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MFAT METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

# SOME SPECTRAL RESULTS FOR CERTAIN POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. This paper examines the spectral properties of specific classes of positive operators arising from matrices associated with the linear complementarity problem. Such operators occupy a central position in diverse domains of mathematics and physics, including operator theory, functional analysis, and quantum mechanics. A thorough understanding of their spectral behavior is fundamental for exploring the dynamics and stability of systems governed by these operators. P-matrix is one of the important types of matrices appearing in linear complementarity problems. In this paper, with the help of spectral results we have given a factorization for P-matrices, as the product of two non-trivial P-matrices. We also focus on elucidating spectral properties such as eigenvalues, approximate eigenvalues and spectral values associated with certain positive operators.

## 1. Introduction

Eigenvalues and eigenvectors, key concepts in linear algebra, have wide-ranging applications in both science and engineering. Certain properties of a matrix can be analyzed if the eigenvalues of the matrix are known to us. For example, if a matrix has no non-zero eigenvalue, then it is invertible. The spectrum is the infinite-dimensional analogue of the set of matrix eigenvalues. Several core results of matrix theory are extended to linear operators on a Hilbert space, where the proofs are typically quite different and one often needs additional assumptions on the operators.

An $n \times n$ real matrix $A$ (it is denoted by $A \in \mathbb{R}^{n\times n}$) is called a P-matrix ($P_0$-matrix) if all its principal minors are positive (non-negative). The matrix $A \in \mathbb{R}^{n\times n}$ reverses the sign of the vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i \leq 0$, for all $i \in \langle n \rangle$, where $x_i$ is the $i$-th component of the vector $x \in \mathbb{R}^n$ and $\langle n \rangle = \{1, 2, 3, \dots, n\}$. Fiedler and Pták [3] have shown that $A$ is a P-matrix if and only if $A$ has sign non-reversal property, that is, if $x \neq 0 \in \mathbb{R}^n$, then there exists some index $j$ for which we have $x_j(Ax)_j > 0$. Cottle et al. [2] have shown that given a real square matrix $A$, the linear complementarity problem $LCP(A, q)$ has a unique solution for each vector $q \in \mathbb{R}^n$ if and only if $A$ is a P-matrix. The P-matrices encompass such notable classes as the Hermitian positive definite matrices, the M-matrices, the totally positive matrices, the real diagonally dominant matrices with positive diagonal entries, and many more. The study of P-matrices has extended to the notion of P-operator to infinite-dimensional Banach spaces having a Schauder basis by Kannan and Sivakumar [8] and P-operator to infinite-dimensional Hilbert spaces relative to various orthonormal bases by Rashid and Johnson [13].

Spectral theory on Hilbert space forms the cornerstone of modern functional analysis, providing powerful tools for understanding the behavior of linear operators on infinite-dimensional spaces. In particular, the study of P-operators within this framework

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holds significant importance, with wide-ranging applications in diverse fields such as quantum mechanics, signal processing, and partial differential equations. The theory of P-matrices (P-operators) explores transformations that preserve positivity, a concept crucial for modeling various physical phenomena and mathematical processes through linear complementarity problem [1, 7, 11, 14]. In this context, P-property is intimately linked with the notions of order, stability, and growth, offering profound insights into the underlying structure of the space under consideration.

We present a foundational exploration of the spectral theory for P-operators and sufficient operators on Hilbert spaces, detailing their spectral properties, including eigenvalues, eigenvectors, and spectral decompositions. Moreover, we will investigate the connections between P-property and the spectral characteristics of these operators, shedding light on their geometric, algebraic, and analytic properties. Through this exploration, we aim to provide a comprehensive understanding of spectral theory on Hilbert spaces for P-operators, equipping the reader with the necessary tools to tackle advanced topics in functional analysis and its applications.

## 2. PRELIMINARIES

**Theorem 2.1.** [14] *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix. Then the following statements hold.*

1. *The matrix $A + D$ is a P-matrix, for any diagonal matrix $D$ with non-negative entries.*
2. *The inverse $A^{-1}$ of $A$ is a P-matrix.*

**Theorem 2.2.** [3] *Let $A \in \mathbb{R}^{n\times n}$. The following statements are equivalent :*

1. *$A$ is a P-matrix.*
2. *The real eigenvalues of the principal submatrices of $A$ are positive.*

**Theorem 2.3.** [3] *Let $A$ be a real square matrix with all non-positive off-diagonal elements. The following statements are equivalent :*

1. *$A$ is a P-matrix.*
2. *Each real eigenvalue of $A$ is positive.*
3. *The real part of each eigenvalue of $A$ is positive.*

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be $n$ real numbers or complex numbers having both conjugate pairs in the collection. The $k^{th}$ elementary symmetric function is defined by

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \prod_{t=1}^{k} \lambda_{i_t}.$$

The elementary symmetric functions characterize the spectra of P-matrices and P$_0$-matrices as shown in the following result.

**Theorem 2.4.** [5] *The set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a spectrum of some P-matrix iff*

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0, \; k = 1, 2, \ldots, n.$$

In the above result, if $\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) \ge 0, k = 1, 2, \ldots, n$, it characterizes P$_0$-matrices.

**Example 2.5.** Let $A = \begin{pmatrix} -1 & -1 \\ 4 & 3 \end{pmatrix}$ be a matrix whose spectrum is $\{1, 1\}$. Though the symmetric functions $\sigma_k(\{1, 1\})$ are positive, for $k = 1, 2$, the matrix $A$ is not a P-matrix. However, by Theorem 2.4, the set $\{1, 1\}$ is a spectrum of the P-matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a matrix having positive sums of principal minor, then it satisfies

$$\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0, k = 1, 2, \ldots, n.$$

Moreover, the converse is also true as shown in Theorem 2.4: $\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is positive for $k = 1, 2, \ldots, n$ if and only if it is a spectrum of some P-matrix ($P_0$-matrix). Such a set $S$ is called a P-set ($P_0$-set).

