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NEW BEST PROXIMITY POINT RESULTS FOR GENERALIZED MULTIVALUED $F-$PROXIMAL CONTRACTIONS IN PARTIAL METRIC SPACES WITH APPLICATION

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Abstract. In this paper, we introduce generalized multivalued $F-$proximal contraction mappings within the partial metric spaces framework and establish best proximity point results for such mappings. The best proximity point theorem for multivalued $F-$proximal contraction mappings involving $\alpha-$admissibility is also obtained. Several related results in the literature are unified and generalized by our new best proximity point results. We also provide nontrivial examples to support our findings. Finally, we derive an existence of a solution to an integral equation that validates our finding.

1. Introduction

The existence of best proximity points constitutes one of the central themes in optimization theory, particularly in situations where fixed points do not exist. Closely related to this topic is the broader objective of metric fixed point theory, which focuses on determining and estimating solutions to fixed point problems under various structural conditions. Best proximity point theory naturally extends classical fixed point results by providing meaningful solutions for mappings defined between nonintersecting subsets of metric spaces.

A significant advancement in this direction was made by Nadler in 1969 [23], who established fundamental fixed point theorems for multivalued mappings. Since then, the theory of multivalued mappings has attracted considerable attention due to its wide range of applications, particularly in approximation theory and optimization.

In another important development, Wardowski [28] introduced the concept of $F-$contraction mappings and proved a fixed point theorem in complete metric spaces, thereby extending the classical Banach contraction principle [8]. This innovative approach opened new avenues for generalizing contraction conditions through auxiliary control functions. For further developments and related results in this direction, we refer the reader to [1, 2, 4, 7, 11, 12, 14, 16, 19, 20, 22, 29]. Building upon these ideas, Ali et al. [3] introduced the notion of $\alpha - \psi-$proximal multivalued contraction mappings and established several best proximity point results, thereby enriching the theory by combining admissibility conditions with generalized contraction frameworks.

Subsequently, in 2015, Nazari [24] obtained best proximity point results for generalized multivalued contractions by employing the weak $p-$property, which further broadened the scope of the theory. For additional contributions and comprehensive discussions on best proximity point theory, we refer to [6, 9, 10, 13, 15, 25, 26].

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More recently, Jain et al. [17] established best proximity point results for multivalued generalized contractions in partially ordered metric spaces, highlighting the interplay between order structures and proximity theory. Shortly thereafter, Sahin et al. [27] derived best proximity point results for multivalued $F-$contraction mappings in complete metric spaces, further demonstrating the effectiveness of $F-$contractions in addressing proximity problems.

However, many real world problems such as in optimization, control theory, and nonlinear analysis arise in settings where the classical metric structure is too restrictive. In particular, distances may not satisfy the exact triangle inequality or may allow nonzero self-distances, as seen in data approximation and computational geometry. The framework of partial metric spaces provides a natural geometric environment for modeling such situations. At the same time, the concept of an F-proximal contraction generalizes the traditional contraction by introducing an auxiliary function F that controls the rate of contraction more flexibly. This combination allows a richer geometric interpretation: the F-function governs how the proximity between two nonintersecting sets evolves under the mapping, while the partial metric captures the relaxed distance behavior inherent in generalized spaces. Thus, studying F-proximal contractions in partial metric spaces is both conceptually and geometrically motivated, as it bridges gaps between classical fixed point results and practical applications where exact metric properties fail.

Motivated by these advancements, we establish new best proximity point results for generalized multivalued $F-$proximal contractions within the framework of partial metric spaces. The results obtained not only extend but also unify and generalize several existing best proximity point and fixed point theorems reported in the literature. To illustrate the applicability and effectiveness of our theoretical findings, we present concrete examples. Furthermore, as an application of our main results, we demonstrate the existence of a solution to a nonlinear integral equation.
For nonvoid set W, $CB(W)$ denotes the collection of nonvoid closed and bounded subsets of W and $K(W)$ be nonvoid compact subsets of W.

2. Preliminaries

In this section, we present some useful definitions and important results which will be used in the sequel.

Definition 2.1. [21] For nonvoid set W, a function $d_\rho : W \times W \to [0, \infty)$ is called partial metric if for all $x, y, z \in W$ such that the following conditions are satisfied:

(1) $x = y \; if \; and \; only \; if \; d_\rho(x, x) = d_\rho(x, y) = d_\rho(y, y)$;
(2) $d_\rho(x, x) \leq d_\rho(x, y)$;
(3) $d_\rho(x, y) = d_\rho(y, x)$;
(4) $d_\rho(x, y) \leq d_\rho(x, z) + d_\rho(z, y) - d_\rho(z, z)$.

The pair (W, d_ρ) is called a partial metric space.

Definition 2.2. [21] For partial metric space (W, d_ρ) and sequence $\{x_m\}$ in W; then,

(1) $\{x_m\} \to x \in W$ if $d_\rho(x_m, x) \to d_\rho(x, x), \; as \; m \to \infty$;
(2) $\{x_m\}$ is a Cauchy sequence if $\lim_{p,m\to\infty} d_\rho(x_m, x_p) \to \; exists \; and \; is \; finite, \; as$;
(3) (W, d_ρ) is complete if every Cauchy sequence $\{x_m\}$ in W converges to $x \in W$. Moreover,

$$\lim_{m,p\to\infty} d_\rho(x_m, x_p) = \lim_{m\to\infty} d_\rho(x_m, x) = d_\rho(x, x).$$

Example 2.3. Let $W = [0,1]$ and $x_p = \{\frac{1}{p} : p \geq 1\}$. Define $d_\rho : W \times W \to [0,\infty)$ *by* $d_\rho(x,y) = |x-y| + c, \ c > 0$. Obviously, (W, d_ρ) is a partial metric space. So,

$$\lim_{p\to\infty} d_\rho(x_p, 0) = \lim_{p\to\infty} d_\rho(\frac{1}{p}, 0) = \lim_{p\to\infty} \frac{1}{p} + c = d_\rho(0,0).$$

Thus, $\{x_p\}$ converges in W.

Definition 2.4. [28] Assume that the function $F : (0,\infty) \to (-\infty,\infty)$ satisfies the following conditions:

$(F1)$ F is strictly increasing;
$(F2)$ For each sequence $\{\alpha_m\}$ of positive numbers,

$$\lim_{m\to\infty} \alpha_m = 0, \ if \ and \ only \ if \ \lim_{m\to\infty} F(\alpha_m) = -\infty;$$

$(F3)$ There exists $\kappa \in (0,1)$ such that $\lim_{\beta\to 0^+} \beta^\kappa F(\beta) = 0$.

In this paper, we denote with $\digamma$ the set of all functions F that satisfy $(F1) - (F3)$.

Example 2.5. The following functions $F : (0,\infty) \to (-\infty,\infty)$ are in $\digamma$,
1) $F(t) = \ln(t^2 + t)$ 2) $F(t) = \ln t$ 3) $F(t) = \ln t + t$ 4) $F(t) = -\frac{1}{\sqrt{t}}$.

Hausdorff distance under the structure of partial metric spaces have been introduced by Aydi [5].

Definition 2.6. [5] Let (W, d_ρ) be a partial metric space. Let $E, G \in CB(W)$, the partial Hausdorff metric on $CB(W)$ induced by d_ρ is given as follows:

$$H_\rho(E,G) = \max\{D_\rho(E,G), D_\rho(G,E)\},$$

where

$$D_\rho(E,G) = \sup\{D_\rho(\varrho, G) : \varrho \in E\}$$

$$D_\rho(G,E) = \sup\{D_\rho(\upsilon, E) : \upsilon \in G\} \ and$$

$$D_\rho(\upsilon, E) = \inf\{d_\rho(\upsilon, \varrho) : \varrho \in E\}$$

$$d_\rho(E,G) = \inf\{d_\rho(\varrho, \upsilon) : \varrho \in E \ and \ \upsilon \in G\}.$$

Let E and G be nonvoid subsets of (W, d_ρ), we define

$$E_0 := \{\varrho \in E : d_\rho(\varrho, \upsilon) = d_\rho(E,G), \ for \ some \ \upsilon \in G\}$$

$$G_0 := \{\upsilon \in G : d_\rho(\varrho, \upsilon) = d_\rho(E,G), \ for \ some \ \varrho \in E\}.$$

Let $T : E \to CB(G)$ be a non-self multivalued mapping. An element ϱ in E is called a best proximity point of T if $D_\rho(\varrho, T\varrho) = d_\rho(E,G)$.

