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DCCC AND META-LINDELÖF LIKE CHARACTERIZATION OF STAR-LINDELÖF BITOPOLOGICAL SPACES

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Abstract. (τ_1, τ_2)-star-Lindelöfness ensures that for every pair $(\mathcal{U}_1 \subseteq \tau_1, \mathcal{U}_2 \subseteq \tau_2)$ of open covers, a countable subcover of $\mathcal{U}_1$, can spread through $\mathcal{U}_2$ via the star operation to cover the entire bitopological space (X, τ_1, τ_2). Giving a positive answers to the questions of Choudhury et. al. [12], DCCC and meta-Lindelöf like characterization of star-Lindelöf bitopological spaces are presented in this paper. It has been established that a JDCCC bitopological space is both (τ_1, τ_2)-2-star-Lindelöf and (τ_2, τ_1)-2-star-Lindelöf. And if a bitopological space which is both (τ_1, τ_2)-n-star-Lindelöf ($n \in \mathbb{N}$) and (τ_1, τ_2)-meta-Lindelöf, then (X, τ_2) is $(n-1)$-star Lindelöf (0-star-Lindelöfness represents Lindelöfness).

1. Introduction

Kelly [21] presented the idea of bitopological space, or Bi-TS, in 1963. Several mathematicians proceeded on exploring topological concepts into bitopological situations. Generalized neighborhood system and generalized topological space (GTS) were first introduced by Császár [13]. Several recommendations have been made for continuity, homogeneity, and compactness, including for sets that are extended from conventional topological spaces to incorporate GTS. Boonpok [10] developed the idea of bi-generalized topological space, or Bi-GTS, as a generalization of Bi-TS. He studied (m,n)-closed and (m,n)-open sets in Bi-GTS. The idea of different kinds of closed sets [2, 14, 16, 18, 20, 25, 28] and different kinds of open sets [11, 19, 24] in Bi-GTS was also developed by a number of authors.

R. Rishanthini and P. Elango [26] used Boonpok's [10] generalized open sets of bitopological spaces to study the Lindelöfness of bitopological spaces. However, we found that the star operator can be quite important for creating a link between open covers from various topologies. The phrase "star operator" was first defined by E. K. van Douwen [15] in 1991. The star operator is a popular idea in topology that is used to characterize and examine different kinds of covering formations and patterns in topological spaces.

The star of $\Omega \subseteq X$ with respect to $\Psi \subseteq \mathcal{P}(X)$ is the union of all the elements of Ψ that touches Ω. It depicts the idea about how much of the space is covered by the elements of Ψ that touches Ω. It is denoted as

$$St(\Omega, \Psi) = \bigcup \{A \in \Psi : A \cap \Omega \neq \emptyset\}.$$

Iteratively, for $n \in N$,

$$St^{n+1}(\Omega, \Psi) = \bigcup \{A \in \Psi : A \cap St^n(\Omega, \Psi) \neq \emptyset\}.$$

For simplicity, $St^0(\Omega, \Psi) = \Omega$, $St^1(\Omega, \Psi)$ is simply written as $St(\Omega, \Psi)$. Also, $St(a, \Psi)$ is written in place of $St(\{a\}, \Psi)$ when $\Omega = \{a\}$.

Kocinac [22] recently utilized the star operator to study covering attributes and selection principles in bitopological spaces. Few instances of how the star operator is used can be found in [1, 6, 5, 4, 8, 7, 3, 9, 23, 27, 29, 30]. This operator allows for a more accurate interpretation of the interface between the family Ψ and a given set Ω. E.K. van Douwen

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[15] used the star operator to describe covering properties such star compactness and star-Lindelöfness. The interaction of an open set and an open cover via star operator become more interesting when the open sets and the open covers are considered from different topologies defined on the same set. While exploring this interesting feature, Choudhury et. al. [12] has introduced (τ_1, τ_2)-star-Lindelöf space and asked whether a Lindelöf like characterization or a DCCC like characterization for this space is possible or not? Giving a positive answer to their questions we have introduced Joint DCCC property and (τ_1, τ_2)–Lindelöf spaces in this paper and established their connections to (τ_1, τ_2)-star-Lindelöf spaces.