Kellogg [9] proved that elements of a P-set cannot lie in the given wedge around the negative axis. More precisely he proved the following result.

**Theorem 2.6.** [9]

(1) *If* $\mathcal{K} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *is a P-set, then*

$$|arg\ \lambda_i| < \frac{n-1}{n}\pi, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

(2) *If* $\mathcal{K} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_i \neq 0$, $i = 1, 2, \ldots, n$, *is a* $P_0$*-set, then*

$$|arg\ \lambda_i| \leq \frac{n-1}{n}\pi, \quad i = 1, 2, \ldots, n.$$

*Equality holds in the above inequality if and only if*

$$\begin{array}{rcll} \sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) & = & 0, & k = 1, 2, \ldots, n-1, \\ \sigma_n(\lambda_1, \lambda_2, \ldots, \lambda_n) & > & 0. & \end{array}$$

It is proved in [6] that if the number of elements in $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ in the right half plane, or in the left half plane, is given, then the bound of Equation (2.1) can be improved, namely, there exists $\alpha$ such that

$$|arg\ \lambda_i| < \alpha < \frac{n-1}{n}\pi,\ i = 1, 2, \ldots, n.$$

It is also proved that if the set $\mathcal{K} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ has exactly one element in the right half plane, then

$$|arg\ \lambda_i| < \frac{2}{3}\pi,\ i = 1, 2, \ldots, n.$$

The above inequality is independent of $n$. It was conjectured in [6] that if a P-matrix $A$ has exactly $k$ eigenvalues in the left half plane ($k$ is an even integer), then

$$|arg\ \lambda| < \frac{2k+1}{2k+2}\pi, \quad \text{for all } \lambda \in \sigma(A).$$

The following result has given a negative answer to the conjecture.

**Theorem 2.7.** [5] *Let* **C** *be a finite set of complex numbers, consisting of positive numbers and non-real conjugate pairs. Then one can obtain a set with all elementary symmetric functions positive by adding positive numbers to* **C**.

Every P-matrix can have almost all of its eigenvalues in the left half of the complex plane.

**Theorem 2.8.** [5] *There exists a P-matrix all of those eigenvalues, except one when* $n$ *is even or two when* $n$ *is odd, have negative real parts.*

A matrix $A$ all of whose powers are P-matrices is denoted $A \in \mathcal{PM}$ (respectively, $A \in \mathcal{P}_0\mathcal{M}$ if $A^k$ is a $P_0$-matrix for all powers $k$). If we denote the set of eigenvalues with multiple appearances corresponding to algebraic multiplicities of $A$ by $\sigma(A)$, it is then natural to raise the following question :

$$\text{if } A \in \mathcal{P}M,\ \text{does } \lambda \in \sigma(A) \text{ imply } \lambda > 0?$$

The above question has been answered affirmatively by Hershkowitz [5] for $n \leq 4$ and it is not resolved for $n \geq 5$.

**Definition 2.9.** [4] A matrix $A \in \mathbb{R}^{n\times n}$ is called column sufficient (CSU) if $x_i(Ax)_i \leq 0$ for all $i \in \langle n\rangle$, implies $x_i(Ax)_i = 0$ for all $i \in \langle n\rangle$. A matrix $A$ is said to be row sufficient (RSU) if $A^T$ is column sufficient. A matrix that is both row and column sufficient is simply called sufficient (SU). We denote the class of sufficient matrices by **S**.

## 3. Factorization of P-matrices using eigenvalues

**Proposition 3.1.** *Let $A$ be an $n \times n$ real matrix such that $I + A$ is invertible. We define*

$$U(A) = (I + A)^{-1}(I - A).$$

*Then the following statements hold:*

(1) $A = U(U(A)) = (I + U(A))^{-1}(I - U(A))$.
(2) $I + U(A) = 2(I + A)^{-1}$.
(3) *If $A$ is invertible, then* $I - U(A) = 2(I + A^{-1})^{-1}$.

*Proof.* (1) We have $(I + A)U(A) = I - A$. So, $U(A) + AU(A) = I - A$. Hence

$$A(I + U(A)) = I - U(A). \tag{3.2}$$

It is noted that if $(I + U(A))x = 0$, then $x = 0$. Therefore $-1 \notin \sigma(U(A))$. Since $(I + U(A))^{-1}$ and $I + U(A)$ commute, by the Equation (3.2), we get that $A = U(U(A))$.

(2) We have

$$\begin{aligned} I + U(A) &= I + (I + A)^{-1}(I - A) \\ &= (I + A)^{-1}[I + A + I - A] \\ &= 2(I + A)^{-1}. \end{aligned}$$

(3) We have

$$\begin{aligned} I - U(A) &= I - (I + A)^{-1}(I - A) \\ &= (I + A)^{-1}[I + A - I + A] \\ &= 2(I + A)^{-1}A. \end{aligned}$$

If $A$ is invertible, then we have

$$\begin{aligned} I - U(A) &= 2(I + A)^{-1}A \\ &= 2(I + A)^{-1}(A^{-1})^{-1} \\ &= 2[A^{-1}(I + A)]^{-1} \\ &= 2(I + A^{-1})^{-1}. \end{aligned}$$

□

**Theorem 3.2.** *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix. Then $A$ is a product of P-matrices.*

*Proof.* By Theorem 2.3, each real eigenvalue of $A$ is positive. Hence $A$ has no negative real eigenvalues, thus $(I + A)^{-1}$ and $(I - A)$ exist. Therefore $U(A) = (I + A)^{-1}(I - A)$ is well defined. By Proposition 3.1, we have

$$\begin{aligned} I + U(A) &= 2(I + A)^{-1} \\ I - U(A) &= 2(I + A^{-1})^{-1}. \end{aligned}$$

By Theorem 2.1, $I+U(A)$ and $I-U(A)$ are P-matrices. Thus $A = (I-U(A))^{-1}(I-U(A))$ is a factorization of $A$. □

**Definition 3.3.** A matrix $A \in \mathbb{R}^{n\times n}$ is called positive stable if all its eigenvalues have positive real parts.

