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FIXED POINT RESULTS OF A GENERALIZED MODIFIED $F-$CONTRACTION VIA RECTANGULAR $G-\alpha-$ADMISSIBLE MAPPING WITH APPLICATION

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Abstract. In this paper, we introduce the notion of a generalized modified $F-$contraction via rectangular $G-\alpha-$admissible mapping and establish some new fixed point results under Wardowski's $F-$contractive condition $(F1)$ together with an auxiliary function ϕ in the setting of $G-$metric spaces. Our new fixed point results improve, generalize and unify several related result in the existing literature. Moreover, we give some non-trivial examples to verify our results. Finally, we derive the existence of a solution to an integral equation to support our main findings.

1. Introduction

Wardowski [13] introduced the concept of an F-contraction, which constitutes a novel type of contractive mapping, and established a fixed point theorem that serves as a generalization of the classical Banach contraction principle [4]. This contribution opened new directions in fixed point theory by extending the applicability of contractive conditions beyond the traditional setting. Building on this foundation, Piri and Kumam [15] further refined the framework of F-contractions by enlarging the class of admissible functions. Specifically, they modified the original definition of Wardowski by replacing condition $(F3)$ with the weaker assumption $(F3')$, thereby introducing a broader category of F-contractions and demonstrating that fixed point results can still be obtained under these relaxed conditions.

In a parallel development, Mustafa and Sims [20] introduced the notion of G-metric spaces as a generalization of standard metric spaces. Their work not only extended the classical concept of distance but also explored the associated topological structures and fundamental properties of these spaces. The introduction of G-metric spaces provided a fertile ground for the study of fixed point theory in more generalized settings. Indeed, over the last two decades, G-metric spaces have become an active and rapidly growing area of research, with numerous authors investigating their applications to nonlinear analysis, topology, and various branches of applied mathematics.

In 2017, Singh et al. [12] introduced the notion of Hardy-Rogers type generalized F-contractive mappings in the framework of G-metric spaces and established corresponding fixed point results. This contribution demonstrated that the classical Hardy-Rogers contractive condition can be effectively extended to the setting of F-contractions, thereby enriching the theory of fixed points in generalized metric structures. In the same year, Singh et al. [14] further advanced this line of research by formulating the concept of Ćirić-type generalized F-contractions in G-metric spaces and proving several fixed point theorems in this context. These works emphasized the flexibility of the F-contraction approach and its ability to unify and generalize existing contractive mappings within the

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framework of G-metric spaces.

Also in 2017, Aydi et al. [18] proposed a broadened version of F-contractive mappings in the setting of metric spaces. By relaxing and widening the original definition of Wardowski's F-contractions, they were able to establish new fixed point theorems that extend the applicability of these results beyond the earlier restrictive conditions. This line of inquiry stimulated further research, and a number of studies subsequently emerged that focused on generalizing these classes of contractive mappings in diverse spaces such as partial metric spaces, b-metric spaces, and other generalized structures (see [2]-[9], [21]-[25]).

More recently, Vujaković et al. [1] introduced new contractive conditions in the setting of complete metric spaces and established corresponding fixed point theorems. Their results not only improved upon those obtained in [18], but also provided sharper and more comprehensive generalizations, thereby contributing to the ongoing development of fixed point theory in generalized contractive frameworks.

Motivated by the works of the above researchers, we define a generalized modified $F-$contraction mappings recognizing the concept of Singh et al. [12, 14] and Vujaković et al. [1] and obtain some fixed point results under the condition $(F1)$ together with an auxiliary function ϕ in $G-$metric spaces that can not be obtained from the existing results in the setting of associated metric spaces. Moreover, illustrative examples are provided to validate the applicability of our results. As a practical application, we employ one of the main theorems to demonstrate the existence of solutions to an integral equation in complete $G-$metric spaces.

2. Preliminaries

In this section, we collect the background materials to make our presentation as self-contained as possible.

Definition 2.1. [20] Let X be a nonempty set and $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions:
$(G1)$ $G(x, y, z) = 0$ if $x = y = z$;
$(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
$(G3)$ $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots$,(symmetry in all three variables);
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

G is called a generalized or a $G-$metric on X. The pair (X, G) is called a $G-$metric space.

Proposition 2.2. [20] *Every $G-$metric space (X, G) will define a metric space (X, d) by $d(x, y) = G(x, y, y) + G(y, x, x)$, for all $x, y \in X$.*

Definition 2.3. [26] A $G-$metric space (X, G) is said to be symmetric if $G(x, x, y) = G(x, y, y)$ *for all* $x, y \in X$.

Example 2.4. Let $X = [0, \infty)$ and $G(x, y, z) =| x - y | + | y - z | + | x - z |$ be a $G-$metric on X. Then $d(x, y) = G(x, x, y) + G(y, y, x) = 4 | x - y |$ is a metric on X. In addition, G is symmetric on X.

Lemma 2.5. [19] *By the rectangle inequality $(G5)$ together with the symmetry $(G4)$, we have*
$G(x, y, y) = G(y, y, x) \leq G(y, x, x) + G(x, y, x) = 2\, G(y, x, x).$

Definition 2.6. [13] Let $F : (0, \infty) \to (-\infty, \infty)$ be a function satisfying the following conditions:
$(F1)$ F is strictly increasing;
$(F2)$ for all nonnegative sequence $\{x_n\}$, $lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} F(x_n) = -\infty$;
$(F3)$ there exists $0 < k < 1$ such that $\lim_{x\to 0^+} x^k F(x) = 0$.

In this paper, we denote by $\digamma$ the family of all functions satisfying conditions $(F1) - (F3)$.

Definition 2.7. [13] Let (X, d) be a metric space. A self-mapping T on X is called an $F-$contraction if there exists $\tau > 0$ such that for $x, y \in X$
$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y))$, where $F \in \digamma$.

Piri et al. [15] replaced Wardowski's $F-$contractive condition $(F3)$ as follows:
$(F1)$ F is strictly increasing,
$(F3')$ F is continuous on $(0, \infty)$.
Let us denote Γ be the family of all functions $F : [0, \infty) \to (-\infty, \infty)$ satisfying the conditions $(F1)$ and $(F3')$.
Singh et al. [14] defined Ćirić-type generalized $F-$contractive mapping in the setting of $G-$metric spaces and established some fixed point results as follows:

Definition 2.8. [14] A mapping $T : X \to X$ is said to be a Ciric-type generalized $F-$contractive mapping in $G-$metric spaces if $F \in \Gamma$ and there exists $\tau > 0$, such that $G(Tx, Ty, T^2y) > 0 \implies \tau + F(G(Tx, Ty, T^2y) \leq F(\alpha\mu(x, y))$ for all $x, y \in X$, where $0 < \alpha < 1$ and $\mu(x, y) = \max\{G(x, y, Ty), G(x, Tx, Ty), G(x, Tx, T^2x), \frac{1}{2}(G(y, Ty, T^2y) + G(y, T^2x, T^2y))\}$.