2. Preliminaries

For the readers' comfort, an overview of essential concepts is included in this part. No axiom of separation has been addressed during the research process. Throughour the paper, bispaces are bitopological spaces that feature two linked topologies, whereas topological spaces are spaces with their associated topologies. We use [17] for commonly used symbols and ideas.

Definition 2.1. [15] A topological space (X, τ) satisfies the Discrete Countable Chain Condition (DCCC) if every collection of pairwise disjoint, non-empty open sets in τ is at most countable.

A refinement $\mathcal{V}$ of an open cover $\mathcal{U}$ means that every set in $\mathcal{V}$ is contained in some set of $\mathcal{U}$. A refinement $\mathcal{V}$ is point-countable if each point in X belongs to at most countably many sets in $\mathcal{V}$.

Definition 2.2. [15] A topological space (X, τ) is called meta-Lindelöf if every open cover of X has a point-countable open refinement.

Definition 2.3. [12] If for every pair $(\mathcal{G}_1, \mathcal{G}_2)$ of τ_1-open cover and τ_2-open cover of X, there exists a countable subset $\mathcal{W} \subseteq \mathcal{G}_1$ such that $\mathrm{St}^n(\bigcup \mathcal{W}, \mathcal{G}_2) = X$, then the bispace (X, τ_1, τ_2) will be called (τ_1, τ_2)-n-star-Lindelöf. If $n = 1$, i.e., if $\mathrm{St}(\bigcup \mathcal{W}, \mathcal{G}_2) = X$, then we will simply call the bispace to be (τ_1, τ_2)-star-Lindelöf.

3. Main Result

In general topology, discrete countable chain condition (in short DCCC) plays an important role in the illustration of higher order star-Lindelöf spaces. Thus we present the following definition.

Definition 3.1. A bitopological space (X, τ_1, τ_2) is said to satisfy joint discrete countable chain condition (in short JDCCC) if any collection of non-empty, pairwise disjoint sets that are either τ_1-open or τ_2-open is at most countable.

Theorem 3.2. *Every JDCCC bitopological space is both (τ_1, τ_2)-2-star-Lindelöf and (τ_2, τ_1)-2-star-Lindelöf.*

Proof. Let $\mathcal{U}_1 \subseteq \tau_1$ and $\mathcal{U}_2 \subseteq \tau_2$ be two covers of the bitopological space (X, τ_1, τ_2). Consider a pairwise disjoint family $\mathcal{V}$ of non-empty subsets of $\mathcal{U}_1 \cup \mathcal{U}_2$. i.e., for every $U_\alpha, U_\beta \in \mathcal{V}$, $U_\alpha \cap U_\beta = \emptyset$. By JDCCC property, the maximal collection of pairwise disjoint sets taken from $\mathcal{U}_1 \cup \mathcal{U}_2$ is at most countable. So, we can assume that $\mathcal{V} = \{U_n : n \in \mathbb{N}\}$ where $U_n \in \mathcal{U}_1 \cup \mathcal{U}_2$ for all $n \in \mathbb{N}$.

We divide $\mathcal{V}$ into two sets $\mathcal{M} = \{M : M \in \mathcal{U}_1 \cap \mathcal{V}\}$ and $\mathcal{L} = \mathcal{V} \setminus \mathcal{M}$. Clearly both $\mathcal{M} \subseteq \mathcal{U}_1$ and $\mathcal{L} \subseteq \mathcal{U}_2$ are countable and $\mathcal{V} = \mathcal{M} \cup \mathcal{L}$. We assume $\mathcal{L} = \{L_i : i \in \mathbb{N}\}$. We select a $U_i \in \mathcal{U}_1$ such that $L_i \cap U_i \neq \emptyset$ for each $i \in \mathbb{N}$. Therefore, $L_i \subseteq St(U_i, \mathcal{U}_2)$ for each $i \in \mathbb{N}$. So $\mathcal{W} = \mathcal{M} \cup \{U_i : i \in \mathbb{N}\} \subset \mathcal{U}_1$ is countable.