**Proposition 3.4.** *Let $A \in \mathbb{R}^{n\times n}$ be a P-matrix and let $D \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix. Then $AD$ is positive stable.*

*Proof.* Since $A$ is a P-matrix, it has no negative real eigenvalues. As $D$ is a diagonal matrix with positive diagonal entries, every eigenvalue of $AD$ has positive real parts. □

The following result describes the factorization of a P-matrix into a product of positive stable matrices which are P-matrices as well.

**Theorem 3.5.** *Let $A \in \mathbb{R}^{n\times n}$. If $A$ is a P-matrix, then there exist positive diagonal matrices $S$ and $T$ such that $S^{-1}AT$ is a product of positive stable P-matrices.*

*Proof.* By Theorem 2.1, $I + U(A)$ and $I - U(A)$ are P-matrices. Thus by Proposition 3.4, $(I + U(A))S$ and $(I - U(A))T$ are positive stable matrices. Thus,

$$\begin{aligned} S^{-1}AT &= S^{-1}(I+U(A))^{-1}(I-U(A))T \\ &= [(I+U(A))S]^{-1}(I-U(A))T. \end{aligned}$$

□

## 4. Spectral results for certain positive operators

We start this section with different spectra of operators on a Hilbert space. We denote $\mathcal{B}(\mathcal{H})$ the collection of all bounded linear operators on a Hilbert space $\mathcal{H}$.

**Definition 4.1.** [12] Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. A scalar $k$ is called an eigenvalue of $T$ if there is a non-zero $x \in \mathcal{H}$ such that $Tx = kx$. In this case, $x$ is called an eigenvector of $T$ corresponding to $k$.

The set of all eigenvalue of $T$ is called the eigenspectrum of $T$. We denote eigenspectrum (or point spectrum) of $T$ by $\sigma_e(T)$.

A scalar $k$ is called an approximate eigenvalue of $T$ if the operator $T-kI$ is not bounded below, that is, for every $\beta > 0$, there is some $x \in \mathcal{H}$ with $||x|| = 1$ and $||Tx - kx|| < \beta$. The set of all approximate eigenvalues of $T$ constitutes the approximate eigenspectrum of $T$, we denote it by $\sigma_a(T)$.

$$\sigma_a(T) = \{k \in \mathbb{K} : (T - kI)x_n \longrightarrow 0 \text{ for some sequence } \{x_n\} \text{ in } \mathcal{H} \text{ with } ||x_n|| = 1 \ \forall n\}.$$

A scalar $k$ is called spectral value of $T$ if the bounded operator $T - kI$ is not invertible in $\mathcal{B}(\mathcal{H})$. The set of all spectral values of $T$ is called the spectrum of $T$. We denote it by $\sigma(T)$.

**Theorem 4.2.** [10] *(Spectral theorem for compact operators). Let $\mathcal{H}$ be an infinite dimensional Hilbert space, and let $T$ be a compact operator.*

1. *Except for the possible value $0$, the spectrum of $T$ is entirely point spectrum; in other words $\sigma(T)\backslash\{0\} = \sigma_e(T)\backslash\{0\}$.*
2. *We have $0 \in \sigma(T)$, and $0 \in \sigma_e(T)$ if and only if $T$ is not injective.*
3. *The point spectrum outside of 0 is countable and has finite multiplicity: for each $\lambda \in \sigma_e(T)\backslash\{0\}$, we have $dim(\lambda I - T) < +\infty$.*
4. *Assume $T$ is normal. Let $\mathcal{H}_0 = ker(T)$, and $\mathcal{H}_1 = ker(T)^{\perp}$. Then $T$ maps $\mathcal{H}_0$ to $\mathcal{H}_0$ and $\mathcal{H}_1$ to $\mathcal{H}_1$; on $\mathcal{H}_1$, which is separable, there exists an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and $\lambda_n \in \sigma_e(T)\backslash\{0\}$ such that*

$$\lim_{n\to 0} \lambda_n = \{0\}$$

*and $Te_n = \lambda_n e_n$ for all $n > 1$.*

*In particular, if* $\{f_i\}_{i\in I}$*, where* $I$ *is an index set, is an arbitrary orthonormal basis of* $\mathcal{H}_0$*, which may not be separable, we have*

$$T\left(\sum_{i\in I}\alpha_i f_i + \sum_{n\geq 1}\alpha_n e_n\right) = \sum_{n\geq 1}\lambda_n\alpha_n e_n$$

*for all scalars* $\alpha_n$, $\alpha_i \in \mathbb{C}$ *for which the vector on the left-hand side lies in* $\mathcal{H}$*, and the series on the right converges in* $\mathcal{H}$*. This can be expressed also as*

$$Tv = \sum_{n\geq 1}\lambda_n\langle v, e_n\rangle e_n. \tag{4.3}$$

**Proposition 4.3.** *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a P-operator relative to an orthonormal basis* $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$*. Then any real eigen value of* $T$ *is positive.*

*Proof.* Let $\lambda \in \sigma_e(T)$. Then there exists $x \neq 0$ such that $Tx = \lambda x$. As $T \in \mathcal{B}(\mathcal{H})$ is a P-operator relative to the orthonormal basis $\mathcal{B}$, there exists some index $j$ for which, $\langle x, e_j\rangle\langle Tx, e_j\rangle > 0$, that is, $\langle x, e_j\rangle\langle \lambda x, e_j\rangle > 0$ implies $\langle x, e_j\rangle\lambda\langle x, e_j\rangle > 0$. Therefore, $\lambda\langle x, e_j\rangle^2 > 0$, this implies $\lambda > 0$.

