19),

$$\mathcal{H}_5(\lambda,u)(t) = r + \int_T^t \phi^{-1}\big(\phi(r') + \lambda N_g(u)(s) - \lambda N_g(u)(T)\big)ds,$$

for Problem (3.20),

$$\mathcal{H}_6(\lambda,u)(t) = r' + \int_T^t \phi^{-1}\big(\phi(r) + \lambda N_g(u)(s)\big)ds$$

and for Problem (3.21),

$$\mathcal{H}_7(\lambda,u)(t) = r + \int_0^t \phi^{-1}\big(\phi(r') + \lambda N_g(u)(s) - \lambda N_g(u)(T)\big)ds.$$

Note here that $\alpha_{\lambda,u} = G_\phi\big(r'-r, \lambda N_g(u)\big)$ is uniquely determined by Lemma 2.6 and

$$\phi^{-1}(-\alpha_{0,u}) = \frac{r'-r}{T}, \quad \forall u \in \mathcal{C}([0,T],\mathbb{R}).$$

From the assumptions, the function $\mathcal{H}_i$ is continuous and completely continuous for all $i \in \{1,2,3,4,5,6,7\}$. Also, observe that, the fixed points of $\mathcal{H}_1(\lambda,.)$, $\mathcal{H}_2(\lambda,.)$, $\mathcal{H}_3(\lambda,.)$, $\mathcal{H}_4(\lambda,.)$, $\mathcal{H}_5(\lambda,.)$, $\mathcal{H}_6(\lambda,.)$ and $\mathcal{H}_7(\lambda,.)$ are respectively solutions of (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21). Indeed, for example, for Problem (3.15), if $u = \mathcal{H}_1(\lambda,u)$, then, for all $t \in [0,T]$, we have

$$u(t) = \phi^{-1}\left(\phi\big(r\big) + \lambda N_g(u)(t)\right).$$

Thus, in particular for $t=0$ one has $u(0) = r$. Also by derivation, we get

$$(\phi(u(t)))' = \lambda g(u)(t) \text{ for almost all } t \in [0,T].$$

Hence u is a solution of (3.15).

Proposition 3.4. *For all $i \in \{1,2,3,4,5,6,7\}$, there exists $R_i > 0$ such that*

$$\mathit{deg}(id - \mathcal{H}_i(\lambda,\cdot),\mathcal{U}_i) = 1$$

for every $\lambda \in [0,1]$, where $\mathcal{U}_i = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R_i\}$.

Proof. Fix $R_1 > a$, $R_2 > a$, $R_3 > |r| + aT$, $R_4 > |r| + aT$, $R_5 > |r| + 2aT$, $R_6 > |r'| + 2aT$ and $R_7 > |r| + aT$. First, for all $i \in \{1,2,3,4,5,6,7\}$, note that $\|u\|_\infty < R_i$ for any fixed point u of $\mathcal{H}_i(\lambda,.)$. Now, set

$$\mathcal{U}_i = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\|_\infty < R_i\}, \quad \forall i \in \{1,2,3,4,5,6,7\}.$$

For all $i \in \{1,2,3,4,5,6,7\}$, we have

$$u \neq \mathcal{H}_i(\lambda,u), \quad \forall (\lambda,u) \in [0,1] \times \partial\mathcal{U}_i.$$

Then, by the homotopy property of the topological degree, for all $i \in \{1,2,3,4,5,6,7\}$, one has

$$\deg(id - \mathcal{H}_i(\lambda,\cdot),\mathcal{U}_i) = \deg(id - \mathcal{H}_i(0,\cdot),\mathcal{U}_i), \quad \forall \lambda \in [0,1].$$

On the other hand, let $i \in \{1,2,3,4,5,6,7\}$. Set $A_i(t,u) = t\mathcal{H}_i(0,u)$, for all $t \in [0,1]$ and $u \in \overline{\mathcal{U}_i}$. For all $s \in [0,T]$, we have

$$|A_i(t,u)(s)| < R_i.$$

Hence, for every $t \in [0,1]$, one has $0 \notin \big(id - A_i(t,\cdot)\big)(\partial\mathcal{U}_i)$. Then

$$\deg(id - A_i(1,\cdot),\mathcal{U}_i) = \deg(id - A_i(0,\cdot),\mathcal{U}_i),$$

which gives

$$\deg(id - \mathcal{H}_i(\lambda,\cdot),\mathcal{U}_i) = \deg(id - \mathcal{H}_i(0,\cdot),\mathcal{U}) = \deg(id,\mathcal{U}_i) = 1, \quad \forall \lambda \in [0,1].$$

□

It follows from the last proposition that, for every $\lambda \in [0,1]$, for all $i \in \{1,2,3,4,5,6,7\}$, $H_i(\lambda,\cdot)$ has a fixed point. So, every problem among Problems (3.15), (3.16), (3.17), (3.18), (3.19), (3.20) and (3.21) has a solution for $\lambda = 1$. Hence every problem among Problems (1.1), (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) has at least one solution u on $[0,T]$.