Theorem 2.9. [14] *Let (X, G) be a complete $G-$metric space and $T : X \to X$ be a Ciric-type generalized $F-$contractive mapping. Then T has a fixed point in X. Moreover, if $2\alpha \leq 1$, then fixed point of T is unique.*

Samet et al. [16] introduced the following mappings:

Definition 2.10. [16] Let X be a nonempty set and $\alpha : X \times X \to [0, \infty)$ be a function. A self-mapping T on X is called

(i) $\alpha - admissible\ if\ for\ all\ x, y \in X,\ \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$,
(ii) a triangular $\alpha-$admissible if it is $\alpha-$admissible and if for all $x, y, z \in X$
$(\alpha(x, y) \geq 1\ and\ \alpha(y, z) \geq 1) \implies \alpha(x, z) \geq 1$.

Lemma 2.11. [17] *Let T be a triangular $\alpha-$admissible mapping on a nonempty set X. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$. Then $\alpha(x_m, x_n) \geq 1$ for $m, n \geq 0$ with $m < n$.*

There have been lots of generalizations of both Wardowski's fixed point result and the notion of $F-$ contraction. Let Φ be the set of mappings $\phi : [0, \infty) \to [0, \infty)$ complying with conditions:
(a) ϕ is nondecreasing,
(b) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.
This set is also known as a set of $c-$comparison functions. Observe that $\phi(t) < t$ for all $t > 0$ and ϕ is continuous at 0.

In 2021, Vujaković et al. [1] introduced new contraction mapping and established some fixed point results in complete metric spaces.

Definition 2.12. [1] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a modified $F-$contraction via triangular $\alpha-$admissible if there exists $\tau > 0$ such that for all $x, y \in X$ with $\alpha(x,y)d(Tx,Ty) > 0 \implies \tau + F(\alpha(x,y)d(Tx,Ty)) \leq F(\phi(d(x,y)))$, where the mappings $F \in \digamma$ and $\phi \in \Phi$.

Theorem 2.13. [1] *Let (X, d) be a complete metric space and $T : X \to X$ be a modified $F-$contraction via triangular $\alpha-$admissible mapping. Suppose that*

(i) *T is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*
Then T has a fixed point.

In the following theorem, Vujakovic et al. [1] replaced condition (iii) of Theorem 2.13 with the condition (H).
(H) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \to x \in X$ *as* $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k.

Theorem 2.14. [1] *Let (X, d) be a complete metric space and $T : X \to X$ be a modified $F-$contraction via triangular $\alpha-$admissible mapping. Suppose that*

(i) *T is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *(H) holds.*

Then T has a fixed point.

Let condition (U) holds, i.e., $\alpha(x, y) \geq 1$ for all $x, y \in Fix(T)$, where $Fix(T)$ denotes the set of all fixed points of T. Then, Aydi et al. [18], adding condition (U) to the hypotheses of Theorem 2.13 (resp. Theorem 2.14) ensures that x is the unique fixed point of T.

Definition 2.15. [10] Let (X, G) be a $G-$metric space, let T be a self-mapping on X, and
let $\alpha : X^3 \to [0, \infty)$ be a function. We say that T is a $G - \alpha-$admissible mapping if $x, y, z \in X$,

$$\alpha(x, y, z) \geq 1 \implies \alpha(Tx, Ty, Tz) \geq 1.$$

Definition 2.16. [11] Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha : X \times X \times X \to [0, \infty)$ be a function. We say that T is a rectangular $G - \alpha-$admissible mapping if

$$(T1) \quad \alpha(x, y, z) \geq 1 \implies \alpha(Tx, Ty, Tz) \geq 1, \ x, y, z \in X,$$

$$(T2) \begin{cases} \alpha(x, y, y) \geq 1 \\ \alpha(y, z, z) \geq 1 \implies \alpha(x, z, z) \geq 1, x, y, z \in X. \end{cases}$$

3. Main Results

In this section, we define a generalized modified $F-$contraction via rectangular $G-\alpha-$admissible mapping and establish some fixed point results in $G-$metric spaces.

Definition 3.1. Let (X, G) be a $G-$metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function with $\phi(t) < t$ for all $t > 0$ and continuous at 0, $\alpha : X^3 \to [0, \infty)$ be a function, and let $F : [0, \infty) \to (-\infty, \infty)$ be a strictly increasing function. A mapping $T : X \to X$ is said to be a generalized modified $F-$contraction via rectangular $G - \alpha-$admissible if there exists $\tau > 0$ such that for $x, y \in X$ with

$$\alpha(x, y, Ty)G(Tx, Ty, T^2y) > 0 \implies \tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) \leq F(\phi(\mu(x, y))), \tag{3.1}$$

where

$$\mu(x,y)=\max\left\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),G(y,Ty,T^2y),G(y,T^2x,T^2y)\right\}. \tag{3.2}$$

Theorem 3.2. *Let (X,G) be a complete $G-$metric space and $T:X\to X$ be a generalized modified $F-$contraction via rectangular $G-\alpha-$admissible mapping. Suppose that*

(i) *T is rectangular $G-\alpha-$admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0,T^2x_0)\geq 1$;*
(iii) *T is continuous.*

Then T has a fixed point in X.

Proof. By (ii), there exists x_0 in X such that $\alpha(x_0,Tx_0,T^2x_0)\geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_n=Tx_{n-1}\ for\ all\ n\geq 1.$$

If there exists some positive integer n_0 such that $x_{n_0}=Tx_{n_0}$, then x_{n_0} is a fixed point of T.
Now, we shall assume that $x_n\neq x_{n+1}$ for all $n\geq 0$. So, we have $G(x_n,x_{n+1},x_{n+2})\neq 0$ for all $n\geq 0$. Due to (ii) and T is $G-\alpha-$admissible, we have $\alpha(x_0,x_1,x_2)=\alpha(x_0,Tx_0,T^2x_0)\geq 1$ which in turn implies that $\alpha(x_1,x_2,x_3)=\alpha(Tx_0,Tx_1,T^2x_1)\geq 1$. Inductively, we get

$$\alpha(x_n,x_{n+1},x_{n+2})\geq 1\ for\ all\ n\geq 0.$$

Using 3.1, 3.2, and $\alpha(x_{n-1},x_n,x_{n+1})\geq 1$, putting $x=x_{n-1}$ and $y=x_n$ for $n\geq 1$, we obtain the following:

$$\begin{aligned}\alpha(x_{n-1},x_n,Tx_n)G(Tx_{n-1},Tx_n,T^2x_n)>0 &\implies\\ \tau+F(\alpha(x_{n-1},x_n,Tx_n)G(Tx_{n-1},Tx_n,T^2x_n))&\leq F(\phi(\mu(x_{n-1},x_n))),\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\mu(x_{n-1},x_n)&=\max\left\{G(x_{n-1},x_n,Tx_n),G(x_{n-1},Tx_{n-1},Tx_n),G(x_{n-1},Tx_{n-1},T^2x_{n-1})\right.\\ &\qquad\left.,G(x_n,Tx_n,T^2x_n),G(x_n,T^2x_{n-1},T^2x_n)\right\}\\ &=\max\left\{G(x_{n-1},x_n,x_{n+1}),G(x_{n-1},x_n,x_{n+1}),G(x_{n-1},x_n,x_{n+1}),G(x_n,x_{n+1},x_{n+2})\right.\\ &\qquad\left.,G(x_n,x_{n+1},x_{n+2})\right\}\\ &=\max\{G(x_{n-1},x_n,x_{n+1}),G(x_n,x_{n+1},x_{n+2})\}.\end{aligned}$$

If $\mu(x_{n-1},x_n)=G(x_n,x_{n+1},x_{n+2})$, then 3.3 becomes

$$\tau+F(\alpha(x_{n-1},x_n,x_{n+1})G(x_n,x_{n+1},x_{n+2}))\leq F(\phi(G(x_n,x_{n+1},x_{n+2}))).$$

Using the properties of F and ϕ, it follows that $G(x_n,x_{n+1},x_{n+2})<G(x_n,x_{n+1},x_{n+2})$, which is a contradiction.
Hence, $\mu(x_{n-1},x_n)=G(x_{n-1},x_n,x_{n+1})$ for all $n\geq 1$. Now, 3.3 becomes

$$\tau+F(\alpha(x_{n-1},x_n,x_{n+1})\ G(x_n,x_{n+1},x_{n+2}))\leq F(\phi(G(x_{n-1},x_n,x_{n+1})))\ for\ all\ n\geq 1. \tag{3.4}$$

Using 3.4 and properties of F and ϕ, we get

$$G(x_n,x_{n+1},x_{n+2})<G(x_{n-1},x_n,x_{n+1})\ for\ all\ n\geq 1.$$

Hence, $\{G(x_{n-1},x_n,x_{n+1})\}$ is nonincreasing and bounded below. So, there exists $r\geq 0$ such that

$$\lim_{n\to\infty}G(x_{n-1},x_n,x_{n+1})=r.$$

Assume $r > 0$ and using 3.4, we have

$$\tau + F(G(x_n, x_{n+1}, , x_{n+2})) \leq F(G(x_{n-1}, x_n, x_{n+1})). \tag{3.5}$$

Furthermore, by the property of F and taking limit as $n \to \infty$ in 3.5, we get

$$\tau + F(r) \leq F(r),$$

which is a contradiction. Hence, we have

$$\lim_{n\to\infty} G(x_{n-1}, x_n, x_{n+1}) = 0. \tag{3.6}$$

Now, we need to show that $\{x_n\}$ is a Cauchy sequence in X. Suppose that $\{x_n\}$ in X is not a Cauchy sequence, then there exists an $\epsilon > 0$ such that

$$G(x_{m_k}, x_{m_k+1}, x_{n_k}) \geq \epsilon, \tag{3.7}$$

where $\{m_k\}$ and $\{n_k\}$ are sequences of positive integers with $n_k > m_k > k$ for all positive integers k. In addition, n_k is chosen as the smallest integer satisfying 3.7. Hence, we have

$$G(x_{m_k}, x_{m_k+1}, x_{n_k-1}) < \epsilon. \tag{3.8}$$

Applying the rectangular inequality and using 3.7 and 3.8, we get

$$\begin{aligned}\epsilon \leq G(x_{m_k}, x_{m_k+1}, x_{n_k}) &\leq G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + G(x_{n_k-1}, x_{m_k}, x_{m_k+1}) \\ &\leq 2G(x_{n_k-1}, x_{n_k}, x_{n_k}) + G(x_{m_k}, x_{m_k+1}, x_{n_k-1}) \\ &< 2G(x_{n_k-1}, x_{n_k}, x_{n_k}) + \epsilon. \end{aligned}\tag{3.9}$$

Taking the limit as $k \to \infty$ in 3.9 and using 3.6 , we obtain

$$\lim_{n\to\infty} G(x_{m_k}, x_{m_k+1}, x_{n_k}) = \epsilon. \tag{3.10}$$

Again, from the rectangle inequality, we have

$$\begin{aligned}G(x_{m_k}, x_{m_k+1}, x_{n_k}) &\leq G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) + G(x_{m_k-1}, x_{m_k+1}, x_{n_k}) \\ &\leq G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) + G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}) \\ &\leq 2G(x_{m_k-1}, x_{m_k}, x_{m_k}) + 2G(x_{n_k-1}, x_{n_k}, x_{n_k}) + G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}). \end{aligned}\tag{3.11}$$

Taking the limit as $k \to \infty$ in 3.11 and using 3.6 and 3.7, we have

$$\epsilon \leq \lim_{n\to\infty} G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}). \tag{3.12}$$

Furthermore, with the similar approach as above, we have

$$\begin{aligned}&G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}) \\ &\leq G(x_{n_k-1}, x_{n_k}, x_{n_k}) + G(x_{m_k-1}, x_{m_k}, x_{m_k}) + G(x_{n_k}, x_{m_k}, x_{m_k+1}). \end{aligned}\tag{3.13}$$

Taking the limit as $k \to \infty$ in 3.13 and using 3.6 and 3.10,

$$\lim_{k\to\infty} G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}) \leq \epsilon. \tag{3.14}$$

Using 3.12 and 3.14 , we get

$$\lim_{k\to\infty} G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}) = \epsilon. \tag{3.15}$$

Similarly, we have

$$G(x_{n_k-1}, x_{m_k-1}, x_{m_k+1}) \leq 2G(x_{m_k}, x_{m_k+1}, x_{m_k+1}) + G(x_{m_k-1}, x_{m_k}, x_{n_k-1}) \tag{3.16}$$

and

$$\begin{aligned}&G(x_{m_k-1}, x_{m_k}, x_{n_k-1}) \\ &\leq G(x_{m_k-1}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k+1}, x_{m_k+1}) + G(x_{m_k}, x_{m_k+1}, x_{n_k-1}). \end{aligned}\tag{3.17}$$