$$\text{Now, } \big(\bigcup \mathcal{M}\big) \cup \{\bigcup_{i\in\mathbb{N}} St(U_i, \mathcal{U}_2)\} \subseteq St\big(\big(\big(\bigcup \mathcal{M}\big) \cup \big(\bigcup_{i\in\mathbb{N}} U_i\big)\big), \mathcal{U}_2\big) = St\big(\bigcup \mathcal{W}, \mathcal{U}_2\big).$$

$$\text{So, } \big(\bigcup \mathcal{M}\big) \cup \{\bigcup_{i\in\mathbb{N}} L_i\} \subseteq St\big(\bigcup \mathcal{W}, \mathcal{U}_2\big).$$

$$\text{Therefore, } \big(\bigcup \mathcal{M}\big) \cup \big(\bigcup \mathcal{L}\big) \subseteq St\big(\bigcup \mathcal{W}, \mathcal{U}_2\big).$$

$$\text{i.e., } \bigcup \mathcal{V} \subseteq St\big(\bigcup \mathcal{W}, \mathcal{U}_2\big).$$

$$\text{So, } St\big(\bigcup \mathcal{V}, \mathcal{U}_2\big) \subseteq St\big(St\big(\bigcup \mathcal{W}, \mathcal{U}_2\big), \mathcal{U}_2\big) = St^2\big(\bigcup \mathcal{W}, \mathcal{U}_2\big).$$

By the maximality of $\mathcal{V}$, every element of $\mathcal{U}_2$ must intersect an element of $\mathcal{V}$. Thus $St\big(\bigcup \mathcal{V}, \mathcal{U}_2\big) = X$. So, $X \subseteq St^2\big(\bigcup \mathcal{W}, \mathcal{U}_2\big)$. But, $St^2\big(\bigcup \mathcal{W}, \mathcal{U}_2\big) \subseteq X$. Therefore, $St^2\big(\bigcup \mathcal{W}, \mathcal{U}_2\big) = X$. Thus $\mathcal{W}$ witnesses the (τ_1, τ_2)-2-star-Lindelöfness of the bitopological space (X, τ_1, τ_2).

Similarly, if we take, $\mathcal{M} = \{M : M \in \mathcal{U}_2 \cap \mathcal{V}\}$, then we will arrive to the conclusion that (X, τ_1, τ_2) is a (τ_2, τ_1)-2-star-Lindelöf space. □

The idea of meta-Lindelöfess, which is a generalization of Lindelöfness in topology, is crucial to comprehending the composition and properties of topological space. While Lindelöf requires covering a space from an open cover with a countable number of open sets, meta-Lindelöfness weakens this constraint and concentrates on point-countable covers. This motivates us for the following definitions.

Definition 3.3. A family $\mathcal{E}$ will be called $\mathcal{F}$-set-countable, if for every $E \in \mathcal{E}, E \cap F \neq \emptyset$ for countably many $F \in \mathcal{F}$.

Definition 3.4. In a bispace (X, τ_1, τ_2) if for every pair $(\mathcal{U}_1, \mathcal{U}_2)$ of τ_1-open cover and τ_2-open cover, $\mathcal{U}_2$ has a τ_2-open refinement $\mathcal{V}$ such that $\mathcal{U}_1$ is $\mathcal{V}$-set-countable, then (X, τ_1, τ_2) is called a (τ_1, τ_2)-meta-Lindelöf space.

Theorem 3.5. *If a bispace (X, τ_1, τ_2) is (τ_1, τ_2)-star-Lindelöf and (τ_1, τ_2)-Lindelöf, then (X, τ_2) is Lindelöf.*