□

4. Illustrative examples

Example 4.1. Consider the following ϕ-Laplacian differential inclusions with different boundary conditions

$$\begin{cases} \left(x(t)^5\right)' \in F(t,x(t)), & \text{a.e. on } [0,\frac{\pi}{2}];\\ x(0)=0; \end{cases} \tag{4.22}$$

$$\begin{cases} \left(x(t)^5\right)' \in F(t,x(t)), & \text{a.e. on } [0,\frac{\pi}{2}];\\ x(\frac{\pi}{2})=1; \end{cases} \tag{4.23}$$

where $F:[0,\frac{\pi}{2}]\times\mathbb{R}\to 2^{\mathbb{R}}$ is a multi-valued map defined by

$$F(t,x)=\{v\in\mathbb{R}: f_1(t,x)\le v\le f_2(t,x)\}, \forall (t,x)\in\left[0,\frac{\pi}{2}\right]\times\mathbb{R},$$

and $f_1, f_2:[0,\frac{\pi}{2}]\times\mathbb{R}\to\mathbb{R}$ are single-valued functions such that, for each $(t,x)\in\left[0,\frac{\pi}{2}\right]\times\mathbb{R}$,

$$f_1(t,x)=\frac{5\cos(t)\sin^4(t)(1+\sin^2(t))}{1+x^2}-1$$

and

$$f_2(t,x)=\frac{5\cos(t)\sin^4(t)(1+\sin^2(t))}{1+x^2}+1.$$

It is clear that F has nonempty compact values and is mesurable. For each $t\in[0,\frac{\pi}{2}]$, $f_1(t,.)$ and $f_2(t,.)$ are continuous on $\mathbb{R}$. Then, for each $t\in[0,\frac{\pi}{2}]$, $f_1(t,.)$ is upper semi-continuous on $\mathbb{R}$ and $f_2(t,.)$ is lower semi-continuous on $\mathbb{R}$. Hence, for each $t\in[0,\frac{\pi}{2}]$, $F(t,.)$ is lower semi-continuous on $\mathbb{R}$. Moreover, for almost all $t\in[0,\frac{\pi}{2}]$ and all $x\in\mathbb{R}$

$$\begin{aligned}\|F(t,x)\| &= \sup\Big\{|y| : y\in[f_1(t,x),f_2(t,x)]\Big\}\\ &\le max\left\{|f_1(t,x)|,|f_2(t,x)|\right\}\\ &\le \left|\frac{5\cos(t)\sin^4(t)(1+\sin^2(t))}{1+x^2}\right|+1\\ &\le 10\sin^4(t)+1.\end{aligned}$$

Set

$$m(t)=10\sin^4(t)+1,\quad \forall t\in\left[0,\frac{\pi}{2}\right].$$

Then, for almost all $t\in[0,\frac{\pi}{2}]$ and all $x\in\mathbb{R}$

$$\|F(t,x)\|\le m(t),$$

with $m\in L^1([0,\frac{\pi}{2}],\mathbb{R}^+)$. In this example $\phi(x)=x^5$. It is clear that ϕ is an homeomorphism. We conclude that all assumptions of Theorem 3.3 are verified, thus Problem (4.22) (resp. (4.23)) has a least one solution on $[0,\frac{\pi}{2}]$. Set

$$u(t)=\sin(t), \forall t\in[0,\frac{\pi}{2}].$$

u is a solution of Problem (4.22). Indeed, for almost all $t \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} \big(\phi(u(t))\big)' &= \big(u(t)^5\big)' \\ &= 5u'(t)u(t)^4 \\ &= 5\cos(t)\sin^4(t) \\ &= \frac{5\cos(t)\sin^4(t)(1+\sin^2(t))}{1+\sin^2(t)} \\ &= \frac{5\cos(t)\sin^4(t)(1+\sin^2(t))}{1+u(t)^2} \end{aligned}$$

Hence, for almost all $t \in [0, \frac{\pi}{2}]$

$$\big(\phi(u(t))\big)' \in [f_1(t,u(t)), f_2(t,u(t))] = F(t,u(t)).$$

In addition $u(0) = 0$. On the other hand, since $u(\frac{\pi}{2}) = 1$, u is also a solution of Problem (4.23).