□

In general, we prove the following result.

**Theorem 4.4.** *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a P-operator relative to an orthonormal basis* $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$*. Then every eigenvalue* $\lambda \in \sigma_e(T)$ *satisfies*

$$\mathrm{Re}(\lambda) > 0.$$

*Proof.* Suppose, to the contrary, that there exists an eigenvalue $\lambda \in \sigma_e(T)$ with $\mathrm{Re}(\lambda) \leq 0$. Then there exists a nonzero vector $x \in H$ such that

$$Tx = \lambda x.$$

Consider the coordinate-wise products:

$$\langle x, e_i\rangle\langle Tx, e_i\rangle = \langle x, e_i\rangle\langle \lambda x, e_i\rangle = \lambda|\langle x, e_i\rangle|^2, \text{ for all } i.$$

Since $\mathrm{Re}(\lambda) \leq 0$, it follows that for each $i$,

$$\mathrm{Re}\left(\langle x, e_i\rangle\langle Tx, e_i\rangle\right) = \mathrm{Re}(\lambda)|\langle x, e_i\rangle|^2 \leq 0.$$

This implies

$$\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0 \quad \text{for all } i,$$

with equality possibly for some $i$. By the P-operator property, it gives $x = 0$, contradicting the choice of an eigenvector. Thus, no eigenvalue of $T$ can have non-positive real part. □

**Example 4.5.** Consider the matrix

$$A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Then $A$ is a P-matrix. The eigenvalues of $A$ are given by

$$\lambda = 1 \pm i.$$

**Proposition 4.6.** *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a P-operator relative to an orthonormal basis* $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$*. Then any real element in* $\sigma_a(T)$ *is positive.*

*Proof.* Let $\lambda \in \sigma_a(T)$ be real. Then there exists a sequence $\{x_n\} \in \mathcal{H}$ with $||x_n|| = 1$ and $Tx_n - \lambda x_n \longrightarrow 0$ as $n \longrightarrow \infty$, that is, $(T - \lambda I)x_n \longrightarrow 0$ as $n \longrightarrow \infty$, This implies that $Tx_n \longrightarrow \lambda x$ as $n \longrightarrow \infty$, where $x_n \longrightarrow x$ as $n \longrightarrow \infty$. As $T$ is a P-operator relative to the orthonormal basis $\mathcal{B}$ and $||x_n|| = 1$, there exists $j_n$ such that $\langle x_n, e_{j_n}\rangle\langle Tx_n, e_{j_n}\rangle > 0$, that is, $\langle x, e_{j_n}\rangle\lambda\langle x, e_{j_n}\rangle > 0$ an $n \longrightarrow \infty$. Thus $\lambda\langle x, e_{j_n}\rangle^2 > 0$, hence $\lambda > 0$. □

**Theorem 4.7.** *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator and let $\mathcal{H}_0 = ker(T)$ and $\mathcal{H}_1 = ker(T)^{\perp}$. If $T'$, the restriction of $T$ to $\mathcal{H}_1$, is a P-operator relative to an orthonormal basis $\mathcal{B}' = \{e_i\}_{i=1}^{\infty}$, where $\mathcal{B}'$ is the sequence of eigenvectors of $T$ with $Te_i = \lambda_i e_i$, Then there exists a positive operator $R \in K(\mathcal{H})$ such that $R^2 = T$, which is denoted $\sqrt{T'}$ or $(T')^{1/2}$. It is unique among positive bounded operators.*

*Proof.* We have $Te_i = \lambda_i e_i$, where $\lambda_i \in \sigma_e(T)\backslash\{0\}$. As $T'$ is a P-operator relative to $\mathcal{B}'$, we have

$$\begin{aligned} \lambda_i &= \lambda_i\langle e_i, e_i\rangle, \\ &= \langle \lambda_i e_i, e_i\rangle, \\ &= \langle Te_i, e_i\rangle, \\ &= \langle e_i, e_i\rangle\langle Te_i, e_i\rangle > 0 \ \forall i. \end{aligned}$$

Now we define

$$R(u_0 + u_1) = \sum_{i\geq 1} \sqrt{\lambda_i}\alpha_i e_i.$$

For any $u_0 \in \mathcal{H}_0$ and $u_1 = \sum_{i\geq 1} \alpha_i e_i \in \mathcal{H}_1$, we can see that $R$ is a diagonal operator with coefficients $\sqrt{\lambda_i}$. Now by Theorem 4.2, we have $\sqrt{\lambda_i} \longrightarrow 0$ as $i \longrightarrow \infty$. This shows that $R$ is a well defined compact operator. Thus for every $u \in \mathcal{H}_1$, we have

$$\begin{aligned} R^2(u) &= R\left(\sum_{i\geq 1} \sqrt{\lambda_i}\alpha_i e_i\right), \\ &= \sum_{i\geq 1} \lambda_i\alpha_i e_i, \\ &= T'u \quad \forall u \in \mathcal{H}_1. \end{aligned}$$

