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SOLVABILITY OF A CAYLEY INCLUSION INVOLVING H-MONOTONE IN BANACH SPACES

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Abstract. In this paper, a new class of H-monotone in Banach spaces is considered and studied. The resolvent operator and Cayley approximation operator associated with the H-monotone are defined, and the Lipschitz continuity of Cayley approximation operator is also established. An application involves the solvability of a class of generalized Cayley inclusions with H-monotone in Banach spaces. By utilizing the technique of resolvent, an iterative algorithm is developed for solving such a class of generalized Cayley inclusions in Banach spaces. The convergence of the iterative sequence generated by the algorithm is proven under certain suitable conditions. The results are justified by means of a numerical example analytically and graphically using Python(matplotlib).

1. Introduction

It is now the understood fact that variational inequality theory has emerged as a highly effective tool for analyzing a broad range of linear and nonlinear problems across various fields of pure and applied sciences including mathematical programming, optimization theory, engineering, elasticity theory, equilibrium problems in mathematical economy, and game theory, see for example [9, 10, 22, 26]. A key focus within variational inequality theory is the advancement of efficient iterative algorithms for computing approximate solutions. Several researchers have made advances in the approximation solvability of variational inclusions, see for example [4, 5, 8, 12, 13, 15]. Among the most effective numerical techniques used for addressing variational inequalities in Hilbert spaces is the projection method and its various adaptations. However, the traditional projection method relies heavily on the inner product property of Hilbert spaces, rendering it unsuitable for variational inequalities in Banach spaces. This limitation prompts the exploration of alternative methods to develop iterative algorithms for approximating solutions of variational inequalities in Banach spaces. Several researchers have discussed the approximate solvability of various classes variational inequalities/inclusions and their variant forms by involving different types of operators such as H, G, A, H-η, $H(\phi,\eta)$-operators, see for example [3, 4, 7, 11, 18, 23].

In the recent past, numerical computation of zeros or fixed points of nonlinear mappings involved in variational inequalities/inclusions have been studied by several researchers, see for example, [17, 28, 30, 32]. Zhang et al. [31] studied an iteration procedure for approximating the solution of the inclusion problem and a fixed point of a nonexpansive mapping in Hilbert spaces. Peng et al. [24] studied a viscosity approximation method for finding a solution of a variational inclusion involving maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem and a fixed point of a non expansive mapping. It is remarked that most of the work has been done in the setting of Hilbert spaces. Therefore, a natural question of extending these problems in the Banach space arises.

Motivated by the research work going in this direction. In this paper, a new class of H-monotone operators in Banach spaces is considered and studied. The resolvent operator

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and Cayley approximation operator associated with the H-monotone operator are defined, and Lipschitz continuity of Cayley approximation operator is also demonstrated. An application involves the solubility of a class of generalized Cayley inclusion s with H-monotone operators in Banach spaces. By utilizing the technique of resolvent operator, an iterative algorithm is developed for solving such a class of generalized Cayley inclusion s in Banach spaces. The convergence of the iterative sequence generated by the algorithm is proven under certain suitable conditions. As a matter of justification, we have constructed a numerical example in view of different operators considered and shown the convergence of the sequences generated to the exact solution graphically. For plotting the graph, we have used Python(Metplotlib). The results presented in this paper improve and extend some known results in the literature, see for example, [1, 3, 8, 11, 13, 15, 20, 21, 23, 26] and the related references cited therein.

2. Preliminaries

Throughout this work, we assume X to be a Banach space, 2^X denotes the family of all subsets of X and X^* the topological dual of X. whose norm is denoted by $\|.\|$, and $\langle ., . \rangle$ is the duality pairing between X and X^*. The normalized duality mapping $\mathcal{J} : X \to 2^{X^*}$ is defined by

$$\mathcal{J}(x) = \{f \in X^* : \langle f, x\rangle = \|f\|\|x\|, \|f\| = \|x\|, \quad \forall x \in X\}.$$

We remark that all the Hilbert spaces are uniformly smooth so in that case J reduces to an identity mapping.