Taking limit as $k \to \infty$ in 3.16 and 3.17 and using 3.8 and 3.15, we obtain

$$\lim_{k\to\infty} G(x_{m_k-1}, x_{m_k}, x_{n_k-1}) = \epsilon. \tag{3.18}$$

Since T is rectangular $G-\alpha-$admissible, we have

$$\alpha(x_{m_k-1}, x_{n_k-1}, x_{n_k}) \geq 1. \tag{3.19}$$

Using 3.1 with $x = x_{m_k-1}$ and $y = x_{n_k-1}$, we get
$\alpha(x_{m_k-1}, x_{n_k-1}, x_{n_k})G(x_{m_k}, x_{n_k}, x_{n_k+1}) > 0$
$\Longrightarrow \tau + F(\alpha(x_{m_k-1}, x_{n_k-1}, x_{n_k})G(x_{m_k}, x_{n_k}, x_{n_k+1})) \leq F(\phi(\mu(x_{m_k-1}, x_{n_k-1}))$.
It follows that by 3.19 and properties of F and ϕ, we obtain

$$\tau + F(G(x_{m_k}, x_{n_k}, x_{n_k+1})) \leq F(\mu(x_{m_k-1}, x_{n_k-1}), \tag{3.20}$$

where

$$\begin{aligned} &\mu(x_{m_k-1}, x_{n_k-1}) \\ &= \max\big\{G(x_{m_k-1}, x_{n_k-1}, x_{n_k}), G(x_{m_k-1}, x_{m_k}, x_{n_k}), G(x_{m_k-1}, x_{m_k}, x_{m_k+1}), \\ &\qquad G(x_{n_k-1}, x_{n_k}, x_{n_k+1}), G(x_{n_k-1}, x_{m_k+1}, x_{n_k+1})\big\}. \end{aligned}$$

Taking limit as $k \to \infty$ in 3.20 and using 3.6, 3.8, 3.10, 3.15 and 3.18 and F is strictly increasing, we get

$$\tau + F(\epsilon) \leq F(\epsilon),$$

which is a contradiction.
Hence, $\{x_n\}$ is a Cauchy sequence in a complete $G-$metric space X. Thus, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Since T is continuous, we get

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T[\lim_{n\to\infty} x_n] = Tu.$$

Therefore, T has a fixed point u in X. □

In the following we get a corollary of Theorem 3.2.

Corollary 3.3. *Let (X, G) be a complete $G-$metric space and $T : X \to X$ be a given mapping. Suppose that there exists $\tau > 0$ such that $G(Tx, Ty, T^2y) > 0 \Longrightarrow \tau + F(G(Tx, Ty, T^2y) \leq F(\phi(\mu(x, y)))$,*
for all $x, y \in X$, where F, ϕ and $\mu(x, y)$ are as defined in Theorem 3.2 and T is continuous. Then T has a fixed point in X.

Proof. The result follows from Theorem 3.2 by setting $\alpha(x, y, Ty) = 1$ for all x, y in X. □

In the following theorems we replace condition (iii) of Theorem 3.2 with :
(H) for any sequence $\{x_n\}$ in X with $\alpha(x_{n-1}, x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ such that $x_n \to u$ as $n \to \infty$ we have $\alpha(x_{n-1}, u, u) \geq 1$.

Theorem 3.4. *Let (X, G) be a complete $G-$metric space and $T : X \to X$ be a generalized modified $F-$contraction via rectangular $G-\alpha-$admissible mapping. Suppose that*

(i) *T is rectangular $G-\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \geq 1$;*
(iii) *(H) holds.*

Then T has a fixed point in X.

Proof. Following the proof of Theorem 3.2, we have that the sequence $\{x_n\}$ which is defined by
$x_n = Tx_{n-1}$ for all $n \geq 1$, converges to some element u in X. That is,

$$\lim_{n\to\infty} x_n = u.$$

Assume that $Tu \notin \{x_n\}$ in X for some positive integer k with $n \geq k$. Since by Theorem 3.4 (iii) and
$\alpha(x_{n-1}, x_n, x_{n+1}) \geq 1$ for all $n \geq 1$, we have $\alpha(x_{n-1}, u, u) \geq 1$. Hence, for all $n \geq k$ and letting $x = x_{n-1}$ and $y = u$ in 3.1 and 3.2, we have
$\alpha(x_{n-1}, u, Tu)G(Tx_{n-1}, Tu, T^2u) > 0 \implies$
$\tau + F(\alpha(x_{n-1}, u, Tu)G(Tx_{n-1}, Tu, T^2u)) \leq F(\phi(\mu(x_{n-1}, u)))$.
Using the properties of F and ϕ, we have
$\tau + F(G(x_n, Tu, T^2u) \leq F(\mu(x_{n-1}, u))$ and $G(x_n, Tu, T^2u) < \mu(x_{n-1}, u)$, where

$$\mu(x_{n-1}, u) = \max\big\{G(x_{n-1}, u, Tu), G(x_{n-1}, x_n, Tu), G(u, x_{n-1}, T^2u), G(x_{n-1}, x_n, x_{n+1}) \\ , G(u, Tu, T^2u)\big\}.$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mu(x_{n-1}, u) = \max\{G(u, u, Tu), G(u, u, T^2u)\}.$$

Suppose that

$$\lim_{n\to\infty} \mu(x_{n-1}, u) = G(u, u, Tu),$$

then

$$G(u, Tu, T^2u) < G(u, u, Tu)$$

and from $(G3)$, we get

$$G(u, Tu, T^2u) < G(u, u, Tu) \leq G(u, Tu, T^2u)$$

for $Tu \neq T^2u$, which is a contradiction. Thus

$$\lim_{n\to\infty} \mu(x_{n-1}, u) = G(u, u, T^2u).$$

Consequently, we have

$$G(u, Tu, T^2u) < G(u, u, T^2u)$$

and again from $(G3)$, we get

$$G(u, Tu, T^2u) < G(u, u, T^2u) \leq G(u, Tu, T^2u)$$

for $Tu \neq T^2u$, which is also a contradiction. Therefore, $Tu = T^2u = u$. Hence T has a fixed point in X. □

In the following we get a corollary of Theorem 3.4.