Example 4.2. Consider the following ϕ-Laplacian differential inclusions with different boundary conditions

$$\begin{cases} \big(x'(t)^7 + x'(t)^5 + x'(t)^3\big)' \in F(t,x(t)), & \text{a.e. on } [0,2]; \\ x(0) = 1, \ x(2) = 5; \end{cases} \tag{4.24}$$

$$\begin{cases} \big(x'(t)^7 + x'(t)^5 + x'(t)^3\big)' \in F(t,x(t)), & \text{a.e. on } [0,2]; \\ x(0) = 1, \ x'(0) = 0; \end{cases} \tag{4.25}$$

$$\begin{cases} \big(x'(t)^7 + x'(t)^5 + x'(t)^3\big)' \in F(t,x(t)), & \text{a.e. on } [0,2]; \\ x(2) = 5, \ x'(2) = 4; \end{cases} \tag{4.26}$$

$$\begin{cases} \big(x'(t)^7 + x'(t)^5 + x'(t)^3\big)' \in F(t,x(t)), & \text{a.e. on } [0,2]; \\ x'(0) = 0, \ x(2) = 5; \end{cases} \tag{4.27}$$

and

$$\begin{cases} \big(x'(t)^7 + x'(t)^5 + x'(t)^3\big)' \in F(t,x(t)), & \text{a.e. on } [0,2]; \\ x(0) = 1, \ x'(2) = 4; \end{cases} \tag{4.28}$$

where $F : [0,2] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multi-valued map defined by

$$F(t,x) = \{v \in \mathbb{R} : f_1(t,x) \le v \le f_2(t,x)\}, \forall (t,x) \in [0,2] \times \mathbb{R},$$

and $f_1, f_2 : [0,2] \times \mathbb{R} \to \mathbb{R}$ are single-valued functions such that, for each $(t,x) \in [0,2] \times \mathbb{R}$,

$$f_1(t,x) = \frac{t^2(896t^4 + 160t^2 + 24)\left(2 + e^{(t^2+1)^2}\right)}{2 + e^{x^2}} - \frac{1}{2}$$

and

$$f_2(t,x) = \frac{t^2(896t^4 + 160t^2 + 24)\left(2 + e^{(t^2+1)^2}\right)}{2 + e^{x^2}} + \frac{1}{2}.$$

It is clear that F has nonempty compact values and is measurable. As above, for each $t \in [0,2]$, $F(t,.)$ is lower semi-continuous on $\mathbb{R}$. Moreover, for almost all $t \in [0,2]$ and all

$x \in \mathbb{R}$

$$\begin{aligned} \|F(t,x)\| &= \sup\Big\{|z| : z \in [f_1(t,x), f_2(t,x)]\Big\} \\ &\leq max\,\{|f_1(t,x)|, |f_2(t,x)|\} \\ &\leq \left|\frac{t^2(896t^4+160t^2+24)\left(2+e^{(t^2+1)^2}\right)}{2+e^{x^2}}\right| + \frac{1}{2} \\ &\leq \frac{t^2}{3}(896t^4+160t^2+24)\left(2+e^{(t^2+1)^2}\right)+\frac{1}{2}. \end{aligned}$$

Set

$$m(t) = \frac{t^2}{3}(896t^4+160t^2+24)\left(2+e^{(t^2+1)^2}\right)+\frac{1}{2}, \quad \forall t \in [0,2].$$