The modulus of smoothness of the Banach space X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup \Big[\frac{\|x + y\| - \|x - y\|}{2} - 1 : \ \|x\| \leq 1, \|y\| \leq t \Big].$$

The Banach space X is called uniformly smooth if $\lim\limits_{t\to 0} \frac{\rho_X(t)}{t} = 0$.

Proposition 2.1. [25] *Let X be a Banach space and $\mathcal{J}$ be the normalized duality mapping from X into $X,^*$ then for all $x, y \in X$, we have*

(1) $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, \mathcal{J}(x - y)\rangle$,

(2) $\langle x - y, \mathcal{J}(x) - \mathcal{J}(y)\rangle \leq 2d^2 \rho_X(4\|x - y\|/d)$, where $d = \sqrt{\frac{\|x\|^2 + \|y\|^2}{2}}$.

Definition 2.2. [16] Let $T : X \to X^*$ and $g : X \to X$ be two single-valued mappings then:

(1) T is monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in X;$$

(2) T is r-Strongly monotone, if there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|, \ \forall x, y \in X;$$

(3) T is s-Lipschitz continuous, if there exists a constant $s > 0$, such that

$$\|Tx - Ty\| \leq s\|x - y\|, \ \forall x, y \in X;$$

(4) g is k-strongly accretive, if there exists a constant $k > 0$ such that

$$\langle \mathcal{J}(x - y), g(x) - g(y)\rangle \geq k\|x - y\|^2. \ \forall x, y \in X.$$

Definition 2.3. [20] A set valued mapping $A : X \to 2^{X^*}$ is said to be

(1) Monotone if, $x, y \in X, u \in Ax$ and $v \in Ay$,

$$\langle u - v, x - y\rangle \geq 0;$$

(2) Maximal monotone if, for any $x \in X$, $u \in Ax$,
$\langle u - v, x - y \rangle \geq 0$ implies $v \in A(y)$;
(3) λ- strongly monotone if, for any $x, y \in X, u \in Ax$ and $v \in Ay$,

$$\langle u - v, x - y \rangle \geq \lambda \|x - y\|^2.$$

Definition 2.4. [16] Let $H : X \to X$ be a strictly monotone operator and $M : X \to 2^X$ be an H-monotone operator.

(1) The resolvent operator $\mathcal{R}_\lambda^{H,M} : X \to X$ is defined by

$$\mathcal{R}_\lambda^{H,M}(x) = (H + \lambda M)^{-1}(x), \ \forall x \in X.$$

(2) The Cayley operator $\mathcal{C}_\lambda^{H,M}(x) : X \to X$ is defined as:

$$\mathcal{C}_\lambda^{H,M}(x) = \{2\mathcal{R}_\lambda^{H,M} - H\}(x), \ \forall x \in X.$$

Theorem 2.5. [16] *Let $H : X \to X^*$ be a strongly monotone operator and $M : X \to 2^{X^*}$ be H-monotone. Then the resolvent operator $(H + \lambda M)^{-1}$ is single-valued.*

Theorem 2.6. [16] *Let $H : X \to X^*$ be a r-strongly monotone, s-Lipschitz continuous and $M : X \to 2^{X^*}$ be an H-monotone operator. Then the resolvent operator $\mathcal{R}_\lambda^{H,M} : X \to X$ is Lipschitz continuous with constant $\frac{1}{r}$, that is,*

$$\|\mathcal{R}_\lambda^{H,M}(x) - \mathcal{R}_\lambda^{H,M}(y)\| \leq \frac{1}{r}\|x - y\|.$$

Theorem 2.7. *Let H and M be same as in Theorem 2.6, then the Cayley operator is L-Lipschitz continuous, where $L = \dfrac{rs + 2}{r}$.*