Corollary 3.5. *Let (X, G) be a complete $G-$metric space and $T : X \to X$ be a mapping. Suppose that there exists $\tau > 0$ such that $G(Tx, Ty, T^2y) > 0 \implies \tau + F(G(Tx, Ty, T^2y)) \leq F(\phi(\mu(x, y))$, for all $x, y \in X$, where F, ϕ and $\mu(x, y)$ are defined as in Theorem 3.4. Then T has a fixed point in X.*

Proof. The result follows from Theorem 3.4 by setting $\alpha(x, y, Ty) = 1$ for all x, y in X. □

Theorem 3.6. *Let (X, G) be a complete $G-$metric space, $T : X \to X$ be a mapping and $F : [0, \infty) \to (-\infty, \infty)$ be a strictly increasing function. Let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\phi(t) < t$ for all $t > 0$ and continuous at 0. Suppose that there exists $\tau > 0$ such that*

(i) $\alpha(x,y,Ty)G(Tx,Ty,T^2y)>0 \implies$
$\tau+F(\alpha(x,y,Ty)G(Tx,Ty,T^2y))\leq F(\phi(aG(x,y,Ty)+bG(x,Tx,Ty)$
$+cG(x,Tx,T^2x)+dG(y,Ty,T^2y)+eG(y,T^2x,T^2y))),$
for all $x,y\in X$, $a,b,c,d,e\geq 0$ *and* $a+b+c+d+e\leq 1$*;*

(ii) T *is rectangular* $G-\alpha-$*admissible;*

(iii) *there exists* $x_0\in X$ *such that* $\alpha(x_0,Tx_0,T^2x_0)\geq 1$*;*

(iv) T *is continuous or* (H) *holds.*

Then T *has a fixed point in* X*.*

Proof. Using properties of F, for all $x,y\in X$, we get

$$\alpha(x,y,Ty)G(Tx,Ty,T^2y)>0 \implies$$
$$\begin{aligned}\tau+F(\alpha(x,y,Ty)G(Tx,Ty,T^2y))&\leq F(\phi(aG(x,y,Ty)+bG(x,Tx,Ty)+cG(x,Tx,T^2x)\\&\quad+dG(y,Ty,T^2y)+eG(y,T^2x,T^2y)))\\&\leq F(\phi((a+b+c+d+e)\max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\\&\quad G(y,Ty,T^2y),G(y,T^2x,T^2y)\}))\\&\leq F(\phi(\max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),G(y,Ty,T^2y),G(y,T^2x,T^2y)\})).\end{aligned}$$

The rest of the proof follows from Theorems 3.2 and 3.4. □

Theorem 3.7. *Let* (X,G) *be a complete* $G-$*metric space,* T *be a self-mapping on* X *and* $F:[0,\infty)\to(-\infty,\infty)$ *be a strictly increasing function. Let* $\phi:[0,\infty)\to[0,\infty)$ *be a nondecreasing function with* $\phi(t)<t$ *for all* $t>0$ *and continuous at* 0*. Suppose that there exists* $\tau>0$ *such that*

(i) $\alpha(x,y,Ty)G(Tx,Ty,T^2y)>0 \implies$
$$\tau+F(\alpha(x,y,Ty)G(Tx,Ty,T^2y))$$
$$\leq F(\phi(\max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}))$$
for all $x,y\in X$*;*

(ii) T *is rectangular* $G-\alpha-$*admissible;*

(iii) *there exists* $x_0\in X$ *such that* $\alpha(x_0,Tx_0,T^2x_0)\geq 1$*;*

(iv) T *is continuous or* (H) *holds.*

Then T *has a fixed point in* X*.*

Proof. Since,
$\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\leq \max\{G(y,Ty,T^2y),G(y,T^2x,T^2y)\}$,
using (i), we have

$$\alpha(x,y,Ty)G(Tx,Ty,T^2y)>0 \implies$$
$$\tau+F(\alpha(x,y,Ty)G(Tx,Ty,T^2y))$$
$$\leq F(\phi(\max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),\frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}))$$
$$\leq F(\phi(\max\{G(x,y,Ty),G(x,Tx,Ty),G(x,Tx,T^2x),G(y,Ty,T^2y),G(y,T^2x,T^2y)\}))$$

for all $x,y\in X$. The rest of the proof follows from Theorems 3.2 and 3.4. □

Corollary 3.8. *Let* (X,G) *be a complete* $G-$*metric space,* $T:X\to X$ *be a mapping on* X *and* $F:[0,\infty)\to(-\infty,\infty)$ *be a strictly increasing function. Let* $\phi:[0,\infty)\to[0,\infty)$ *be a nondecreasing function with* $\phi(t)<t$ *for all* $t>0$ *and continuous at* 0*. Suppose that there exists* $\tau>0$ *such that*
$G(Tx,Ty,T^2y)>0 \implies$

$\tau + F(G(Tx, Ty, T^2y)) \le F(\phi(\max\{G(x, y, Ty), G(x, Tx, Ty), G(x, Tx, T^2x)$
$, \frac{G(y,Ty,T^2y)+G(y,T^2x,T^2y)}{2}\}))$
for all $x, y \in X$. Then T has a fixed point in X.

Proof. The result follows from Theorem 3.7, by setting $\alpha(x, y, Ty) = 1$ for all x, y in X. □

Theorem 3.9. *Adding the following conditions: (U) for all $x, y \in Fix(T)$, $\alpha(x, y, y) \ge 1$ and G is symmetric on $Fix(T)$ to the hypotheses of Theorem 3.2 and Theorem 3.4. Then T has a unique fixed point in X.*

Proof. Let $x, y \in Fix(T)$ with $x \ne y$. Then from 3.1, we have
$\tau + F(\alpha(x, y, y)G(x, y, y)) \le F(\phi(\mu(x, y)))$
where

$$\begin{aligned}\mu(x, y) &= \max\{G(x, y, y), G(x, x, y), G(x, x, x), G(y, y, y), G(y, x, y)\}\\ &= \max\{G(x, y, y), G(x, x, y)\} = G(x, y, y),\end{aligned}$$

since G is symmetric. Thus

$$\tau + F(\alpha(x, y, y)G(x, y, y)) \le F(\phi(G(x, y, y))).$$

From the properties of F and ϕ, and $\alpha(x, y, y) \ge 1$, we have

$$G(x, y, y) < G(x, y, y),$$

which is a contradiction. Hence $x = y$. Therefore T has a unique fixed point in X. □

The following result deals about existence and uniqueness of fixed point without the assumption of G is symmetric on $Fix(T)$ unlike Theorem 3.9.

Theorem 3.10. *Let (X, G) be a complete $G-$metric space, T be a self-mapping on X and $F : [0, \infty) \to (-\infty, \infty)$ be a strictly increasing function. Let $\phi : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\phi(t) < t$ for all $t > 0$ and continuous at 0. Suppose that there exists $\tau > 0$ such that*

(i) $\alpha(x, y, Ty)G(Tx, Ty, T^2y) > 0 \implies$
$\tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) \le F(\phi(\max\{G(x, y, Ty), \frac{G(x,Tx,Ty)+G(x,Tx,T^2x)}{2}$
$, G(y, Ty, T^2y), G(y, T^2x, T^2y)\}))$*, for all $x, y \in X$;*
(ii) *T is rectangular $G - \alpha-$admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, T^2x_0) \ge 1$;*
(iv) *T is continuous or (H) holds;*
(v) *for all $x, y \in Fix(T), \alpha(x, y, y) \ge 1$.*

Then T has a unique fixed point in X.