Therefore $R^2 = T'$. We next show the uniqueness. Let $R \in B(\mathcal{H})$ be such that $R^2 = T'$. Then $RT' = RR^2 = R^2R = T'R$. That is, $R$ and $T'$ commute. It follows that any non-zero eigenvalue $\lambda_i$ of $T'$, $R$ induces the operator

$$R_i : ker(T' - \lambda_i I) \longrightarrow ker(T' - \lambda_i I)$$

given by $R_i = \sqrt{\lambda_i} I$ on the finite-dimensional $\lambda_i$ - eigenspace of $T'$, we can see that $R_i$ are P-operators relative to the orthonormal basis of $ker(T' - \lambda_i I)$ and it satisfies $R_i^2 = \lambda_i iI$. Also note that

$$\begin{aligned} \|R(u)\|^2 &= \langle R(u), R(u)\rangle, \\ &= \langle R^2(u), u\rangle, \\ &= \langle T'(u), u\rangle. \end{aligned}$$

Thus

$$\begin{aligned} ker(R) &= \{u \in \mathcal{H} : R(u) = 0\}, \\ &= \{u \in \mathcal{H} : ||R(u)||^2 = 0\}, \\ &= \{u \in \mathcal{H} : \langle T'(u), u\rangle = 0\}. \end{aligned}$$

Now by the expression 4.3 of the Theorem 4.2, we have

$$T'u \;=\; \sum_{i\geq 1}\lambda_i\langle u, e_i\rangle e_i.$$

Thus

$$\begin{aligned}\langle T'u, u\rangle &= \langle\sum_{i\geq 1}\lambda_i\langle u, e_i\rangle e_i, u\rangle\\ &= \sum_{i\geq 1}\lambda_i|\langle u, e_i\rangle|^2.\end{aligned}$$

In terms of orthonormal basis of eigenvectors $\{e_i\}_{i=1}^{\infty}$ and the positivity $\lambda_i > 0$ of eigenvalues, we have $\langle T'u, u\rangle = 0$ if and only if $u$ is perpendicular to span of $\{e_i\}_{i=1}^{\infty}$. Thus the P-operator $R$ is uniquely determined on each eigenspace of $T'$ and $ker(T')$. By the spectral theorem, this implies that $R$ is unique. □

Next we define the concept of $k$th elementary symmetric function in infinite dimensional cases as follows:

**Definition 4.8.** Let $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots)$ denote the $k$th symmetric function of the numbers $\lambda_1, \lambda_2, \lambda_3 \ldots$, given by

$$\sigma_k(\lambda_1, \lambda_2, \lambda_3, \cdots) = \sum_{1\leq i_1<i_2<\cdots<i_k}\lambda_{i_1}\cdot\lambda_{i_2}\cdots\lambda_{i_k}.$$

**Theorem 4.9.** *The set* $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ *is an eigen spectrum of some P-operator* $T \in \mathcal{B}(\mathcal{H})$ *relative to an orthonormal basis* $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ *if and only if*

$$\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots) > 0, \;\; k = 1, 2, 3, \ldots$$

*Proof.* Suppose that the set $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ is an eigen spectrum of some P-operator relative to an orthonormal basis $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$. As $\lambda_i \in \sigma_e(T)$, for each $i$, we have $\lambda_i > 0$ by Proposition 4.3. As $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots)$ is the result of sums of the products of $\lambda_i$, we have $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots) > 0$ for each $k$.

Conversely, suppose that the set $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ satisfies $\sigma_k(\lambda_1, \lambda_2, \lambda_3, \ldots) > 0$ for each $k$. Let us define the operator $T \in \mathcal{B}(\mathcal{H})$ by

$$Tx = \sum_{i=1}^{\infty}\alpha\sigma_i\langle x, e_i\rangle e_i,$$

where $\alpha > 0$ so small, so that $T$ is bounded and $\sigma_i$ denotes $\sigma_i = \sigma_i(\lambda_1, \lambda_2, \lambda_3, \ldots)$. Then $T$ is the required P-operator relative to the orthonormal basis $\mathcal{B}$. □

**Theorem 4.10.** *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a P-operator relative to an orthonormal basis* $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$. *Then* $T$ *satisfies*

$$\inf_{x\in\mathcal{H}^+}\sup_{e_i\in\mathcal{B}}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} \geq 0$$

*where* $\mathcal{H}^+ = \{x \in \mathcal{H} : \langle x, e_i\rangle \neq 0$ *for all* $i\}$. *Moreover, if* $T$ *satisfies* $Tu = \rho(T)u$ *for some* $u \in \mathcal{H}^+$, *then*

$$\inf_{x\in\mathcal{H}^+}\sup_{e_i\in\mathcal{B}}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} = \sup_{e_i\in\mathcal{B}}\inf_{x\in\mathcal{H}^+}\frac{\langle Tx, e_i\rangle}{\langle x, e_i\rangle} = \rho(T).$$

*Proof.* As $T$ is a P-operator relative to an orthonormal basis $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$, there exists an index $j$ such that $\langle x, e_j\rangle\langle Tx, e_j\rangle > 0$, thus $\frac{\langle x,e_j\rangle\langle Tx,e_j\rangle}{\langle x,e_j\rangle^2} > 0$, that is, $\frac{\langle Tx,e_j\rangle}{\langle x,e_j\rangle} > 0$, so for any $x \in \mathcal{H}^+$, $\sup_{e_i\in\mathcal{H}}\frac{\langle Tx,e_j\rangle}{\langle x,e_j\rangle} > 0$. Thus if we take infimum of these positive numbers we get that $\inf_{x\in\mathcal{H}^+}\sup_{e_i\in\mathcal{B}}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle} \geq 0$. Now

$$\inf_{x\in\mathcal{H}^+}\sup_{e_i\in\mathcal{B}}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle}\leq\sup_{e_i\in\mathcal{B}}\frac{\langle Tu,e_i\rangle}{\langle u,e_i\rangle}=\sup_{e_i\in\mathcal{B}}\frac{\rho(T)\langle u,e_i\rangle}{\langle u,e_i\rangle}=\rho(T). \tag{4.4}$$