Proof. Let $x, y \in X^*$, it follows that

$$\begin{aligned}
\|\mathcal{C}_\lambda^{H,M}(x) - \mathcal{C}_\lambda^{H,M}(y)\| &= 2\|(\mathcal{R}_\lambda^{H,M}(x) - H(x) - (\mathcal{R}_\lambda^{H,M}(y) - H(y)\| \\
&\leq 2\{\|\mathcal{R}_\lambda^{H,M}(x) - \mathcal{R}_\lambda^{H,M}(y)\| + \|H(x) - H(y)\|\} \\
&= (\frac{2}{r} + s)\|x - y\| \\
&= (\frac{2 + rs}{r})\|x - y\|.
\end{aligned}$$

That is,

$$\|\mathcal{C}_\lambda^{H,M}(x) - \mathcal{C}_\lambda^{H,M}(y)\| \leq L\|x - y\|. \tag{2.1}$$

□

3. Formulation of problem

Let X be Banach space and $A, f, \mathcal{C}_\lambda^{H,M}, g : X \to X$ be single-valued mappings. Let $M : X \to 2^{X^*}$ be a set-valued mapping, we consider the following Cayley inclusion (in short, CI), Find $x \in X$ such that:

$$0 \in A(x - f(x)) + \mathcal{C}_\lambda^{H,M}(x) + M(g(x)). \tag{3.2}$$

Some special cases of CI (3.2):
Case I: If $\mathcal{C}_\lambda^{H,M}(x) \equiv 0$, then CI (3.2) reduces to the following problem: Find $x \in X$, such that;

$$0 \in A(x - f(x)) + M(g(x)). \tag{3.3}$$

Problem (3.3) was introduced and studied by Luo and Huang [21].

Case II: If $\mathcal{C}_\lambda^{H,M}(x) \equiv 0$ and $f(x) \equiv 0$, then CI (3.2) reduces to the following problem: Find $x \in X$ such that;

$$0 \in A(x) + M(g(x)). \tag{3.4}$$

Problem (3.4) was introduced and studied by Xia and Huang [27].

We remark that for suitable choices of different mappings and the underlying space in the problem CI (3.2), we can obtain many known and new results in the literature. see for example [1, 6].

Theorem 3.1. *Let $A, f, \mathcal{C}_\lambda^{H,M}, g : X \to X$ be single-valued mappings. An element $x \in X$ is the solution of problem CI (3.2) if and only if,*

$$g(x) = \mathcal{R}_\lambda^{H,M}[g(x) - \lambda\{A(x - f(x)) + \mathcal{C}_\lambda^{H,M}(x)\}], \tag{3.5}$$

where $\lambda > 0$ is a constant, $R_\lambda^{H,M}(x) = (H + \lambda M)^{-1}(x)$ is the resolvent operator of M and H is r-strongly monotone.

Proof. By the definition of $R_\lambda^{H,M}(x)$, we know that (3.2) holds,

$$\begin{aligned}
&\iff \quad Hg(x) - \lambda\ [A(x - f(x)) + \mathcal{C}_\lambda^{H,M}(x)\ ] \in\ (H + \lambda M\ )g(x)\\
&\iff \quad \{A(x - f(x)) + \mathcal{C}_\lambda^{H,M}(x)\} \in Mg(x)\\
&\iff \quad 0 \in A(x - f(x)) + \mathcal{C}_\lambda^{H,M}(x) + M(g(x)).
\end{aligned}$$

□

Based on Theorem 3.1, we construct the following iterative algorithm for solving CI (3.2).

4. Iterative Algorithm and Convergence Analysis

Iterative Algorithm 4.1. *Given $x_0 \in X$, define iterative sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = x_n - g(x_n) + \mathcal{R}_\lambda^{H,M}[Hg(x_n) - \lambda\{A(x_n - f(x_n)) + \mathcal{C}_\lambda^{H,M}(x_n)\}].$$

If $\mathcal{C}_\lambda^{H,M}(x) \equiv 0$, then 4.1 reduces to the following algorithm.

Iterative Algorithm 4.2. *Given $x_0 \in X$, define iterative sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = x_n - g(x_n) + \mathcal{R}_\lambda^{H,M}[Hg(x_n) - \lambda\{A(x_n - f(x_n))\}].$$

Iterative Algorithm 4.2 gives the approximate solution of the (3.3).