Proof. Let $\mathcal{U}_2$ be an arbitrary τ_2-open cover and $\mathcal{U}_1$ be a τ_1-open cover of X. Since (X, τ_1, τ_2) is (τ_1, τ_2)-Lindelöf, $\mathcal{U}_2$ has a τ_2-open refinement $\mathcal{V}$ such that $\mathcal{U}_1$ is $\mathcal{V}$-set countable. But (X, τ_1, τ_2) is also (τ_1, τ_2)-star-Lindelöf. So, there exists a countable $\mathcal{U}' = \{U_1, U_2, \cdots, U_k, \cdots\} \subseteq \mathcal{U}_1$ such that $St(\bigcup U_{i\in\mathbb{N}}, \mathcal{V}) = X$. So, $\{St(U_i, \mathcal{V}) : i \in \mathbb{N}\}$ is a countable τ_2 open cover of X. But $\mathcal{U}_1$ is $\mathcal{V}$-set-countable. So, $\{V_{i,j} \in \mathcal{V} : U_i \cap V_{i,j} \neq \emptyset\}$ is countable for each $i \in \mathbb{N}$. Thus, $\mathcal{V}' = \{V_{i,j} \in \mathcal{V} : i, j \in \mathbb{N} \text{ and } U_i \cap V_{i,j} \neq \emptyset\}$ is a countable τ_2-open cover of X and $\mathcal{V}' \subseteq \mathcal{V}$. Since $\mathcal{V}$ is a refiement of $\mathcal{U}_2$, for every $V_{i,j} \in \mathcal{V}'$, we can choose $W_{i,j} \in \mathcal{U}_2$ such that $V_{i,j} \subseteq W_{i,j}$ for each $i, j \in \mathbb{N}$. Thus $\mathcal{W} = \{W_{i,j} : V_{i,j} \subseteq W_{i,j}$ and $V_{i,j} \in \mathcal{V}'\}$ is a countable τ_2-open subcover of $\mathcal{U}_2$ covering X. Hence (X, τ_2) is Lindelöf. □

Theorem 3.6. *If a bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-2-star-Lindelöf and (τ_1, τ_2)-Lindelöf, then (X, τ_2) is star-Lindelöf.*

Proof. Let $\mathcal{U}_1$ be a τ_1-open cover and $\mathcal{U}_2$ be a τ_2-open cover for a bitopological space (X, τ_1, τ_2). But (X, τ_1, τ_2) is (τ_1, τ_2)–Lindelöf. So, $\mathcal{U}_2$ has a τ_2-open refinement $\mathcal{V}$ such that $\mathcal{U}_1$ is $\mathcal{V}$-set-finite. But (X, τ_1, τ_2) is also (τ_1, τ_2)-2-star-Lindelöf. So, there exists a countable $\mathcal{U}' = \{U_1, U_2, ..., U_k\} \subseteq \mathcal{U}_1$ such that $St^2(\bigcup_{i\in\mathbb{N}} U_i, \mathcal{V}) = X$. Therefore, $\{St^2(U_i, \mathcal{V}) : i \in \mathbb{N}\}$ is a countable τ_2-covering of X. But $\mathcal{U}_1$ is $\mathcal{V}$-set-finite. So, $\{V_{i,j} \in \mathcal{V} : U_i \cap \mathcal{V}_{ij} \neq \emptyset\}$ is countable for each $i \in \mathbb{N}$. So, $\bigcup_{j\in\mathbb{N}} St(V_{i,j}, \mathcal{V}) = St^2(U_i, \mathcal{V})$ for each $i \in \mathbb{N}$. Therefore,

$St(\bigcup_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} V_{i,j},\mathcal{V}) = \bigcup_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} St(V_{i,j},\mathcal{V}) = \bigcup_{i\in\mathbb{N}} St^2(U_i,\mathcal{V}) = St^2(\cup_{i\in\mathbb{N}}U_i,\mathcal{V}) = X$. So, $St(\bigcup_{i\in\mathbb{N}}\bigcup_{j\in\mathbb{N}} V_{i,j},\mathcal{V}) = X$. We take, $\mathcal{V}' = \{V_{i,j} : i,j \in \mathbb{N}\}$. We can choose a $W_{i,j} \in \mathcal{U}_2$ such that $V_{i,j} \subseteq W_{i,j}$ for each $i,j \in \mathbb{N}$. Thus $\mathcal{W}' = \{W_{i,j} : i,j \in \mathbb{N}\}$ is a countable subset of $\mathcal{U}_2$ with $X = St(\bigcup\mathcal{W}',\mathcal{V}) \subseteq St(\bigcup\mathcal{W}',\mathcal{U}_2)$. Therefore, $St(\bigcup\mathcal{W}',\mathcal{U}_2) = X$. Hence, (X,τ_2) is a star-Lindelöf space. □

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