Also we have,

$$\sup_{e_i\in\mathcal{B}}\inf_{x\in\mathcal{H}^+}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle}\geq\frac{\langle Tu,e_i\rangle}{\langle u,e_i\rangle}=\frac{\rho(T)\langle u,e_i\rangle}{\langle u,e_i\rangle}=\rho(T). \tag{4.5}$$

Now by the well known result that for a function $\psi : X\times Y\longrightarrow X\times Y$, where $X$ and $Y$ are any two sets, we have

$$\inf_{x\in X}\sup_{y\in Y}\psi\geq\sup_{y\in Y}\inf_{x\in X}\psi$$

and from equations (4.4) and (4.5) we get

$$\rho(T)\leq\sup_{e_i\in\mathcal{B}}\inf_{x\in\mathcal{H}^+}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle}\leq\inf_{x\in\mathcal{H}^+}\sup_{e_i\in\mathcal{B}}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle}\leq\rho(T). \tag{4.6}$$

Hence

$$\inf_{x\in\mathcal{H}^+}\sup_{e_i\in\mathcal{B}}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle}=\sup_{e_i\in\mathcal{B}}\inf_{x\in\mathcal{H}^+}\frac{\langle Tx,e_i\rangle}{\langle x,e_i\rangle}=\rho(T).$$

□

**Theorem 4.11.** *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$, $S,T\in\mathcal{B}(\mathcal{H})$ and $D$ be a diagonal operator such that $De_i=d_{ii}e_i$, where $0\leq d_{ii}\leq 1$. Then the following statements hold good :*

1. *If $ST^{-1}$ is a P-operator relative to the orthonormal basis $\mathcal{B}$, then $0\notin\sigma_e(DT+(I-D)S)$.*
2. *If $S^{-1}T$ is a P-operator relative to the orthonormal basis $\mathcal{B}$, then $0\notin\sigma_e(TD+S(I-D))$.*

*Proof.* (1) Let $ST^{-1}$ be a P-operator relative to $\mathcal{B}$. Suppose that $0\in\sigma_e(DT+(I-D)S)$. Then there exists some $0\neq x\in\mathcal{H}$ such that $0=(DT+(I-D)S)x=(D+(I-D)ST^{-1})Tx$. Set $y=Tx$. Then $y\neq 0$, since $T$ is invertible. Now, $Dy+(I-D)ST^{-1}y=0$ implies $\langle De_i,e_i\rangle\langle y,e_i\rangle=-\langle (I-D)e_i,e_i\rangle\langle ST^{-1}y,e_i\rangle$. If $\langle y,e_i\rangle\geq 0$, then $\langle (ST^{-1}y),e_i\rangle\leq 0$ so that $\langle y,e_i\rangle\langle (ST^{-1}y),e_i\rangle\leq 0$. If $\langle y,e_i\rangle\leq 0$, then $\langle (ST^{-1}y),e_i\rangle\geq 0$ so that $\langle y,e_i\rangle\langle (ST^{-1}y),e_i\rangle\leq 0$. That is, $\langle y,e_i\rangle\langle (ST^{-1}y),e_i\rangle\leq 0$ for all $i$, a contradiction to $ST^{-1}$ being a P-operator relative to $\mathcal{B}$. Thus $0\notin\sigma_e(DT+(I-D)S)$.

(2) Let $S^{-1}T$ be a P-operator relative to $\mathcal{B}$. Suppose that $0\in\sigma_e(TD+S(I-D))$. Then there exists some $0\neq x\in\mathcal{H}$ such that $0=(TD+S(I-D))x=S(S^{-1}TD+(I-D))x$. As $S$ is invertible, we have $S^{-1}TDx+(I-D)x=0$. Now, $\langle (I-D)e_i,e_i\rangle\langle x,e_i\rangle=-\langle De_i,e_i\rangle\langle S^{-1}Tx,e_i\rangle$. If $\langle x,e_i\rangle\geq 0$, then $\langle (S^{-1}Tx),e_i\rangle\leq 0$ so that $\langle x,e_i\rangle\langle (S^{-1}Tx),e_i\rangle\leq 0$. If $\langle x,e_i\rangle\leq 0$, then we get $\langle (S^{-1}Tx),e_i\rangle\geq 0$ so that $\langle x,e_i\rangle\langle (S^{-1}Tx),e_i\rangle\leq 0$. That is, $\langle x,e_i\rangle\langle (S^{-1}Tx),e_i\rangle\leq 0$ for all $i$, a contradiction to $S^{-1}T$ being a P-operator relative to $\mathcal{B}$. Thus $0\notin\sigma_e(TD+S(I-D))$.

□

Next we discuss some spectral results for operator form of sufficient matrices, called sufficient operators in Hilbert space settings defined as follows.

**Definition 4.12.** Let $\mathcal{B}=\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Then $T\in\mathcal{B}(\mathcal{H})$ is said to be:

1. C-sufficient relative to the given orthonormal basis $\mathcal{B}$ if for $x\in\mathcal{H}$, the inequalities $\langle x,e_i\rangle\langle Tx,e_i\rangle\leq 0$ for all $i$ imply that $\langle x,e_i\rangle\langle Tx,e_i\rangle=0$ for all $i$.