Now, we give some sufficient conditions which guarantee the convergence of the approximate solution obtained through iterative sequence generated by the Iterative Algorithm 4.1.

Theorem 4.3. *Let X be a Banach space, and X^* be the dual space of X.*

1. *Let $g : X \to X$ be a k_1-strongly monotone and δ_1-Lipschitz continuous mapping,*
2. *$f : X \to X$ be k_2-strongly monotone and δ_2-Lipschitz continuous mapping,*
3. *$H : X \to X^*$ be strongly monotone with constant r and Lipchitz continuous with constant s and let $A : X \to X$ be ζ- Lipchitz continuous. If the following conditions are satisfied:*

$$\lambda > \frac{(rt_1 + s\delta_1) - r}{\zeta t_2 + L}, \qquad r > rt_1 + s\delta_1, \tag{4.6}$$

$$\text{where} \quad t_1 = \sqrt{(1 - 2k_1 + 64C\delta_1^2)} \quad \text{and} \quad t_2 = \sqrt{(1 - 2k_2 + 64C\delta_2^2}.$$

Then the iterative sequence $\{x_n\}$ generated by the Algorithm 4.1, converge to the unique solution of CI (3.2).

Proof. By iterative Algorithm 4.1 and Theorem 3.1

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|x_n - g(x_n) + \mathcal{R}_\lambda^{H,M}[g(x_n) - \lambda\{A(x_n - f(x_n)) + \mathcal{C}_\lambda^{H,M}(x_n)\}] \\
&-(x_{n-1} - g(x_{n-1}) + \mathcal{R}_\lambda^{H,M}[g(x_{n-1}) - \lambda\{A(x_{n-1} - f(x_{n-1})) + \mathcal{C}_\lambda^{H,M}(x_{n-1})\}]\| \\
&\leq \|x_n - x_{n-1} - g(x_n) + g(x_{n-1})\| + \frac{1}{r}\|g(x_n) - \lambda\{A(x_n - f(x_n)) + \mathcal{C}_\lambda^{H,M}(x_{n-1}) \\
&\quad -g(x_{n-1}) - \lambda\{A(x_{n-1} - f(x_{n-1})) + \mathcal{C}_\lambda^{H,M}(x_{n-1})\| \\
&\leq \|x_n - x_{n-1} - g(x_n) + g(x_{n-1})\|| + \frac{1}{r}\|Hg(x_n) - Hg(x_{n-1})\| \\
&+\frac{\lambda}{r}\|A(x_n - f(x_n)) - A(x_{n-1} - f(x_{n-1}))| + \frac{\lambda}{r}\|\mathcal{C}_\lambda^{H,M}(x_n) - \mathcal{C}_\lambda^{H,M}(x_{n-1})\|.
\end{aligned} \tag{4.7}$$

Since $g : X \to X$ is k_1-strongly accretive, δ_1-Lipschitz continuous mapping and X is Banach space, by Proposition 2.1 we have

$$\begin{aligned}
&\|x_n - x_{n-1} - g_(x_n) + g_(x_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\langle \mathcal{J}(x_n - x_{n-1}) - (g(x_n) + g(x_{n-1}))\rangle \\
&\quad -(g(x_n) + g(x_{n-1})) \\
&= \|x_n - x_{n-1}\|^2 - 2\langle \mathcal{J}(x_n - x_{n-1}), (g(x_n) + g(x_{n-1}))\rangle \\
&\quad +2\langle \mathcal{J}(x_n - x_{n-1}) - (g(x_n) + g(x_{n-1}))\rangle \\
&\quad -\mathcal{J}(x_n - x_{n-1}) - (g(x_n) + g(x_{n-1})) \\
&\leq \|x_n - x_{n-1}\|^2 - 2k_1\|x_n - x_{n-1}\|^2 \\
&\quad +4d^2\rho_X(4\|g(x_n) - g(x_{n+1})\|/d) \\
&\leq (1-2k_1)\|x_n - x_{n-1}\|^2 + C\|g(x_n) - g(x_{n+1})\|^2 \\
&\leq (1 - 2k_1 + 64C\delta_1^2)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{4.8}$$