Definition 2.7. [18] Let E and G be two nonvoid subsets of a metric space (W, d_ρ) with E_0 is nonvoid. The pair (E,G) is said to have the weak $p-$property if and only if

$$\begin{aligned} d_\rho(\varrho_1, \nu_1) &= d_\rho(E,G) \\ d_\rho(\varrho_2, \nu_2) &= d_\rho(E,G) \\ \Rightarrow d_\rho(\varrho_1, \varrho_2) &\leq d_\rho(\nu_1, \nu_2), \end{aligned}$$

for all $\varrho_1, \varrho_2 \in E$ *and* $\nu_1, \nu_2 \in G$.

The contractive mappings shown below were first presented by Ali et al. [3].

Definition 2.8. [3] Let E and G be two nonvoid subsets of a metric space (W, d_ρ) and $\alpha : E \times E \to [0, \infty)$ be a function. Then $T : E \to 2^G$ *is* $\alpha-$proximal admissible mapping if

$$\begin{array}{lll} \alpha(\varrho_1, \varrho_2) & \geq 1 & \\ d_\rho(\upsilon_1, \omega_1) & = & d_\rho(E, G) \\ d_\rho(\upsilon_2, \omega_2) & = & d_\rho(E, G) \\ \Rightarrow \alpha(\upsilon_1, \upsilon_2) & \geq 1, & \end{array}$$

for all $\varrho_1, \varrho_2, \upsilon_1, \upsilon_2 \in E$ *and* $\omega_1 \in T\varrho_1$ *and* $\omega_2 \in T\varrho_2$.

Obviously, if $E = G$, T is $\alpha-$admissible.

Remark 2.9. Let E be a compact subset of a partial metric space (W, d_ρ) and $\omega \in W$, then there exists $\varrho \in E$ such that

$$d_\rho(\omega, \varrho) = d_\rho(\omega, E).$$

Now, we present our new findings. First, we introduce generalized multivalued $F-$proximal contraction mapping in partial metric spaces and prove some proximity point theorems for such mappings.

3. Main Result

Definition 3.1. Let (W, d_ρ) be a partial metric space and E, G be nonvoid closed subsets of (W, d_ρ). Then $T : E \to CB(G)$ is called generalized multivalued $F-$proximal contraction mapping if there exist $\sigma > 0$ *and* $F \in \digamma$ such that

$$H_\rho(Tx, Ty) > 0 \implies \sigma + F(H_\rho(Tx, Ty)) \leq F(\Delta(x, y))$$

$$\Delta(x, y) = \max\Big\{d_\rho(x, y),\ D_\rho(x, Tx) - d_\rho(E, G),\ D_\rho(y, Ty) - d_\rho(E, G), \frac{D_\rho(x, Ty) + D_\rho(y, Tx)}{2} - d_\rho(E, G)\Big\}$$

for all $x, y \in E$.

Theorem 3.2. *Let (W, d_ρ) be a complete partial metric space, E and G be nonvoid closed subsets of (W, d_ρ) such that E_0 is nonvoid and $T : E \to CB(G)$ is a mapping. Suppose that*

(i) *the pair (E, G) has the weak $p-$property;*
(ii) *there exist $x_0, x_1 \in E_0$ and $y_0 \in Tx_0$ such that $d_\rho(x_1, y_0) = d_\rho(E, G)$;*
(iii) *T is generalized multivalued $F-$proximal contraction and $Tx \subseteq G_0$ for all $x \in E_0$;*
(iv) *$F(\inf P) = \inf F(P)$ for all $P \subset (0, \infty)$ with $\inf P > 0$.*

Then, T has a best proximity point in E.

Proof. By (ii), there exist $x_0, x_1 \in E_0$ *and* $y_0 \in Tx_0$ such that $d_\rho(x_1, y_0) = d_\rho(E, G)$. If $y_0 \in Tx_1$, then $D_\rho(x_1, Tx_1) = d_\rho(E, G)$. Suppose that $y_0 \notin Tx_1$, then $D_\rho(y_0, Tx_1) > 0$. Also, we have that

$$D_\rho(y_0, Tx_1) \leq H_\rho(Tx_0, Tx_1). \tag{3.1}$$

Using (iv) and (3.1), we get

$$\inf_{x \in Tx_1} F_\rho(d_\rho(y_0, Tx_1)) = F(D_\rho(y_0, Tx_1)) \leq F(H_\rho(Tx_0, Tx_1)). \tag{3.2}$$

By (3.2), there exists $y_1 \in Tx_1$ such that

$$F(d_\rho(y_0, y_1)) \leq F(H_\rho(Tx_0, Tx_1)). \tag{3.3}$$

From $y_1 \in Tx_1 \subseteq G_0$, there exists $x_2 \in E_0$ such that

$$d_\rho(x_2, y_1) = d_\rho(E, G). \tag{3.4}$$

Using (i), (ii) and (3.4), we get

$$d_\rho(x_1, x_2) \le d_\rho(y_0, y_1). \tag{3.5}$$

Using $(F1)$ and (3.5), we have

$$F(d_\rho(x_1, x_2)) \le F(d_\rho(y_0, y_1)).$$

Similarly, if $y_1 \in Tx_2$, then by (3.4), $D_\rho(x_2, Tx_2) = d_\rho(E, G)$. Assume that $y_1 \notin Tx_2$, then $D_\rho(y_1, Tx_2) > 0$. Also, we know that

$$F(D_\rho(y_1, Tx_2)) \le F(H_\rho(Tx_1, Tx_2)). \tag{3.6}$$

Using (iv) and (3.6), we get

$$\inf_{x \in Tx_2} F(d_\rho(y_1, Tx_2)) = F(D_\rho(y_1, Tx_2)) \le F(H_\rho(Tx_1, Tx_2)). \tag{3.7}$$

By (3.7), there exists $y_2 \in Tx_2$ such that

$$F(d_\rho(y_1, y_2)) \le F(H_\rho(Tx_1, Tx_2)). \tag{3.8}$$

From $y_2 \in Tx_2 \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d_\rho(x_3, y_2) = d_\rho(E, G). \tag{3.9}$$

Using (i), (3.4) and (3.9), we get

$$d_\rho(x_2, x_3) \le d_\rho(y_1, y_2). \tag{3.10}$$

Using $(F1)$, (3.8) and (3.10), we get

$$F(d_\rho(x_2, x_3)) \le F(d_\rho(y_1, y_2)) \le F(H_\rho(Tx_1, Tx_2)).$$

Maintaining this procedure, there exist sequences $\{x_m\}$ in E and $\{y_m\}$ in G such that $y_m \in Tx_m$,

$$d_\rho(x_{m+1}, y_m) = d_\rho(E, G)$$
$$d_\rho(x_m, x_{m+1}) \le d_\rho(y_{m-1}, y_m),$$

and

$$F(d_\rho(x_m, x_{m+1})) \le F(d_\rho(y_{m-1}, y_m)) \le F(H_\rho(Tx_{m-1}, Tx_m)) \tag{3.11}$$

for all $m \ge 1$. Using (iii) and (3.11), we have

$$F(d_\rho(x_m, x_{m+1})) \le F(d_\rho(y_{m-1}, y_m)) \le F(H_\rho(Tx_{m-1}, Tx_m)) \le F(\Delta(x_{m-1}, x_m) - \sigma$$

$$\Delta(x_{m-1}, x_m)$$

$$= \max\Big\{d_\rho(x_{m-1}, x_m),\ D_\rho(x_{m-1}, Tx_{m-1}) - d_\rho(E, G),\ D_\rho(x_m, Tx_m) - d_\rho(E, G),$$
$$\frac{D_\rho(x_{m-1}, Tx_m) + D_\rho(x_m, Tx_{m-1})}{2} - d_\rho(E, G)\Big\}$$