(2) R-sufficient relative to the given orthonormal basis $\mathcal{B}$ if the adjoint of $T$ is a C-sufficient operator relative to $\mathcal{B}$.
(3) sufficient if it is both a C-sufficient and R-sufficient operator relative to the given orthonormal basis $\mathcal{B}$.

**Example 4.13.** Consider the Hilbert space $\mathcal{H} = \ell^2$, the space of square summable sequences and $\mathcal{B}_1 = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathcal{H}$. Let $T$ be an operator on $\mathcal{H}$ defined by:

$$T(x_1, x_2, x_3, \dots) = (0, x_1 + x_2, 0, x_3 + x_4, 0, x_5 + x_6, 0, \dots).$$

Then $T$ is a bounded linear operator on $\mathcal{H}$. Now consider another orthonormal basis $\mathcal{B}_2 = \{Ue_i\}_i^{\infty}$ of $\mathcal{H}$, where $U$ is the unitary operator on $\mathcal{H}$ given by

$$U(x_1, x_2, x_3, \dots) = \Big(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} + \frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}, \dots\Big).$$

Then we see that the operator $T$ is a C-sufficient operator relative to the orthonormal basis $\mathcal{B}_2$. Because here the inequalities $\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle \leq 0$ for all $i$ imply that $(\frac{x_i}{\sqrt{2}} + \frac{x_{i+1}}{\sqrt{2}})^2 \leq 0$ and $(\frac{x_{i+1}^2}{2} - \frac{x_i^2}{2}) \leq 0$, solving these inequalities together will get $x_i = -x_{i+1}$, but in that case $T(x_1, x_2, x_3, \dots) = (0, 0, 0, \dots)$, thus we get $\langle x, Ue_i\rangle\langle Tx, Ue_i\rangle = 0$ for all $i$. This shows that $T$ is a C-sufficient operator relative to $\mathcal{B}_2$.

The operator $T$ is not a C-sufficient operator relative to the orthonormal basis $\mathcal{B}_1$. Because, here $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all $i$ imply that $x_1x_2 + x_2^2 \leq 0$, thus we can observe that for the element $(-5, \sqrt{2}, 0, 0, \dots)$ will satisfy that $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all $i$, but $\langle x, e_2\rangle\langle Tx, e_2\rangle \neq 0$, thus by definition, $T$ is not a C-sufficient operator relative to $\mathcal{B}_1$.

**Example 4.14.** Let us consider the Hilbert space $\mathcal{H} = \ell^2$ and $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathcal{H}$. Let $T \in B(\mathcal{H})$ be defined by

$$T(x_1, x_2, x_3, \dots) = (x_1, x_1, x_3, x_3, x_5, x_5, \dots).$$

The adjoint of the operator $T$ is given by

$$T^*(y_1, y_2, y_3, \dots) = (y_1 + y_2, y_3 + y_4, y_5 + y_6, \dots).$$

$T$ is a C-sufficient operator relative to the given orthonormal basis $\mathcal{B}$. As $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all $i$ imply that $\langle x, e_i\rangle\langle Tx, e_i\rangle = 0$ for all $i$. But $T$ is not R-sufficient operator, because the operator $T^*$ is not C-sufficient operator relative to the given orthonormal basis $\mathcal{B}$, for the non-zero element $(2, -3, 0, 0, 0, \dots)$, satisfies $\langle x, e_i\rangle\langle T^*x, e_i\rangle \leq 0$ for all $i$, but $\langle x, e_i\rangle\langle T^*x, e_i\rangle \neq 0$ when $i = 2$. If we let $S$ be the operator on $\mathcal{H}$ given by $S = T^*$, then $S^* = T$. As $S^*$ is C-sufficient relative to the orthonormal basis $\mathcal{B}$, the operator $S$ is a R-sufficient operator relative to the orthonormal basis $\mathcal{B}$. Also, we have seen that $S = T^*$ is not C-sufficient relative to the orthonormal basis $\mathcal{B}$.

Next, we give an example of a sufficient operator relative to an orthonormal basis.

**Example 4.15.** Let us consider the Hilbert space $\mathcal{H} = \ell^2$ and $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Let $T$ be an operator on $\mathcal{H}$ defined by

$$T(x_1, x_2, x_3, x_4, x_5, \dots) = (x_1, x_1 + x_2, x_3, x_3 + x_4, x_5, \dots).$$

Then $T$ is a bounded linear operator on $\mathcal{H}$ and the adjoint of the operator $T$ is given by

$$T^*(y_1, y_2, y_3, y_4, y_5, \dots) = (y_1 + y_2, y_2, y_3 + y_4, y_4, \dots).$$

Then $T$ is a C-sufficient operator relative to the given orthonormal basis $\mathcal{B}$. Also, $T$ is a R-sufficient operator relative to the given orthonormal basis $\mathcal{B}$. Thus $T$ is a sufficient operator relative to the given orthonormal basis $\mathcal{B}$.

**Definition 4.16.** Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$. Let $T \in \mathcal{B}(\mathcal{H})$. The term $\lambda$-eigenvector denotes an eigenvector $u$ corresponding to the eigenvalue $\lambda$ of $T$, i.e., $Tu = \lambda u$. If such an eigenvector $u$ is such that $\langle u, e_i\rangle \neq 0$ for all $i$, then $u$ is called a strictly nonzero eigenvector. On the other hand, if $\langle u, e_i\rangle = 0$ for at least one $i$, then $u$ is called a partly zero eigenvector.