Then, for almost all $t \in [0,2]$ and all $x \in \mathbb{R}$

$$\|F(t,x)\| \leq m(t),$$

with $m \in L^1([0,2], \mathbb{R}^+)$. In this example $\phi(x) = x^7 + x^5 + x^3$. It is clear that ϕ is an homeomorphism. We conclude that all assumptions of Theorems 3.3 are verified, thus Problem (4.24) (resp. (4.25), (4.26), (4.27) and (4.28)) has a least one solution on $[0,2]$. Set

$$u(t) = t^2+1, \forall t \in [0,2].$$

u is a solution of Problem (4.24). Indeed, for almost all $t \in [0,2]$,

$$\begin{aligned} \big(\phi(u'(t))\big)' &= \big(u'(t)^7 + u'(t)^5 + u'(t)^3\big)' \\ &= 7u''(t)u'(t)^6 + 5u''(t)u'(t)^4 + 3u''(t)u'(t)^2 \\ &= t^2(896t^4+160t^2+24) \\ &= \frac{t^2(896t^4+160t^2+24)\left(2+e^{(t^2+1)^2}\right)}{2+e^{(t^2+1)^2}} \\ &= \frac{t^2(896t^4+160t^2+24)\left(2+e^{(t^2+1)^2}\right)}{2+e^{u(t)^2}}. \end{aligned}$$

Hence, for almost all $t \in [0,2]$

$$\big(\phi(u'(t))\big)' \in [f_1(t,u(t)), f_2(t,u(t))] = F(t,u(t)).$$

In addition $u(0) = 1$ and $u(2) = 5$. It is clear that u satisfies the conditions of Problems (4.25), (4.26), (4.27) and (4.28). Then u is a solution of Problem (4.25) (resp. (4.26), (4.27) and (4.28)).

Acknowledgements

The authors would like to thank the referee for his careful and thorough reading of the paper.

References

[1] M. Aitalioubrahim, N. Askouraye and A. Tebbaa, *Neumann boundary value problems for lower semi-continuous non-convex differential inclusions with ϕ-Laplacian*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **51** (2025), no. 1, 31–41, doi:10.30546/2409-4994.2025.51.1.1006.

[2] M. Aitalioubrahim and N. Askouraye, *On two and three and four point boundary value problems for differential inclusions with ϕ-Laplacian*, Miskolc Math. Notes **25** (2024), no. 2, 579–592, doi:10.18514/MMN.2024.4425.

[3] M. Aitalioubrahim and A. Tebbaa, *Boundary value problem for nonconvex differential inclusions with ϕ-Laplacian*, Discuss. Math. Differ. Incl. Control Optim. **43** (2023), 49–61, doi:10.7151/dmdico.1241.

[4] M. Aitalioubrahim and N. Ayyadi, *Existence of solutions for lower semi-continuous differential inclusions with the method of upper and lower solutions*, Nonlinear Stud. **25** (2018), no. 2, 333–343.

[5] M. Aitalioubrahim and A. Tebbaa, *Boundary value problems for differential inclusions with $\phi-$Laplacian*, J. Dyn. Control Syst. **28** (2022), no. 3, 505–516, doi:10.1007/s10883-021-09544-3.

[6] Y. J. Cho and Y. Q. Chen, *Topological degree theory and applications*, Chapman Hall/CRC Math., 2006, doi:10.1201/9781420011487.

[7] A. Colombo and G. Bressan, *Extensions and selections of maps with decomposable values*, Studia Math. **90** (1988), no. 1, 69–86, doi:10.4064/sm-90-1-69-86.

[8] L. Ferracuti and F. Papalini, *Boundary value problems for strongly non-linear multivalued equations involving different $\phi-$Laplacians*, Adv. Differential Equations. **14** (2009), no. 5-6, 541–566, doi:10.57262/ade/1355867259.

[9] M. Frigon and A. Granas, *Théorèmes d'existence pour des inclusions différentielles sans convexité*, C. R. Math. Acad. Sci. Paris, **310** (1990), 819-822.

[10] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monogr. Math. **14**, 2003, doi:10.1007/978-0-387-21593-8.

[11] P. Jebelean and C. Serban, *Boundary value problems for discontinuous perturbations of singular $\phi-$Laplacian operator*, J. Math. Anal. Appl. **431** (2015), no. 1, 662-681, doi:10.1016/j.jmaa.2015.06.004.

[12] N.S. Papageorgiou, C. Vetro and F. Vetro, *Nonlinear multivalued Duffing systems*, J. Math. Anal. Appl. **468** (2018), no. 1, 376–390, doi:10.1016/j.jmaa.2018.08.024.

Najib Askouraye: najib.ask22@gmail.com
University Sultan Moulay Slimane, Faculty polydisciplinary, BP 145, Khouribga, Morocco.

Myelkebir Aitalioubrahim: aitalifr@hotmail.com
University Sultan Moulay Slimane, Faculty polydisciplinary, BP 145, Khouribga, Morocco.

Received 30/12/2024; Revised 07/01/2026