$$\le \max\Big\{d_\rho(x_{m-1}, x_m),\ d_\rho(x_{m-1}, y_{m-1}) - d_\rho(E, G),\ d_\rho(x_m, y_m) - d_\rho(E, G),$$
$$\frac{d_\rho(x_{m-1}, y_m) + d_\rho(x_m, y_{m-1})}{2} - d_\rho(E, G)\Big\}$$

$$\le \max\Big\{d_\rho(x_{m-1}, x_m),\ d_\rho(x_{m-1}, x_m) + d_\rho(y_{m-1}, x_m) - d_\rho(x_m, x_m) - d_\rho(E, G),$$
$$d_\rho(x_{m+1}, x_m) + d_\rho(x_{m+1}, y_m) - d_\rho(x_{m+1}, x_{m+1}) - d_\rho(E, G),$$
$$\frac{d_\rho(x_{m-1}, x_{m+1}) + d_\rho(x_{m+1}, y_m) - d_\rho(x_{m+1}, x_{m+1}) + d_\rho(x_m, y_{m-1})}{2} - d_\rho(E, G)\Big\}$$

$$\leq \max\Bigg\{d_\rho(x_{m-1},x_m),\ d_\rho(x_{m-1},x_m)+d_\rho(E,G)-d_\rho(E,G),$$
$$d_\rho(x_{m+1},x_m)+d_\rho(E,G)-d_\rho(E,G),$$
$$\frac{d_\rho(x_{m-1},x_m)+d_\rho(E,G)+d_\rho(x_{m+1},x_m)+d_\rho(E,G)}{2}-d_\rho(E,G)\Bigg\}$$

$\leq \max\Bigg\{d_\rho(x_{m-1},x_m), d_\rho(x_{m+1},x_m)\Bigg\}.$

Suppose that $\max\{d_\rho(x_{m-1},x_m), d_\rho(x_{m+1},x_m)\} = d_\rho(x_{m+1},x_m)$. Consequently, we get

$$F(d_\rho(x_{m+1},x_m)) \leq F(d_\rho(x_{m+1},x_m)) - \sigma$$

which is not true. Thus, we have

$$F(d_\rho(x_{m+1},x_m)) \leq F(d_\rho(x_{m-1},x_m)) - \sigma.$$

If there exists a nonnegative integer p such that $x_p = x_{p+1}$, then $D_\rho(x_p, Tx_p) = d_\rho(E,G)$. Suppose that $x_m \neq x_{m+1}$ *for all* $m \geq 0$. Then

$$\begin{aligned} F(d_\rho(x_m,x_{m+1})) &\leq F(d_\rho(x_{m-1},x_m)) - \sigma \\ &\leq F(d_\rho(x_{m-2},x_{m-1})) - 2\sigma \leq ... \leq F(d_\rho(x_0,x_1)) - m\sigma. \end{aligned} \tag{3.12}$$

Taking limit as $m \to \infty$ in (3.12), we have

$$\lim_{m\to\infty} F(d_\rho(x_m,x_{m+1})) = -\infty. \tag{3.13}$$

Using (3.13) and $(F2)$, we get

$$\lim_{m\to\infty} d_\rho(x_m,x_{m+1}) = 0. \tag{3.14}$$

Let us denote $d_m = d_\rho(x_m,x_{m+1})$ *for all* $m \geq 0$. Using $(F3)$ and (3.14), there exists $\kappa \in (0,1)$ such that

$$\lim_{m\to\infty} d_m^\kappa F(d_m) = 0. \tag{3.15}$$

Using (3.12) and (3.15), we get

$$d_m^\kappa(F(d_m) - F(d_0)) \leq -d_m^\kappa m\sigma \leq 0. \tag{3.16}$$

Taking limit as $m \to \infty$ in (3.16), we have

$$\lim_{m\to\infty} m d_m^\kappa = 0. \tag{3.17}$$

By (3.17), there exists a positive integer m_1 such that $md_m^\kappa \leq 1$ *for all* $m \geq m_1$, that is,

$$d_m \leq \frac{1}{m^{\frac{1}{\kappa}}}. \tag{3.18}$$

Now, we will prove that a sequence $\{x_m\}$ is Cauchy. Consider nonnegative integers p, m such that $p > m \geq m_1$. Using the triangular inequality and (3.18), we get

$$d_\rho(x_m,x_p) \leq \Big(d_\rho(x_m,x_{m+1}) + d_\rho(x_{m+1},x_{m+2}) + \ldots + d_\rho(x_{p-1},x_p)\Big) - \sum_{q=m+1}^{p-1} d_\rho(x_q,x_q)$$

$$\leq \big(d_m + d_{m+1} + \ldots + d_{p-1}\big) \leq \sum_{q=m}^{p-1} \frac{1}{q^{\frac{1}{\kappa}}} \leq \sum_{q=m}^{\infty} \frac{1}{q^{\frac{1}{\kappa}}}.$$

$$\textit{Since } \sum_{q=m}^{\infty} \frac{1}{q^{\frac{1}{\kappa}}} \textit{ converges to } 0, \textit{ we have}$$

$$\lim_{p,m\to\infty} d_\rho(x_m,x_p) = 0.$$

Thus, $\{x_m\}$ is a Cauchy sequence in W. Since W is complete, there exists $u \in W$ such that

$$\lim_{m\to\infty} d_\rho(x_m, u) = \lim_{p,m\to\infty} d_\rho(x_m, x_p]) = d_\rho(u, u) = 0.$$

Since $d_\rho(x_m, x_{m+1}) \leq d_\rho(y_{m-1}, y_m) \leq d_\rho(x_{m-1}, x_m)$. Hence $\{y_m\}$ is a Cauchy sequence and $\lim_{m\to\infty} y_m = v$. In addition,

$$d_\rho(u, v) \leq d_\rho(u, x_{m+1}) + d_\rho(x_{m+1}, y_m) - d_\rho(x_{m+1}, x_{m+1})$$
$$\leq d_\rho(u, x_{m+1}) + d_\rho(x_{m+1}, y_m) = d_\rho(u, x_{m+1}) + d_\rho(E, G) \to d_\rho(E, G) \ as\ m \to \infty.$$

Thus, $d_\rho(u, v) = d_\rho(E, G)$. If $v \in Tu$, then $D_\rho(u, Tu) = d_\rho(E, G)$. Suppose that $v \notin Tu$, so, there exists a positive integer m_1 such that $y_m \notin Tu\ for\ all\ m \geq m_1$. By (iii), we get