Proof. The existence of a fixed point follows from Theorems 3.2 and 3.4. Let $x, y \in Fix(T)$ with $x \ne y$. Then from (i), we get
$\tau + F(\alpha(x, y, y)G(x, y, y)) \le F(\phi(\max\{G(x, y, y), \frac{G(x,x,y)+G(x,x,x)}{2}, G(y, y, y), G(y, x, y)\}))$.
Consequently, we obtain

$$\begin{aligned}\tau + F(G(x, y, y)) &\le F(\max\{G(x, y, y), \frac{G(x, x, y)}{2}\}))\\ &\le F(\max\{G(x, y, y), \frac{2G(x, y, y)}{2}\})) = F(G(x, y, y)).\end{aligned}$$

Using properties of F, ϕ and $\tau > 0$, we have

$$G(x, y, y) < G(x, y, y),$$

which is a contradiction. Thus, $x = y$. Hence T has a unique fixed point in X. □

In the following we get a corollary of our main results.

Corollary 3.11. *Let (X, G) be a complete $G-$metric space and $T : X \to X$ be a mapping. Suppose that there exist $F : [0, \infty) \to (-\infty, \infty)$ which is strictly increasing function and $\tau > 0$ such that*

(i) $\alpha(x, y, Ty)G(Tx, Ty, T^2y) > 0 \implies \tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) \leq F(\phi(k\mu(x, y)))$ *for all* $x, y \in X$ *and* $k \in (0, \frac{1}{2}]$*;*
(ii) T *is rectangular* $G - \alpha-$*admissible;*
(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0, T^2x_0) \geq 1$*;*
(iv) T *is continuous or* (H) *holds;*
(v) *for all* $x, y \in Fix(T), \alpha(x, y, y) \geq 1$.

Then T has a unique fixed point in X.

Proof. The existence of a fixed is assured by Theorem 3.2 and Theorem 3.4. Now, we shall prove that T has a unique fixed point in X. Let $x, y \in Fix(T)$ with $x \neq y$. Then from (i), we obtain
$\alpha(x, y, y)G(x, y, y) > 0 \implies \tau + F(\alpha(x, y, y)G(x, y, y)) \leq F(\phi(k\mu(x, y)))$,
where

$$\begin{aligned}\mu(x, y) &= \max\{G(x, y, y), G(x, x, y), G(x, x, x), G(y, y, y), G(y, x, y)\}\\ &= \max\{G(x, y, y), G(x, x, y)\}\\ &\leq \max\{G(x, y, y), 2G(x, y, y)\}\\ &= 2G(x, y, y).\end{aligned}$$

As a result, we obtain
$\tau + F(\alpha(x, y, y)G(x, y, y)) \leq F(\phi(2kG(x, y, y)))$. Using properties of $F1$, ϕ, $k \leq \frac{1}{2}$ and $\alpha(x, y, y) \geq 1$, we have

$$G(x, y, y) < G(x, y, y),$$

which is a contradiction. Thus, $x = y$. Hence T has a unique fixed point in X. □

In the following we give examples in support of our results.

Example 3.12. Let $X = \{0, 1, 2\}$ and $T : X \to X$ be a mapping defined by $T0 = 0, T1 = T2 = 1$. Take $\alpha : X \times X \times X \to [0, \infty)$ as $\alpha(1, 2, 1) = \alpha(2, 1, 1) = \alpha(1, 1, 1) = 1$, and $\alpha(x, y, z) = 0$ for $(x, y, z) \notin \{(1, 2, 1),\ (2, 1, 1), (1, 1, 1)\}$. Consider
$G(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\}$, where $d(x, y) = |x - y|$.
Then $G(x, y, z)$ is a $G-$metric on X and T is rectangular $G - \alpha-$admissible. Comparing the following three contractive conditions:

$$G(Tx, Ty, T^2y) > 0 \implies \tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) \leq F(\phi(\mu(x, y))). \quad (3.21)$$

$$G(Tx, Ty, T^2y) > 0 \implies \tau + \alpha(x, y, Ty)F(G(Tx, Ty, T^2y)) \leq F(\phi(\mu(x, y))). \quad (3.22)$$

$$\alpha(x, y, Ty)G(Tx, Ty, T^2y) > 0 \implies \tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) \leq F(\phi(\mu(x, y))). \quad (3.23)$$

(Condition 3.23 is our approach).
Now,we have that $G(Tx, Ty, T^2y) > 0$ for $(x, y, Ty) \in \{(0, 1, 1), (0, 2, 1)\}$. In both cases we get
$\alpha(x, y, Ty)G(Tx, Ty, T^2y) = 0$.
It means that for $(x, y, Ty) \in \{(0, 1, 1), (0, 2, 1)\}$, in 3.21 we get
$\tau + F(\alpha(x, y, Ty)G(Tx, Ty, T^2y)) = \tau + F(0) \leq F(\mu(x, y)) \leq F(2)$.
By the property of F, we have $F(0) \leq F(2)$, which is true.
Again, since $\alpha(x, y, Ty)G(Tx, Ty, T^2y) = 0$ whenever

$$(x, y, Ty) \in \{(1, 2, 1), (2, 1, 1), (1, 1, 1)\},$$

the given example does not support the condition 3.23. Furthermore, for $(x, y, Ty) \in \{(0,1,1),(0,2,1)\}$ 3.22 becomes
$\tau + 0 \times F(G(Tx, Ty, T^2y)) \leq F(\phi(\mu(x,y))$, which in turn implies,

$$\tau \leq F(\phi(2)).$$

It is true because $F(\phi(2)) > 0$. Hence, the given example supports the condition 3.22 for each strictly increasing function $F : [0,\infty) \to (-\infty,\infty)$.