By the Lipschitz continuity of H and g we have

$$\|Hg(x_n) - Hg(x_{n-1})\| \leq s\delta_1\|x_n - x_{n-1}\|. \tag{4.9}$$

By using Theorem 2.7 we have

$$\|\mathcal{C}_\lambda^{H,M}(x_n) - \mathcal{C}_\lambda^{H,M}(x_{n-1})\| \leq L\|x_n - x_{n-1}\|. \tag{4.10}$$

By using the Lipschitz continuity of A we have

$$\begin{aligned}
\|A(x_n - f(x_n)) - A(x_{n-1} - f(x_{n-1}))\| &\leq \zeta\|(x_n - f(x_n)) - (x_{n-1} - f(x_{n-1}))\| \\
&= \zeta\|(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1})\|.
\end{aligned} \tag{4.11}$$

Since $f : X \to X$ is k_2-strongly accretive, δ_2-Lipschitz continuous mapping

$$\begin{aligned}
&\|(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 + 2\langle \mathcal{J}(x_n - x_{n-1}) - (f_(x_n) + f_(x_{n-1}))\rangle \\
&\quad -(f_(x_n) + f_(x_{n-1})) \\
&= \|x_n - x_{n-1}\|^2 - 2\langle \mathcal{J}(x_n - x_{n-1}), (f(x_n) + f(x_{n-1}))\rangle \\
&\quad +2\langle \mathcal{J}(x_n - x_{n-1}) - (f(x_n) + f(x_{n-1}))\rangle \\
&\quad -\mathcal{J}(x_n - x_{n-1}) - (f(x_n) + f(x_{n-1})) \\
&\leq \|x_n - x_{n-1}\|^2 - 2k_2\|x_n - x_{n-1}\|^2 \\
&\quad +4d^2\rho_X(4\|f(x_n-) - f(x_{n+1})\|/d) \\
&\leq (1-2k_2)\|x_n - x_{n-1}\|^2 + C\|f(x_n) - f(x_{n+1})\|^2 \\
&\leq (1 - 2k_2 + 64C\delta_2^2)\|x_n - x_{n-1}\|^2.
\end{aligned} \tag{4.12}$$

Using (4.11) and (4.12), one has

$$\|A(x_n - f(x_n)) - A(x_{n-1} - f(x_{n-1}))\| \leq \zeta\sqrt{(1 - 2k_1 + 64C\delta_1^2)}\|x_n - x_{n-1}\|^2.$$

$$\|x_{n+1} - x_n\| \leq \mu\|x_n - x_{n-1}\|,$$

where

$$\mu = \sqrt{(1 - 2k_1 + 64C\delta_1^2)} + \frac{\lambda}{r}\Big(\zeta\sqrt{(1 - 2k_2 + 64C\delta_2^2}\Big) + \frac{1}{r}\Big(L + s\delta_1\Big). \tag{4.13}$$

From (4.6) and (4.13) , we have $0 < \mu < 1$ and so x_n is a Cauchy sequence. Let $x_n \to x$ as $n \to \infty$. It follows from 4.1 that

$$g(x) = \mathcal{R}_\lambda^{H,M}[Hg(x) - \lambda\{A(x - f(x)) + \mathcal{C}_\lambda^{H,M}(x)\}]. \tag{4.14}$$

Hence by Theorem 3.1, x is the solution of CI (3.2). Next, we proceed for the uniqueness of the solution. For this, let x^* be another solution of CI (3.2) then Theorem 3.1 implies that

$$g(x^*) = \mathcal{R}_\lambda^{H,M}[Hg(x^*) - \lambda\{A(x^* - f(x^*)) + \mathcal{C}_\lambda^{H,M}(x^*)\}]. \tag{4.15}$$