$$F(D_\rho(y_m, Tu)) \leq F(H_\rho(Tx_m, Tu)) \leq F(\Delta(x_m, u)) - \sigma$$

$$\Delta(x_m, u) = \max\Big\{d_\rho(x_m, u),\ D_\rho(x_m, Tx_m) - d_\rho(E, G),\ D_\rho(u, Tu) - d_\rho(E, G),$$
$$\frac{D_\rho(x_m, Tu) + D_\rho(u, Tx_m)}{2} - d_\rho(E, G)\Big\}$$
$$\leq \max\Big\{d_\rho(x_m, u), d_\rho(x_m, x_{m+1}) + D_\rho(x_{m+1}, Tx - m) - d_\rho(x_{m+1}, x_{m+1}) - d_\rho(E, G),$$
$$D_\rho(u, Tu) - d_\rho(E, G),$$
$$\frac{D_\rho(x_m, Tu) + d_\rho(u, x_{m+1}) + D_\rho(x_{m+1}, Tx_m) - d_\rho(x_{m+1}, x_{m+1})}{2} - d_\rho(E, G)\Big\}$$
$$= \max\Big\{d_\rho(x_m, u),\ d_\rho(x_m, x_{m+1}) - d_\rho(x_{m+1}, x_{m+1}) - d_\rho(E, G) + d_\rho(E, G)),$$
$$D_\rho(u, Tu) - d_\rho(E, G),$$
$$\frac{D_\rho(x_m, Tu) + d_\rho(u, x_{m+1}) - d_\rho(x_{m+1}, x_{m+1}) + d_\rho(E, G)}{2} - d_\rho(E, G)\Big\}$$
$$\leq \max\Big\{d_\rho(x_m, u),\ d_\rho(x_m, x_{m+1}),\ d_\rho(y_m, u) + D_\rho(y_m, Tu) - d_\rho(y_m, y_m) - d_\rho(E, G),$$
$$\frac{d_\rho(x_m, y_{m-1}) + d_\rho(y_{m-1}, y_m) + D_\rho(y_m, Tu) + d_\rho(u, x_{m+1}) + d_\rho(E, G)}{2} - d_\rho(E, G)\Big\}$$
$$= \max\Big\{d_\rho(x_m, u), d_\rho(x_m, x_{m+1}), D_\rho(y_m, Tu),$$
$$\frac{d_\rho(E, G) + d_\rho(y_{m-1}, y_m) + D_\rho(y_m, Tu) + d_\rho(u, x_{m+1}) + d_\rho(E, G)}{2} - d_\rho(E, G)\Big\}$$
$$= \max\Big\{d_\rho(x_m, u), d_\rho(x_m, x_{m+1}), D_\rho(y_m, Tu),$$
$$\frac{d_\rho(y_{m-1}, y_m) + D_\rho(y_m, Tu) + d_\rho(u, x_{m+1})}{2}\Big\}$$
$$\leq \max\Big\{d_\rho(x_m, u), d_\rho(x_m, x_{m+1}), D_\rho(y_m, Tu), d_\rho(y_{m-1}, y_m) + d_\rho(u, x_{m+1})\Big\}$$
$$\leq D_\rho(v, Tu)\ as\ m \to \infty.$$

Thus, $F(D_\rho(v, Tu)) \leq F(D_\rho(v, Tu)) - \sigma$, which is not true. Therefore $v \in Tu$. It follows that $D_\rho(u, Tu) = d_\rho(E, G)$. Hence u is a best proximity point of T. □

Theorem 3.3. *Let (W, d_ρ) be a complete partial metric space, E and G be nonvoid closed subsets of (W, d_ρ) such that E_0 is nonvoid and $T : E \to K(G)$ is a mapping. Suppose that*

(i) *the pair* (E, G) *has the weak* $p-$*property;*
(ii) *there exist* $x_0, x_1 \in E_0$ *and* $y_0 \in Tx_0$ *such that* $d_\rho(x_1, y_0) = d_\rho(E, G)$*;*
(iii) T *is generalized multivalued* $F-$*proximal contraction and* $Tx \subseteq G_0$ *for all* $x \in E_0$.

Then, T *has a best proximity point in* E.

Proof. By (ii), there exist $x_0, x_1 \in E_0$ *and* $y_0 \in Tx_0$ such that $d_\rho(x_1, y_0) = d_\rho(E, G)$. If $y_0 \in Tx_1$, then $D_\rho(x_1, Tx_1) = d_\rho(E, G)$. Suppose that $y_0 \notin Tx_1$, then $D_\rho(y_0, Tx_1) > 0$. Also, we know that

$$D_\rho(y_0, Tx_1) \le H_\rho(Tx_0, Tx_1). \tag{3.19}$$

Since Tx_1 is compact and (3.19), there exists $y_1 \in Tx_1$ such that

$$d_\rho(y_0, y_1) = D_\rho(y_0, Tx_1) \le H_\rho(Tx_0, Tx_1). \tag{3.20}$$

Since $y_1 \in Tx_1 \subseteq B_0$, there exists $x_2 \in E_0$ such that

$$d_\rho(x_2, y_1) = d_\rho(E, G). \tag{3.21}$$

Using (i), we get

$$d_\rho(x_1, x_2) \le d_\rho(y_0, y_1). \tag{3.22}$$

Similarly, if $y_1 \in Tx_2$, then by (3.21) $D_\rho(x_2, Tx_2) = d_\rho(E, G)$. Suppose that $y_1 \notin Tx_2$, then $D_\rho(y_1, Tx_2) > 0$. Also, we know that

$$D_\rho(y_1, Tx_2) \le H_\rho(Tx_1, Tx_2). \tag{3.23}$$

Since Tx_2 is compact and (3.23), there exists $y_2 \in Tx_2$ such that

$$d_\rho(y_1, y_2) = D_\rho(y_1, Tx_2) \le H_\rho(Tx_1, Tx_2). \tag{3.24}$$

From $y_2 \in Tx_2 \subseteq G_0$, there exists $x_3 \in A_0$ such that

$$d_\rho(x_3, y_2) = d_\rho(A, B). \tag{3.25}$$

Using (i), (3.21) and (3.25), we get

$$d_\rho(x_2, x_3) \le d_\rho(y_1, y_2). \tag{3.26}$$

Using $(F1)$, (3.24) and (3.26), we get

$$F(d_\rho(x_2, x_3)) \le F(d_\rho(y_1, y_2)) \le F(H_\rho(Tx_1, Tx_2)).$$

Maintaining this procedure, we can find a sequence $\{x_m\}$ in E and $\{y_m\}$ in G such that $y_m \in Tx_m$,

$$d_\rho(x_{m+1}, y_m) = d_\rho(E, G)$$
$$d_\rho(x_m, x_{m+1}) \le d_\rho(y_{m-1}, y_m),$$

and

$$F(d_\rho(x_m, x_{m+1})) \le F(d_\rho(y_{m-1}, y_m)) \le F(H_\rho(Tx_{m-1}, Tx_m))$$

for all $m \ge 1$.

The rest of the proof can be done as in the Theorem 3.2. □

If $d_\rho(E, G) = 0$ in Theorem 3.2, we get the following fixed point results.

Corollary 3.4. *Let* (W, d_ρ) *be a complete partial metric space,* E *be nonvoid closed subset of* (W, d_ρ) *and* $T : E \to CB(E)$ *is a multivalued mapping. Suppose that*

(i) *there exist* $\sigma > 0$ *and* $F \in \digamma$ *such that*

$$H_\rho(Tx, Ty) > 0 \implies \sigma + F(H_\rho(Tx, Ty)) \le F(\Delta(x, y))$$
$$\Delta(x, y) = \max\left\{d_\rho(x, y), D_\rho(x, Tx), D_\rho(y, Ty), \frac{D_\rho(x, Ty) + D_\rho(y, Tx)}{2}\right\}$$

for all $x, y \in E$;

(ii) $F(\inf P) = \inf F(P)$ *for all* $P \subset (0, \infty)$ *with* $\inf P > 0$.

Then, T *has a fixed point in* E.

Corollary 3.5. *Let (W, d_ρ) be a complete partial metric space, E be nonvoid closed subset of (W, d_ρ) and $T : E \to CB(E)$ is a multivalued mapping. Suppose that*

(i) *there exist $\sigma > 0$ and $F \in \digamma$ such that*

$$d_\rho(Tx, Ty) > 0 \implies \sigma + F(d_\rho(Tx, Ty)) \leq F(\Delta(x, y))$$
$$\Delta(x, y) = \alpha d_\rho(x, y) + \beta d_\rho(x, Tx) + \gamma d_\rho(y, Ty) + \delta(d_\rho(x, Ty) + d_\rho(y, Tx))$$

$0 \leq \alpha, \beta, \gamma, \delta \leq 1,\ \alpha + \beta + \gamma + 2\delta \leq 1$, *for all $x, y \in E$,*

(ii) $F(\inf P) = \inf F(P)$ *for all $P \subset (0, \infty)$ with* $\inf P > 0$.

Then, T has a fixed point in E.

If $W = E = G$ in Corollary 3.4, we get the following.

Corollary 3.6. *Let (W, d_ρ) be a complete partial metric space and $T : W \to W$ is a mapping. Suppose that there exist $\sigma > 0$ and $F \in \digamma$ such that*

$$d_\rho(Tx, Ty) > 0 \implies \sigma + F(d_\rho(Tx, Ty)) \leq F(\Delta(x, y))$$
$$\Delta(x, y) = \max\left\{d_\rho(x, y), d_\rho(x, Tx), d_\rho(y, Ty), \frac{d_\rho(x, Ty) + d_\rho(y, Tx)}{2}\right\}$$

for all $x, y \in W$.
Then, T has a fixed point in W.