Example 3.13. Consider the sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= 2, \\ s_2 &= 2 + 2^2, \\ s_3 &= 2 + 2^2 + 2^3, \\ &\vdots \\ s_n &= 2 + 2^2 + \cdots + 2^n = 2(2^n - 1). \end{aligned}$$

Let $X = \{s_n : n \in \mathbb{N}\}$,

$$G(x,y,z) = \begin{cases} 0 & , \textit{if and only if } x = y = z, \\ \max\{x,y,z\} & , \textit{otherwise}, \end{cases}$$

$$\textit{and } \alpha(x,y,z) = \begin{cases} e^{x-y} & , x \leq y, \\ 1 & , x \geq y, \\ 0 & , \textit{otherwise}. \end{cases}$$

Then (X, G) is a complete $G-$metric space.
Define the mapping $T : X \to X$ by $Ts_1 = s_1$ and $Ts_n = s_{n-1}$ for every $n > 1$. Obviously T is rectangular $G - \alpha-$admissible.
Now, we want to show that T is a generalized modified $F-$contraction via rectangular $G - \alpha-$admissible in the framework of $G-$metric spaces with $F(\theta) = \theta + \ln(\theta),\ \theta > 0$.
First, we have the following:
$\alpha(s_n, s_m, Ts_m)G(Ts_n, Ts_m, T^2s_m) > 0$ if and only if
$(n = 1 \textit{ and } m > 2)$ *or* $(m = 1 \textit{ and } n > 2)$ *or* $(1 < n < m)$ *or* $(1 < m < n)$.
In all the possible cases we will show that

$$\frac{\alpha(x,y,Ty)G(Tx,Ty,T^2y)e^{\alpha(x,y,Ty)G(Tx,Ty,T^2y)-\mu(x,y)}}{\mu(x,y)} \leq e^{-\tau}, \tag{3.24}$$

where $x = s_n,\ y = s_m,\ n, m \geq 1,\ \tau = 8$ and

$$\mu(x,y) = \max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), G(y,Ty,T^2y), G(y,T^2x,T^2y)\}.$$

Case (i) when $n = 1, m > 2$. Then

$$\begin{aligned}
G(Tx, Ty, T^2y) &= G(Ts_n, Ts_m, T^2s_m) = s_{m-1},\\
G(x, y, Ty) &= G(s_n, s_m, Ts_m) = s_m,\\
G(x, Tx, Ty) &= G(s_n, Ts_n, Ts_m) = s_{m-1},\\
G(x, Tx, T^2x) &= G(s_n, Ts_n, T^2s_n) = 0,\\
G(y, Ty, T^2y) &= G(s_m, Ts_m, T^2s_m) = s_m,\\
G(y, T^2x, T^2y) &= G(s_m, T^2s_n, T^2s_m) = s_m,\\
\mu(x, y) &= s_m,\\
\alpha(x, y, z) &= e^{s_1 - s_m}.
\end{aligned}$$

Thus, using 3.24, we get

$$\begin{aligned}
\frac{e^{s_1-s_m} \times s_{m-1} \times e^{e^{s_1-s_m} \times s_{m-1} - s_m}}{s_m} &\leq e^{s_1-s_m} \times e^{s_m(\frac{e^{s_1}}{e^{s_m}}-1)}\\
&\leq e^{s_1-s_m} \times e^{s_m(\frac{e^{s_1}}{s_m}-1)}\\
&= e^{s_1-s_m} \times e^{e^{s_1}-s_m}\\
&= e^{s_1+e^{s_1}-2s_m}\\
&\leq e^{10-2s_m}\\
&\leq e^{-18},\ since\ -(-2s_m + 10) \geq 18\ for\ m > 2.\\
&\leq e^{-8}.
\end{aligned}$$

Hence, T is a generalized modified $F-$contraction via rectangular $G - \alpha-$admissible mapping with τ=8.

Case (ii) When $m = 1,\ n > 2$. Then

$$\begin{aligned}
G(Tx, Ty, T^2y) &= G(Ts_n, Ts_m, T^2s_m) = s_{n-1},\\
G(x, y, Ty) &= G(s_n, s_m, Ts_m) = s_n,\\
G(x, Tx, Ty) &= G(s_n, Ts_n, Ts_m) = s_n,\\
G(x, Tx, T^2x) &= G(s_n, Ts_n, T^2s_n) = s_n,\\
G(y, Ty, T^2y) &= G(s_m, Ts_m, T^2s_m) = 0,\\
G(y, T^2x, T^2y) &= G(s_m, T^2s_n, T^2s_m) = s_{n-2},\\
\mu(x, y) &= s_n,\\
\alpha(x, y, z) &= \alpha(s_n, s_m, z) = 1.
\end{aligned}$$

Thus, from 3.24, we obtain the following

$$\frac{s_{n-1} \times e^{s_{n-1}-s_n}}{s_n} \leq e^{s_{n-1}-s_n} \leq e^{-8}, -(-s_n + s_{n-1}) \geq 8\ for\ n > 2.$$

Case (iii) When $1 < n < m$. Then

$$\begin{aligned} G(Tx, Ty, T^2y) &= G(Ts_n, Ts_m, T^2s_m) = s_{m-1}, \\ G(x, y, Ty) &= G(s_n, s_m, Ts_m) = s_m, \\ G(x, Tx, Ty) &= G(s_n, Ts_n, Ts_m) = s_{m-1}, \\ G(x, Tx, T^2x) &= G(s_n, Ts_n, T^2s_n) = s_n, \\ G(y, Ty, T^2y) &= G(s_m, Ts_m, T^2s_m) = s_m, \\ G(y, T^2x, T^2y) &= G(s_m, T^2s_n, T^2s_m) = s_m, \\ \mu(x, y) &= s_m, \\ \alpha(x, y, z) &= e^{s_n - s_m}. \end{aligned}$$

Again, using 3.24, we have

$$\frac{e^{s_n - s_m} \times s_{m-1} \times e^{(e^{s_n - s_m}) \times (s_{m-1} - s_m)}}{s_m} \leq e^{s_n - s_m} \times e^{(e^{s_n - s_m} - 1)s_m} < e^{s_n - s_m}$$

$$\leq e^{-8}, \; since \; -(-s_m + s_n) \geq 8 \; for \; 1 < n < m.$$

Case (iv) When $1 < m < n$. Then

$$\begin{aligned} G(Tx, Ty, T^2y) &= G(Ts_n, Ts_m, T^2s_m) = s_{n-1}, \\ G(x, y, Ty) &= G(s_n, s_m, Ts_m) = s_n, \\ G(x, Tx, Ty) &= G(s_n, Ts_n, Ts_m) = s_n, \\ G(x, Tx, T^2x) &= G(s_n, Ts_n, T^2s_n) = s_n, \\ G(y, Ty, T^2y) &= G(s_m, Ts_m, T^2s_m) = s_m, \\ G(y, T^2x, T^2y) &= G(s_m, T^2s_n, T^2s_m) = s_m \; or \; s_{n-2} \\ \mu(x, y) &= s_n, \\ \alpha(x, y, z) &= \alpha(s_n, s_m, z) = 1. \end{aligned}$$

Furthermore, from 3.24, we get

$$\frac{s_{n-1} \times e^{s_{n-1} - s_n}}{s_n} \leq e^{s_{n-1} - s_n} \leq e^{-8}, \; since \; -(-s_n + s_{n-1}) \geq 8.$$

Therefore, T is a generalized modified $F-$contraction via rectangular $G - \alpha - admissible$ mapping. Moreover, T is continuous on X, $\alpha(s_1, Ts_1, T^2s_1) = 1 \geq 1 \; and \; s_1 = 2$ is a fixed point of T.