By (4.14) and (4.15) and the similar argument as above, we have

$$\|x - x^*\| \leq \mu\|x - x^*\|,$$

where

$$\mu = \sqrt{(1 - 2k_1 + 64C\delta_1^2)} + \frac{\lambda}{r}\big(\zeta\sqrt{(1 - 2k_2 + 64C\delta_2^2}\big) + \frac{1}{r}\big(L + s\delta_1\big).$$

Since $0 < \mu < 1$, $x = x^*$, therefore, x is the unique solution of problem. □

5. Numerical Example

Now we construct the following numerical example to show the convergence analysis of the sequence x_n to the unique solution of the CI (3.2).

Example 5.1. Let $X = \mathbb{R}$, $A, f, g, H : X \to X$ be a single valued mapping defined as $A(x) = \frac{1}{4}x$, $g(x) = \frac{x}{3}$, $f(x) = \frac{2}{3}x$ and $H(x) = \frac{6}{5}x$ and $M : X \to 2^X$ be a multi-valued defined as $M(x) = \Big\{\frac{1}{10}x\Big\}$.

Now

$$\langle Hx - Hy, x - y\rangle = \Big\langle \frac{6}{5}x - \frac{6}{5}y, x - y\Big\rangle = \frac{6}{5}\|x - y\|^2 \geq \frac{11}{10}\|x - y\|^2$$

and

$$\|Hx - Hy\| = \left\|\frac{6}{5}x - \frac{6}{5}y\right\| = \frac{6}{5}\|x - y\| \leq \frac{13}{10}\|x - y\|.$$

That is, H is strongly monotone with constant $r = \dfrac{11}{10}$ and Lipschitz continuous with constant $s = \dfrac{13}{10}$.

Similarly, we can easily verify that A, g and f are Lipschitz continuous with constants $\zeta = 1$, $\delta_1 = \dfrac{1}{2}$ and $\delta_2 = \dfrac{3}{4}$, respectively. and f and g are strongly accretive with constants $k_1 = \dfrac{1}{4}$ and $k_2 = \dfrac{3}{4}$ respectively. Now, for $\lambda = 1$, the resolvent operator and the Cayley operator is given by

$$\mathcal{R}_\lambda^{H,M}(x) = \{H + \lambda M\}^{-1}(x) = \frac{13}{10}x$$

and

$$\mathcal{C}_\lambda^{H,M}(x) = 2\mathcal{R}_\lambda^{H,M}(x) - H(x) = \frac{22}{65}x.$$

Also

$$\left\|\mathcal{R}_\lambda^{H,M}(x) - \mathcal{R}_\lambda^{H,M}(y)\right\| = \left\|\frac{10}{13}x - \frac{10}{13}y\right\| \leq \frac{1}{(11/10)}\|x - y\|$$

and

$$\left\|\mathcal{C}_\lambda^{H,M}(x) - \mathcal{C}_\lambda^{H,M}(y)\right\| = \left\|\frac{22}{65}x - \frac{22}{65}y\right\| \leq \frac{31}{10}\|x - y\|.$$

This shows that $\mathcal{R}_\lambda^{H,M}$ is Lipschitz continuous with constant $\dfrac{1}{r} = \dfrac{10}{11}$ and $\mathcal{C}_\lambda^{H,M}(x)$ is Lipschitz continuous with constant $L = \left(\dfrac{2 + rs}{r}\right) = \dfrac{31}{10}$.

We can see that all the constants satisfy Condition (4.6). Now we compute the sequence $\{x_n\}$ by the iterative Algorithm 4.1

$$\begin{aligned}
x_{n+1} &= x_n - g(x_n) + \mathcal{R}_\lambda^{H,M}\left[Hg(x_n) - \lambda\{A(x_n - f(x_n)) + \mathcal{C}_\lambda^{H,M}(x_n)\}\right] \\
&= x_n - \frac{x_n}{3} + \mathcal{R}_\lambda^{H,M}\left[\frac{6}{5}\left(\frac{x_n}{3}\right)\left\{\left(\frac{1}{4}\right)(x_n - \frac{2}{3}x_n) + \frac{22}{65}x_n\right\}\right] \\
&= \frac{2}{3}x_n + \mathcal{R}_\lambda^{H,M}\left[\frac{-225}{11700}x_n\right] \\
&= \frac{2}{3}x_n + \frac{13}{10}\left[\frac{-225}{11700}x_n\right] \\
&= \frac{224055}{351000}x_n.
\end{aligned}$$