Corollary 3.7. *Let (W, d_ρ) be a complete partial metric space and $T : W \to W$ is a mapping. Suppose that there exist $\sigma > 0$ and $F \in \digamma$ such that*

$$d_\rho(Tx, Ty) > 0 \implies \sigma + F(d_\rho(Tx, Ty)) \leq F(\Delta(x, y))$$
$$\Delta(x, y) = \alpha d_\rho(x, y) + \beta d_\rho(x, Tx) + \gamma d_\rho(y, Ty) + \delta(d_\rho(x, Ty) + d_\rho(y, Tx))$$

$0 \leq \alpha, \beta, \gamma, \delta \leq 1,\ \alpha + \beta + \gamma + 2\delta \leq 1$, *for all $x, y \in W$.*
Then, T has a fixed point in W.

Definition 3.8. Let (W, d_ρ) be a partial metric space and E, G be nonvoid closed subsets of (W, d_ρ). Then $T : E \to CB(G)$ is called generalized multivalued $F-$proximal contraction mapping if there exist $\sigma > 0$ *and* $F \in \digamma$ such that

$$H_\rho(Tx, Ty) > 0 \implies \sigma + F(H_\rho(Tx, Ty)) \leq F(\Delta(x, y)) - d_\rho(E, G)$$
$$\Delta(x, y) = \max\left\{d_\rho(x, y), D_\rho(x, Tx), D_\rho(y, Ty), \frac{D_\rho(x, Ty) + D_\rho(y, Tx)}{2}\right\}$$

for all $x, y \in E$.

Theorem 3.9. *Let (W, d_ρ) be a complete partial metric space, E and G be nonvoid closed subsets of (W, d_ρ) such that E_0 is nonvoid and $T : E \to K(G)$ is a mapping. Suppose that*

(i) *the pair (E, G) has the weak $p-$property;*

(ii) *there exist $x_0, x_1 \in E_0$ and $y_0 \in Tx_0$ such that $d_\rho(x_1, y_0) = d_\rho(E, G)$;*

(iii) *T is generalized multivalued $F-$proximal contraction and $Tx \subseteq G_0$ for all $x \in E_0$.*

Then, T has a best proximity point in E.

Proof. The proof can be done as in the Theorem 3.2. □

Theorem 3.10. *Let (W, d_ρ) be a complete partial metric space, E and G be nonvoid closed subsets of (W, d_ρ) such that E_0 is nonvoid and $T : E \to CB(G)$ is a mapping. Suppose that*

(i) *the pair (E, G) has the weak $p-$property;*

(ii) *there exist $x_0, x_1 \in E_0$ and $y_0 \in Tx_0$ such that $d_\rho(x_1, y_0) = d_\rho(E, G)$;*

(iii) *T is generalized multivalued $F-$proximal contraction and $Tx \subseteq G_0$ for all $x \in E_0$;*

(iv) $F(\inf P) = \inf F(P)$ *for all $P \subset (0, \infty)$ with* $\inf P > 0$.

Then, T has a best proximity point in E.

Proof. The proof can be done as in the Theorem 3.3. □

Theorem 3.11. *Let (W,d_ρ) be a complete partial metric space, E and G be nonvoid closed subsets of (W,d_ρ) such that E_0 is nonempty and $T: E \to CB(G)$ is a multivalued mapping. Suppose that*

(i) *if $\sigma > 0$ and $F \in \digamma$ with $H_\rho(Tx,Ty) > 0$*
$\implies \sigma + F(\alpha(x,y)H_\rho(Tx,Ty)) \le F(d_\rho(x,y))$ *for all $x,y \in E$;*

(ii) *the pair (E,G) has the weak $p-$property and there exist $x_0,x_1 \in E_0$ and $y_0 \in Tx_0$ such that*
$d_\rho(x_1,y_0) = d_\rho(E,G)$ *and* $\alpha(x_0,x_1) \ge 1$;

(iii) *T is $\alpha-$proximal admissible and $Tx \subseteq G_0$ for all $x \in E_0$;*

(iv) *$F(\inf P) = \inf F(P)$ for all $P \subset (0,\infty)$ with $\inf P > 0$.*

Then, T has a best proximity point in E.

Proof. By (ii), there exist $x_0,x_1 \in E_0$ *and* $y_0 \in Tx_0$ such that $d_\rho(x_1,y_0) = d_\rho(E,G)$ and $\alpha(x_0,x_1) \ge 1$. If $y_0 \in Tx_1$, then $D_\rho(x_1,Tx_1) = d_\rho(E,G)$. Suppose that $y_0 \notin Tx_1$, then $D_\rho(y_0,Tx_1) > 0$. Also, we know that

$$F(D_\rho(y_0,Tx_1)) \le F(H_\rho(Tx_0,Tx_1)). \tag{3.27}$$

Using (iv) and (3.27), there exists $y_1 \in Tx_1$ such that

$$F(d_\rho(y_0,y_1)) = F(D_\rho(y_0,Tx_1)) \le F(H_\rho(Tx_0,Tx_1)).$$

From $y_1 \in Tx_1 \subseteq G_0$, there exists $x_2 \in E_0$ such that

$$d_\rho(x_2,y_1) = d_\rho(E,G) \text{ and } \alpha(x_1,x_2) \ge 1. \tag{3.28}$$

Using (ii) and (3.28), we get

$$d_\rho(x_1,x_2) \le d_\rho(y_0,y_1).$$

Similarly, if $y_1 \in Tx_2$, then by (3.28), we get $D_\rho(x_2,Tx_2) = d_\rho(E,G)$. Suppose that $y_1 \notin Tx_2$, then $D_\rho(y_1,Tx_2) > 0$. Also, we know that

$$F(D_\rho(y_1,Tx_2)) \le F(H_\rho(Tx_1,Tx_2)). \tag{3.29}$$

Using (iv) and (3.29), there exists $y_2 \in Tx_2$ such that

$$F(d_\rho(y_1,y_2)) = F(D_\rho(y_1,Tx_2)) \le F(H_\rho(Tx_1,Tx_2)). \tag{3.30}$$

From $y_2 \in Tx_2 \subseteq G_0$, there exists $x_3 \in E_0$ such that

$$d_\rho(x_3,y_2) = d_\rho(E,G) \text{ and } \alpha(x_2,x_3) \ge 1. \tag{3.31}$$

Using (ii), (3.28) and (3.31), we get

$$d_\rho(x_2,x_3) \le d_\rho(y_1,y_2). \tag{3.32}$$

Using $(F1)$, (3.30) and (3.32), we get

$$F(d_\rho(x_2,x_3)) \le F(d_\rho(y_1,y_2)) \le F(H_\rho(Tx_1,Tx_2)).$$

Maintaining this procedure, there exist sequences $\{x_m\}$ in E and $\{y_m\}$ in G such that $y_m \in Tx_m$,

$$d_\rho(x_{m+1},y_m) = d_\rho(E,G) \text{ and } \alpha(x_m,x_{m+1}) \ge 1$$

$$d_\rho(x_m,x_{m+1}) \le d_\rho(y_{m-1},y_m),$$

and

$$F(d_\rho(x_m,x_{m+1})) \le F(d_\rho(y_{m-1},y_m)) \le F(H_\rho(Tx_{m-1},Tx_m)) \tag{3.33}$$

for all $m \geq 1$.
Using $(F1)$, (i) and (3.33), we have

$$F(d_\rho(x_m, x_{m+1})) \leq F(d_\rho(y_{m-1}, y_m))$$
$$\leq F(\alpha(x_{m-1}, x_m)H_\rho(Tx_{m-1}, Tx_m)) \leq F(d_\rho(x_{m-1}, x_m)) - \sigma.$$

Thus, we have

$$F(d_\rho(x_m, x_{m+1})) \leq F(d_\rho(x_{m-1}, x_m)) - \sigma.$$

Thus, by Theorem 3.9, $\{x_m\}$ is a Cauchy sequence in W. Since W is complete, there exists $u \in W$ such that

$$\lim_{m\to\infty} d_\rho(x_m, u) = \lim_{m,p\to\infty} d_\rho(x_m, x_p) = d_\rho(u, u) = 0.$$