**Remark 4.17.** Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ and $T \in \mathcal{B}(\mathcal{H})$. Let us denote $T_\alpha$ for $\alpha \subseteq \mathbb{N}$, the restriction of $T$ to $H' = span\{e_i : i \in \alpha\}$. Any vector in $H'$ we denote by $x_\alpha$.

**Theorem 4.18.** *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$. Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent.*

1. *$T$ is a C-sufficient operator relative to an orthonormal basis $\mathcal{B}$.*
2. *For any index set $\alpha \subseteq \mathbb{N}$ and non-negative diagonal operator $D_\alpha \neq 0$, if $T_\alpha + D_\alpha$ is not invertible, then every zero eigenvector of $T_\alpha + D_\alpha$ is partly zero eigenvector.*

*Proof.* (1) $\Rightarrow$ (2): Suppose there exists an index set $\alpha \subseteq \mathbb{N}$, a nonnegative diagonal operator $D_\alpha \neq 0$, and a strictly nonzero 0-eigenvector $x_\alpha$ such that $(T_\alpha + D_\alpha)x_\alpha = 0$. We denote an element in $H'$ by $x_\alpha$ whereas an element $y_{\overline{\alpha}}$ represents in $H \backslash H'$. Define a vector $y$ such that $y_\alpha = x_\alpha$ and $y_{\overline{\alpha}} = 0$. Then $\langle y, e_i\rangle\langle Ty, e_i\rangle \leq 0$ for all $i$. It follows from the hypothesis that $\langle y, e_i\rangle\langle Ty, e_i\rangle = 0$ for all $i$. Since $x_\alpha$ is a strictly nonzero vector, we have $-Dx_\alpha = T_\alpha x_\alpha = 0$, hence $D_\alpha = 0$, which is a contradiction. Thus the zero eigenvector of $T_\alpha + D_\alpha$ is a partly zero eigenvector.

(2) $\Rightarrow$ (1): Suppose $\langle y, e_i\rangle\langle Ty, e_i\rangle \leq 0$ for all $i$. Let $\alpha = \{i : \langle x, e_i\rangle \neq 0\}$. If $T_\alpha x_\alpha \neq 0$, then there exists a nonnegative diagonal matrix $D \neq 0$ such that $T_a x_\alpha = -Dx_\alpha$. It follows that there exists a strictly nonzero vector $x_\alpha$, satisfying $(T_\alpha + D)x_\alpha = 0$, which is a contradiction. Hence $T_\alpha x_\alpha = 0$, therefore, $\langle y, e_i\rangle\langle Ty, e_i\rangle \leq 0$ for all $i$. □

Sign-reversing is one of the properties of matrices along with a given vector, which is used to identify different classes of matrices having important applications. The concept of sign-reversing proves to be a valuable technique in delineating specific matrix classes in linear complementarity problems [4]. We define this sign-reversing property for operators on a Hilbert space $\mathcal{H}$ as follows.

**Definition 4.19.** Let $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and $T \in B(\mathcal{H})$. We say that $T$ reverses the sign of the vector $x \in \mathcal{H}$ relative to $\mathcal{B}$ if

$$\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0, \quad \text{for all } i.$$

**Example 4.20.** Let us consider the right shift operator $T_R$ on $\mathcal{H} = \ell^2$, defined by

$$T_R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots)$$

and let $\mathcal{B}_1 = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathcal{H}$. Then $T_R$ reverses the sign of the vector $x = (1, 0, 0, \ldots)$ relative to $\mathcal{B}_1$. Let us consider a unitary operator $U$ on $\ell^2$ given by

$$U(x_1, x_2, x_3, x_4, \ldots) = \Big(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} + \frac{x_4}{\sqrt{2}}, \frac{x_3}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}, \ldots\Big).$$

Then $\mathcal{B}_2 = \{Ue_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Here $\langle x, Ue_1\rangle\langle Tx, Ue_1\rangle = \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} = \frac{1}{2} > 0$, so $T$ does not reverses the sign of the vector $x$ relative to $\mathcal{B}_2$.

**Definition 4.21.** Let $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and $T \in B(\mathcal{H})$. We denote the sign reversing set of $T$ relative to $\mathcal{B}$ by $rev_{\mathcal{B}}(T)$ which is defined by

$$rev_{\mathcal{B}}(T) = \{x : \langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0, \quad \text{for all } i\}.$$

**Example 4.22.** Let $T_R$ be the right shift operator on $\mathcal{H} = \ell^2$ and let $\mathcal{B} = \{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\mathcal{H}$. Then

$$rev_{\mathcal{B}}(T_R) = \{x \in \ell^2 : x_i x_{i+1} \leq 0, \ \text{ for all } i\}.$$

**Theorem 4.23.** *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ and $T$ be a C-sufficient operator relative to $\mathcal{B}$. Then any eigenvector corresponding to a non-zero eigenvalue is not an element of $rev_{\mathcal{B}}(T)$.*

*Proof.* Let $\lambda \neq 0 \in \sigma_e(T)$. Then there exists $x \neq 0 \in \mathcal{H}$ such that $Tx = \lambda x$. If $x \in rev_{\mathcal{B}}(T)$, then $\langle x, e_i\rangle\langle Tx, e_i\rangle \leq 0$ for all $i$. As $T$ is a C-sufficient operator, we have $\langle x, e_i\rangle\langle Tx, e_i\rangle = 0$ for all $i$, that is, $\langle x, e_i\rangle\langle \lambda x, e_i\rangle = 0$ for all $i$. Thus $\lambda\langle x, e_i\rangle^2 = 0$ for all $i$, this implies that $\langle x, e_i\rangle = 0$ for all $i$. Thus $x = 0$, which is a contradiction. Hence $x \notin rev_{\mathcal{B}}(T)$. ☐

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