The Iterative Algorithm 4.1 has been implemented using Python. For the following different initial values $x_0 = 3,\ -3,\ 5,\ -5$, it shows that the sequence x_n converges to the solution of the CI (3.2), $x^* = 0$. In this regard the convergence graph and computational table are respectively shown in (Figure 1 and Table 1), below:

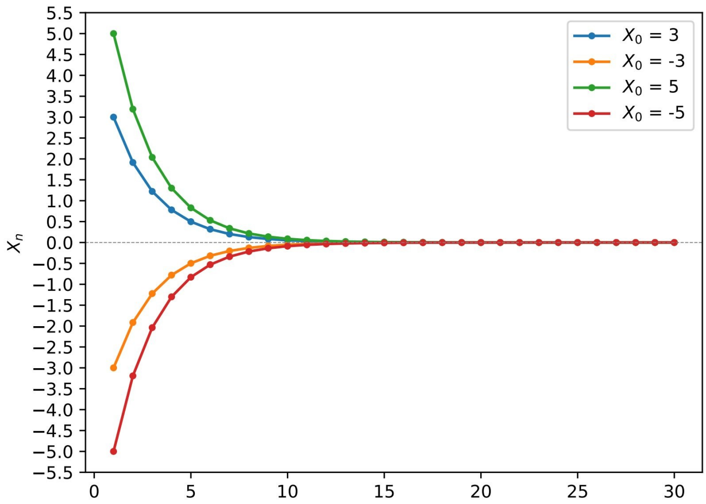

FIGURE 1.

Iteration	$x_0 = 3$	$x_0 = -3$	$x_0 = 5$	$x_0 = -5$
1	3	-3	5	-5
2	1.915	-1.915	3.1917	-3.1917
3	1.2224	-1.2224	2.0374	-2.0374
4	0.7803	-0.7803	1.3005	-1.3005
5	0.4981	-0.4981	0.8302	-0.8302
6	0.318	-0.318	0.5299	-0.5299
7	0.5299	-0.5299	0.3383	-0.3383
8	0.1296	-0.1296	0.2159	-0.2159
9	0.0827	-0.0827	0.1378	-0.1378
10	0.0528	-0.0528	0.0880	-0.0880
11	0.0337	-0.0337	0.0562	-0.0562
12	0.0215	-0.0215	0.0359	-0.0359
13	0.0137	-0.0137	0.0229	-0.0229
14	0.0087	-0.0087	0.0146	-0.0146
15	0.0056	-0.0056	0.0093	-0.0093
16	0.0036	-0.0036	0.0059	-0.0059
17	0.0023	-0.0023	0.0038	-0.0038
18	0.0015	-0.0015	0.0024	-0.0024
19	0.0010	-0.0010	0.0015	-0.0015
20	0.0006	-0.0006	0.0010	-0.0010
21	0.0004	-0.0004	0.0006	-0.0006
22	0.0003	-0.0003	0.0004	-0.0004
23	0.0002	-0.0002	0.0003	-0.0003
24	0.0001	-0.0001	0.0002	-0.0002
25	0	0	0.0001	-0.0001
26	0	0	0	0

TABLE 1.

6. Conclusion

In this paper, we have studied H-monotone associated with general Cayley inclusion within the framework of Banach spaces. Further, we have established existence of solution using fixed point approach through resolvent technique. The applicability of the results and the iterative technique adopted is also demonstrated through numerical example and analyszed numerically and graphically using Python(matplotlib).

Declaration of competing interest

The authors declare that they have no competing interest regarding this research manuscript.

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