Since $d_\rho(x_m, x_{m+1}) \leq d_\rho(y_{m-1}, y_m) \leq d_\rho(x_{m-1}, x_m)$. Hence $\{y_m\}$ is a Cauchy sequence and then, $\lim_{m\to\infty} y_m = v$. Thus, $d_\rho(u, v) = d_\rho(E, G)$. If $v \in Tu$, then $D_\rho(u, Tu) = d_\rho(E, G)$. Suppose that $v \notin Tu$, so, there exists a positive integer m_1 such that $y_m \notin Tu$ *for all* $m \geq m_1$.
Using (i) and $(F1)$, we have

$$F(D_\rho(y_m, Tu)) \leq F(\alpha(x_m, u)H_\rho(Tx_m, Tu)) \leq F(d_\rho(x_m, u)) - \sigma.$$

Consequently, we get

$$\lim_{m\to\infty} F(D_\rho(y_m, Tu)) = -\infty.$$

By $(F2)$, we have

$$\lim_{m\to\infty} D_\rho(y_m, Tu) = 0.$$

Consequently, we get

$$D_\rho(v, Tu) = 0.$$

Hence, $v \in Tu$. Therefore u is a best proximity point. □

Theorem 3.12. *Let (W, d_ρ) be a complete partial metric space, E and G be nonvoid closed subsets of (W, d_ρ) such that E_0 is nonvoid and $T : E \to K(G)$ is a multivalued mapping. Suppose that*

(i) *if $\sigma > 0$ and $F \in \digamma$ with $H_\rho(Tx, Ty) > 0$*
$\implies \sigma + F(\alpha(x, y)H_\rho(Tx, Ty)) \leq F(d_\rho(x, y))$ for all $x, y \in E$;
(ii) *the pair (E, G) has the weak $p-$property and there exist $x_0, x_1 \in E_0$ and $y_0 \in Tx_0$ such that*
$d_\rho(x_1, y_0) = d_\rho(E, G)$ and $\alpha(x_0, x_1) \geq 1$;
(iii) *T is $\alpha-$proximal admissible and $Tx \subseteq G_0$ for all $x \in E_0$.*

Then, T has a best proximity point in E.

Proof. The proof can be done as in the Theorems 3.3 *and* 3.11. □

Example 3.13. Let $W = [0, \infty) \times [0, \infty)$ and a partial metric d_ρ define as

$$d_\rho((\varrho_1, \varrho_2), (\nu_1, \nu_2)) = |\varrho_1 - \nu_1| + |\varrho_2 - \nu_2| + 1$$

for all $\varrho = (\varrho_1, \varrho_2), \nu = (\nu_1, \nu_2) \in W$. Consider $E = \{1\} \times [0, \infty)$ *and* $G = \{0\} \times [0, \infty)$. Then (W, d_ρ) is complete, $E_0 = E, G_0 = G$ *and* $d_\rho(E, G) = 2$. Define the mapping $T : E \to CB(G)$ as

$$T(1, \varrho) = \begin{cases} \{(0,0), (0, \frac{\varrho}{2})\}, & \text{if } 0 \leq \varrho \leq 1, \\ \{(0,0), (0, \frac{\varrho}{\varrho+1})\}, & \varrho > 1. \end{cases}$$

Obviously, $T(1,\varrho) \subseteq G_0$ for all $(1,\varrho) \in E_0$. Let $(1,\varrho_1),(1,\varrho_2) \in E$ *and* $(0,\nu_1),(0,\nu_2) \in G$ such that

$$d_\rho((1,\varrho_1),(0,\nu_1)) = d_\rho(E,G) = 2$$
$$d_\rho((1,\varrho_2),(0,\nu_2)) = d_\rho(E,G) = 2.$$

This means $2+|\varrho_1-\nu_1| = 2 \implies \varrho_1 = \nu_1$ and $2+|\varrho_2-\nu_2| = 2 \implies \varrho_2 = \nu_2$. Then, $d_\rho((1,\varrho_1),(1,\varrho_2)) = |\varrho_1-\varrho_2|+1 = |\nu_1-\nu_2|+1 = d_\rho((0,\nu_1),(0,\nu_2))$. Thus $(E,\ G)$ has the weak $p-$property. Also, we have $\varrho_0 = (1,1), \varrho_1 = (1,0) \in E_0$ *and* $\nu_0 = (0,0) \in T\varrho_0 = \{(0,0),(0,\frac{1}{2})\}$ and $d_\rho((1,0),(0,0)) = 2 = d_\rho(E,G)$. Define $F(\varrho) = \ln\varrho + \varrho,\ \varrho > 0$. Using the given logarithmic function and (iii) of Theorem 3.2, we have

$$\frac{H_\rho(T\varrho,T\nu)}{\Delta(\varrho,\nu)} e^{H_\rho(T\varrho,T\nu)-\Delta(\varrho,\nu)} \leq e^{-\sigma}.$$

Let $(1,\varrho),\ (1,\nu) \in E$, then, we have the following cases:
Case (i): If $0 \leq \nu < \varrho \leq 1$, then

$$H_\rho(T(1,\varrho),T(1,\nu)) = \max\{D_\rho(T(1,\varrho),T(1,\nu)), D_\rho(T(1,\nu),T(1,\varrho))\}.$$
$$D_\rho(T(1,\varrho),T(1,\nu)) = \max\{D_\rho((0,\omega),T(1,\nu)) : (0,\omega) \in T(1,\varrho)\}$$
$$= \max\{D_\rho((0,0),T(1,\nu)), D_\rho((0,\frac{\varrho}{2}),T(1,\nu))\}.$$
$$D_\rho((0,0),T(1,\nu)) = min\{d_\rho((0,0),(0,0)), d_\rho((0,0),(0,\frac{\nu}{2}))\}$$
$$= min\{1,1+\frac{\nu}{2}\} = 1.$$
$$D_\rho((0,\frac{\varrho}{2}),T(1,\nu)) = min\{d_\rho((0,\frac{\varrho}{2}),(0,0)), d_\rho((0,\frac{\varrho}{2}),(0,\frac{\hat{\nu}}{2}))\}$$
$$= min\{1+\frac{\varrho}{2}, 1+\frac{\varrho-\nu}{2}\} = 1+\frac{\varrho-\nu}{2}.$$
$$D_\rho(T(1,\varrho),T(1,\nu)) = 1+\frac{\varrho-\nu}{2}.$$

Similarly, we get

$$D_\rho(T(1,\nu),T(1,\varrho)) = 1+\frac{\varrho-\nu}{2}.$$
$$\implies H_\rho(T(1,\varrho),T(1,\nu)) = 1+\frac{\varrho-\nu}{2}.$$
$$d_\rho((1,\varrho),(1,\nu)) = \varrho-\nu+1 \leq \Delta((1,\varrho),(1,\nu)).$$

Applying all of these in the contraction condition, we have

$$\frac{H_\rho(T(1,\varrho),T(1,\nu))}{\Delta((1,\varrho),(1,\nu))} e^{H_\rho(T(1,\varrho),T(1,\nu))-\Delta((1,\varrho),(1,\nu))} \leq \frac{1+\frac{\varrho-\nu}{2}}{\varrho-\nu+1} e^{1+\frac{\varrho-\nu}{2}-(\varrho-\nu+1)}$$
$$\leq e^{\frac{-1}{2}(\varrho-\nu)} = e^{-\sigma},\ \sigma = \frac{1}{2}(\varrho-\nu) > 0.$$

Case (ii): If $0 \leq \nu \leq 1$ *and* $\varrho > 1$, then

$$D_\rho(T(1,\varrho),T(1,\nu)) = \max\left\{D_\rho((0,0),T(1,\nu)), D_\rho((0,\frac{\varrho}{1+\varrho}),T(1,\nu))\right\}.$$
$$D_\rho((0,0),T(1,\nu)) = min\{d_\rho((0,0),(0,0)), d_\rho((0,0),(0,\frac{\nu}{2}))\} = min\{1,1+\frac{\nu}{2}\} = 1.$$
$$D_\rho((0,\frac{\varrho}{1+\varrho}),T(1,\nu)) = min\left\{d_\rho((0,\frac{\varrho}{1+\varrho}),(0,0)), d_\rho((0,\frac{\varrho}{1+\varrho}),(0,\frac{\varrho}{2}))\right\}$$
$$= min\left\{1+\frac{\varrho}{1+\varrho}, 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}\right\} = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}.$$

$$\implies D_\rho(T(1,\varrho),T(1,\nu)) = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}.$$

Similarly, we get

$$D_\rho(T(1,\nu),T(1,\varrho)) = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}.$$

Hence

$$H_\rho(T(1,\varrho),T(1,\nu)) = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}.$$

$$d_\rho((1,\varrho),(1,\nu)) = 1+\varrho-\nu \leq \Delta((1,\varrho),(1,\nu)).$$

Applying all of these in the contraction condition, we get

$$\frac{H_\rho(T(1,\varrho),T(1,\nu))}{\Delta((1,\varrho),(1,\nu))}e^{H_\rho(T(1,\varrho),T(1,\nu))-\Delta((1,\varrho),(1,\nu))} \leq \frac{1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}}{\varrho-\nu+1}e^{1+\frac{\varrho}{1+\varrho}-\frac{\nu}{2}-(\varrho-\nu+1)}$$

$$\leq e^{\frac{-\varrho^2}{1+\varrho}+\nu} = e^{-\sigma},\ \ \sigma = \frac{\varrho^2}{1+\varrho}-\nu > 0.$$

Case (iii): If $1<\nu<\varrho$, then

$$D_\rho(T(1,\varrho),T(1,\nu)) = \max\left\{D_\rho((0,0),T(1,\nu)), D_\rho((0,\frac{\varrho}{1+\varrho}),T(1,\nu))\right\}.$$

$$D_\rho((0,0),T(1,\nu)) = min\{d_\rho((0,0),(0,0)), d_\rho((0,0),(0,\frac{\nu}{1+\nu}))\}$$

$$= min\{1, 1+\frac{\nu}{1+\nu}\} = 1.$$

$$D_\rho((0,\frac{\varrho}{1+\varrho}),T(1,\nu)) = min\left\{d_\rho((0,\frac{\varrho}{1+\varrho}),(0,0)), d_\rho((0,\frac{\varrho}{1+\varrho}),(0,\frac{\nu}{1+\nu}))\right\}$$

$$= min\left\{1+\frac{\varrho}{1+\varrho}, 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}\right\} = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}.$$

$$\implies D_\rho(T(1,\varrho),T(1,\nu)) = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}.$$

Similarly, we get

$$D_\rho(T(1,\nu),T(1,\varrho)) = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}.$$

Hence

$$H_\rho(T(1,\varrho),T(1,\nu)) = 1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}.$$

$$d_\rho((1,\varrho),(1,\nu)) = 1+\varrho-\nu \leq \Delta((1,\varrho),(1,\nu)).$$

Applying all of these in the contraction condition, we get

$$\frac{H_\rho(T(1,\varrho),T(1,\nu))}{\Delta((1,\varrho),(1,\nu))}e^{H_\rho(T(1,\varrho),T(1,\nu))-\Delta((1,\varrho),(1,\nu))}$$

$$\leq \frac{1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}}{\varrho-\nu+1}e^{1+\frac{\varrho}{1+\varrho}-\frac{\nu}{1+\nu}-(\varrho-\nu+1)}$$

$$\leq e^{\frac{-\varrho^2}{1+\varrho}+\nu-\frac{\nu}{1+\nu}} \leq e^{\frac{-\varrho^2}{1+\varrho}+\frac{\nu^2}{1+\nu}} = e^{-\sigma},$$

$$\sigma = \frac{\varrho^2}{1+\varrho}-\frac{\nu^2}{1+\nu} > 0.$$

Thus, all conditions of Theorem 3.2 are satisfied. Therefore, T has a best proximity point in E and

$D_\rho((1,0),T(1,0)) = 2 = d_\rho(E,G)$.

Example 3.14. Let $W = [0, \infty) \times [0, \infty)$ and a partial metric d_ρ define as

$$d_\rho((\varrho_1, \varrho_2), (\nu_1, \nu_2)) = |\varrho_1 - \nu_1| + |\varrho_2 - \nu_2| + 1$$

for all $\varrho = (\varrho_1, \varrho_2), \nu = (\nu_1, \nu_2) \in W$. Consider $E = \{2\} \times [0, \infty)$ *and* $G = \{1\} \times [0, \infty)$. Then (W, d_ρ) is complete, $E_0 = E, G_0 = G$ *and* $d_\rho(E, G) = 2$. Define the mapping $T : E \to CB(G)$ as

$$T(2, \varrho) = \begin{cases} \{(1, 0), (1, \frac{\varrho}{3})\}, & \text{if} 0 \leq \varrho \leq 1, \\ \{(1, 0), (1, \frac{\varrho}{\varrho + 1})\}, & \varrho > 1. \end{cases}$$

Let $\alpha : E \times E \to [0, \infty)$ be a function defined by

$$\alpha((2, \varrho), (2, \nu)) = \begin{cases} 1, & \text{if} \varrho, \nu \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

Then $T(2, \varrho) \subseteq G_0$ for all $(2, \varrho) \in E_0$. Let $(2, \varrho_1), (2, \varrho_2), (2, \omega_1), (2, \omega_2) \in E$ *and* $(1, \nu_1) \in T(2, \varrho_1),\ (1, \nu_2) \in T(2, \varrho_2)$ such that

$$\alpha((2, \varrho_1), (2, \varrho_2)) \geq 1$$
$$d_\rho((2, \omega_1), (1, \nu_1)) = d_\rho(E, G) = 2$$
$$d_\rho((2, \omega_2), (1, \nu_2)) = d_\rho(E, G) = 2.$$

Thus, we have $\varrho_1,\ \varrho_2 \in [0, 1]$, $2 + |\omega_1 - \nu_1| = 2 \implies \omega_1 = \nu_1$ and $2 + |\omega_2 - \nu_2| = 2 \implies \omega_2 = \nu_2$.
Hence, $\nu_1 \in \{0, \frac{\varrho_1}{3}\}$ and $\nu_2 \in \{0, \frac{\varrho_2}{3}\}$. Therefore, $\alpha((2, \hat{\omega_1}), (2, \omega_2)) = \alpha((2, \nu_1), (2, \nu_2)) \geq 1$, that is, T is $\alpha-$proximal admissible. In addition, $d_\rho((2, \omega_1), (2, \omega_2)) = |\omega_1 - \omega_2| + 1 = |\nu_1 - \nu_2| + 1 = d_\rho((1, \nu_1), (1, \nu_2))$. Thus $(E,\ G)$ has the weak $p-$property. Also, we have $\varrho_0 = (2, 1), \varrho_1 = (2, 0) \in E_0$ *and* $\nu_0 = (1, 0) \in T\varrho_0 = \{(1, 0), (1, \frac{1}{3})\}$ and $d_\rho((2, 0), (1, 0)) = 2 = d_\rho(E, G)$. Define $F(\varrho) = \ln \varrho + \varrho,\ \ \varrho > 0$. Using the given logarithmic function and (i) of Theorem 3.11, we have

$$\frac{\alpha((2, \varrho), (2, \nu)) H_\rho(T(2, \varrho), T(2, \nu))}{d_\rho((2, \varrho), (2, \nu))} e^{\alpha((2,\varrho),(2,\nu)) H_\rho(T(2,\varrho),T(2,\nu)) - d_\rho((2,\varrho),(2,\nu))} \leq e^{-\sigma}.$$

Let $(2, \varrho),\ (2, \nu) \in E$ with $\varrho \neq \nu$ and $\alpha((2, \varrho), (2, \nu)) \geq 1$. Then $\varrho, \nu \in [0, 1]$. For this case, we have

$$H_\rho(T(2, \varrho), T(2, \nu)) = \max\{D_\rho(T(2, \varrho), T(2, \nu)), D_\rho(T(2, \nu), T(2, \varrho))\}.$$

$$D_\rho(T(2, \varrho), T(2, \nu)) = \max\{D_\rho((1, \chi), T(2, \nu)) : (1, \chi) \in T(2, \varrho)\}$$
$$= \max\{D_\rho((1, 0), T(2, \nu)), D_\rho((1, \frac{\varrho}{3}), T(2, \nu))\}.$$

$$D_\rho((1, 0), T(2, \nu)) = min\{d_\rho((1, 0), (1, 0)), d_\rho((1, 0), (1, \frac{\nu}{3}))\}$$
$$= min\{1, 1 + \frac{\nu}{3}\} = 1.$$

$$D_\rho((1, \frac{\varrho}{3}), T(2, \nu)) = min\{d_\rho((1, \frac{\varrho}{3}), (1, 0)), d_\rho((1, \frac{\varrho}{3}), (1, \frac{\nu}{3}))\}$$
$$= min\{1 + \frac{\varrho}{3}, 1 + \frac{|\varrho - \nu|}{3}\} = 1 + \frac{|\varrho - \nu|}{3}.$$

$$D_\rho(T(2, \varrho), T(2, \nu)) = 1 + \frac{|\varrho - \nu|}{3}.$$

Similarly, we get

$$D_\rho(T(2, \nu), T(2, \varrho)) = 1 + \frac{|\varrho - \nu|}{3}.$$

$$\implies H_\rho(T(2,\varrho),T(2,\nu)) = 1+\frac{|\varrho-\nu|}{3}.$$
$$d_\rho((2,\varrho),(2,\nu)) = 1+|\varrho-\nu|.$$

Applying all of these in the contraction condition, we have

$$\frac{\alpha((2,\varrho),(2,\nu))H_\rho(T(2,\varrho),T(2,\nu))}{d_\rho((2,\varrho),(2,\nu))}e^{\alpha((2,\varrho),(2,\nu))H_\rho(T(2,\varrho),T(2,\nu))-d_\rho((2,\hat{\varrho}),(2,\nu))}$$
$$=\frac{1+\dfrac{|\varrho-\nu}{3}}{1+|\varrho-\hat{\nu}|}e^{1+\dfrac{|\varrho-\nu|}{3}-1-|\varrho-\nu|}$$
$$\leq e^{-\dfrac{2|\varrho-\nu|}{3}} = e^{-\sigma}, \sigma = \frac{2|\varrho-\nu|}{3}.$$

Thus, all conditions of Theorem 3.11 are satisfied. Therefore, T has a best proximity point in E and
$D_\rho((2,0),T(2,0)) = d_\rho((2,0),(1,0)) = 2 = d_\rho(E,G)$.

4. Application

In this section, we give an application of our main result to a nonlinear integral equation, that is, we give existence theorem for integral inclusion.
Consider a set of real valued continuous function $W = (C[0,1],\Re)$ defined on $[0,1]$ endowed with
$d_\rho(\varrho,\nu) = \sup_{r\in[0,1]}\left\{e^{-r}|\varrho(r)-\nu(r)|\right\}$ for all $\varrho,\nu\in W$. Then, (W,d_ρ) is a complete partial metric space.
Consider the following integral equation

$$\varrho(r)\in p(r)+\int_0^1\phi(r,\varepsilon,\varrho(\varepsilon))d\varepsilon, \tag{4.34}$$

such that for each continuous $\Psi:[0,1]\times[0,1]\times E\to CB(E)$ there exists $\phi(r,\varepsilon,\varrho)\in\Psi(r,\varepsilon,\varrho)$.
Define a multivalued mapping $T:E\to CB(E)$ as

$$T\varrho(r) = \left\{\omega(r):\omega(r)\in\eta(r)+\int_0^1\Psi(r,\varepsilon,\varrho(\varepsilon))d\varepsilon\right\}. \tag{4.35}$$

Theorem 4.1. *Suppose*

(i) *there exists* $\varrho_0\in E$ *such that* $\varrho_{m+1}\in T\varrho_m$,
(ii) *there exists a continuous function* $\Lambda:[0,1]\times[0,1]\to(-\infty,\infty)$ *such that*

$$\left|\phi(r,\varepsilon,\varrho(\varepsilon))-\phi(r,\varepsilon,\nu(\varepsilon))\right|\leq\sup_{\varepsilon\in[0,1]}\left|\Lambda(\varrho(\varepsilon),\nu(\varepsilon))\right|\left|\varrho(\varepsilon)-\nu(\varepsilon)\right|,$$

for all $r,\varepsilon\in[0,1]$ *and* $|\Lambda(\varrho(\varepsilon),\nu(\varepsilon))|\leq\lambda=\dfrac{1}{2e}$.

Then the integral equation (4.34) *has a solution.*

Proof. Let $F\in\digamma$ such that $F(\theta)=\ln\theta$, $\theta>0$. Applying the given natural logarithm through our Theorem 3.2 condition when $E=G$, we have

$$\frac{H_\rho(T\varrho,T\nu)}{\Delta(\varrho,\nu)}\leq e^{-\sigma}.$$

Then for $\omega\in T\varrho$, we have

$$D_\rho(\omega(r),T\nu(r))\leq d_\rho(\omega(r),\varpi(r)) = \sup_{r\in[0,1]}\{e^{-r}|\omega(r)-\varpi(r)|\},\ \varpi\in T\nu$$

$$
\begin{aligned}
&= \sup_{r\in[0,1]} e^{-r}\Big|\int_0^1 \big(\phi(r,\varepsilon,\varrho(\varepsilon)) - \phi(r,\varepsilon,\nu(\varepsilon))\big)d\varepsilon\Big| \\
&\le \sup_{r\in[0,1]} e^{-r}\int_0^1 d\varepsilon \int_0^1 \Big|\phi(r,\varepsilon,\varrho(\varepsilon)) - \phi(r,\varepsilon,\nu(\varepsilon))\Big|d\varepsilon \\
&\le \sup_{r\in[0,1]} e^{-r}\int_0^1 \sup\big|\Lambda(\varrho(\varepsilon)-\nu(\varepsilon))\big|\big|\varrho(\varepsilon)-\nu(\varepsilon)\big|d\varepsilon \\
&\le \sup_{r\in[0,1]} e^{-r}\int_0^1 \lambda\big|\varrho(\varepsilon)-\nu(\varepsilon)\big|d\varepsilon \\
&\le \sup_{r\in[0,1]} e^{-r}\int_0^1 \lambda e^{\varepsilon}\sup e^{-\varepsilon}\big|\varrho(\varepsilon)-\nu(\varepsilon)\big|d\varepsilon \\
&\le \sup_{r\in[0,1]} e^{-r}\lambda d_\rho(\varrho,\nu)\int_0^1 e^{\varepsilon}d\varepsilon \\
&\le \lambda d_\rho(\varrho,\nu)(e-1).
\end{aligned}
$$

Since $\omega \in T\varrho$ is arbitrary and $\Delta(\varrho,\nu) \ge d_\rho(\varrho,\nu)$, we have

$$D_\rho(T\varrho, T\nu) \le \lambda(e-1)\Delta(\varrho,\nu).$$

Similarly, we get

$$D_\rho(T\nu, T\varrho) \le \lambda(e-1)\Delta(\varrho,\nu).$$

Thus,

$$H_\rho(T\varrho, T\nu) \le \lambda(e-1)\Delta(\varrho,\nu).$$

Consequently, we have

$$\frac{H_\rho(T\varrho,T\nu)}{\Delta(\varrho,\nu)} \le \frac{\lambda(e-1)\Delta(\varrho,\nu)}{\Delta(\varrho,\nu)} = \frac{1}{2e}(e-1) \le \frac{1}{2} = e^{-\sigma} \ \textit{where}\ \sigma = \ln 2.$$

Thus, all conditions of Theorem 3.2 when $E = G$ are satisfied. Hence, T has a fixed point. Therefore, the integral equation (4.34) has a solution. □

5. Conclusion

This paper introduces generalized multivalued $F-$proximal contractions in partial metric spaces and establishes corresponding best proximity point theorems, including results involving $\alpha-$admissibility. The obtained results extend and improve several existing works in the literature, are illustrated by nontrivial examples, and are applied to prove the existence of a solution to an integral equation.

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