Example 3.14. Consider the sequence $\{s_n\}$ defined as

$$\begin{aligned} s_1 &= 3, \\ s_2 &= 3 + 3^2, \\ s_3 &= 3 + 3^2 + 3^3, \\ &\vdots \\ s_n &= 3 + 3^2 + \cdots + 3^n = \frac{3}{2}(3^n - 1). \end{aligned}$$

Let $X = \{s_n : n \in N\}$,

$$G(x,y,z) = \begin{cases} 0, & if\ and\ only\ if\ x = y = z, \\ \max\{x,y,z\}, & otherwise, \end{cases}$$

$$and\ \alpha(x,y,z) = \begin{cases} 1, & x \geq y, \\ 0, & otherwise. \end{cases}$$

Then (X, G) is a complete $G-$metric space.

Define the mapping $T : X \to X$ by $Ts_1 = s_1$ and $Ts_n = s_{n-1}$ for every $n \geq 1$. Obviously T is a rectangular $G - \alpha-$admissible.

Now, we shall prove that T is a generalized modified $F-$contraction via rectangular $G - \alpha - admissible$ in the framework of $G-$metric spaces with $F(\theta) = \theta + ln\theta,\ \theta > 0$. First, we see the following cases:
$\alpha(s_n, s_m, Ts_m)G(Ts_n, Ts_m, T^2 s_m) > 0$ if and only if $(m = 1\ and\ n > 2)\ or\ (1 < m < n)$. In all the possible cases we will show that

$$\frac{\alpha(x,y,Ty)G(Tx,Ty,T^2y)e^{\alpha(x,y,Ty)G(Tx,Ty,T^2y)-\mu(x,y)}}{\mu(x,y)} \leq e^{-\tau}, \tag{3.25}$$

where $x = s_n,\ y = s_m,\ n, m \geq 1,\ \tau = 27$ and

$$\mu(x,y) = \max\{G(x,y,Ty), G(x,Tx,Ty), G(x,Tx,T^2x), G(y,Ty,T^2y), G(y,T^2x,T^2y)\}.$$

Case (i) When $m = 1,\ n > 2$. Then

$$\begin{aligned} G(Tx,Ty,T^2y) &= G(Ts_n, Ts_m, T^2s_m) = s_{n-1}, \\ G(x,y,Ty) &= G(s_n, s_m, Ts_m) = s_n, \\ G(x,Tx,Ty) &= G(s_n, Ts_n, Ts_m) = s_n, \\ G(x,Tx,T^2x) &= G(s_n, Ts_n, T^2s_n) = s_n, \\ G(y,Ty,T^2y) &= G(s_m, Ts_m, T^2s_m) = G(s_1,s_1,s_1) = 0, \\ G(y,T^2x,T^2y) &= G(s_m, T^2s_n, T^2s_m) = G(s_m, s_{n-2}, s_{m-2}) = s_m\ or\ s_{n-2}, \\ \mu(x,y) &= s_n, \\ \alpha(x,y,z) &= 1. \end{aligned}$$

Using 3.25, we get

$$\frac{s_{n-1}e^{s_{n-1}-s_n}}{s_n} \leq e^{-s_n+s_{n-1}} \leq e^{-27},\ \ since\ s_n - s_{n-1} \geq 27\ \ for\ n > 2.$$

Hence, T is a generalized modified $F-$contraction via rectangular $G - \alpha-$admissible with $\tau = 27$.

Condition (ii) When $1 < m < n$. Then

$$\begin{aligned}
G(Tx,Ty,T^2y) &= G(Ts_n,Ts_m,T^2s_m) = s_{n-1},\\
G(x,y,Ty) &= G(s_n,s_m,Ts_m) = s_n,\\
G(x,Tx,Ty) &= G(s_n,Ts_n,Ts_m) = s_n,\\
G(x,Tx,T^2x) &= G(s_n,Ts_n,T^2s_n) = s_n,\\
G(y,Ty,T^2y) &= G(s_m,Ts_m,T^2s_m) = s_m,\\
G(y,T^2x,T^2y) &= G(s_m,T^2s_n,T^2s_m) = G(s_m,s_{n-2},s_{m-2}) = s_m \ or \ s_{n-2},\\
\mu(x,y) &= s_n,\\
\alpha(x,y,z) &= 1.
\end{aligned}$$

Using 3.25, we have

$$\frac{s_{n-1}e^{s_{n-1}-s_n}}{s_n} \leq e^{-s_n+s_{n-1}} \leq e^{-27},\ since\ -(s_n - s_{n-1}) \geq 27\ for\ n > 2.$$

Thus, T is a generalized modified $F-$contraction via rectangular $G-\alpha-$admissible mapping. Moreover, T is continuous on X and $\alpha(s_1,Ts_1,T^2s_1) = 1 \geq 1$ implies $s_1 = 3$ is a fixed point of T.

4. Application

In this section, we give an application of our main result to find a solution of an integral equation. Consider a set of real valued continuous functions $X = C[0,I]$ defined on $[0,I]$ endow with
$G(f,g,h) = \max\big\{\sup_{t\in[0,I]}|f(t)-g(t)|, \sup_{t\in[0,I]}|f(t)-h(t)|, \sup_{t\in[0,I]}|g(t)-h(t)|\big\}$. Then (X,G) is a complete $G-$metric space. Consider the following integral equation

$$x(t) = y(t) + \frac{1}{6I}\int_0^I K(t,s,x(s))ds, \tag{4.26}$$

where $t,s \in [0,I]$, y is a continuous function in X and $K : [0,I]\times[0,I]\times X \to (-\infty,\infty)$ is a given continuous function.

Theorem 4.1. *Let (X,G) be a complete $G-$metric space and define a continuous operator $T : X \to X$ by*

$$Tx(t) = y(t) + \frac{1}{6I}\int_0^I K(t,s,x(s))ds, \tag{4.27}$$

for all $x,y \in X$ with $x \neq y$ and $t,s \in [0,I]$ satisfying the following inequality

$$\begin{aligned}
&\big|K(t,s,x(s)) - K(t,s,y(s))\big|\\
&\leq \max\big\{|x(s)-y(s)|, |x(s)-Ty(s)|, |y(s)-Ty(s)|, |x(s)-Tx(s)|,\\
&|Tx(s)-Ty(s)|, |x(s)-T^2x(s)|, |Tx(s)-T^2x(s)|, |y(s)-T^2y(s)|,\\
&|Ty(s)-T^2y(s)|, |y(s)-T^2x(s)|, |T^2x(s)-T^2y(s)|\big\}.
\end{aligned} \tag{4.28}$$

Then, the integral equation 4.26 has a solution in X.

Proof. Define a function $\alpha : X\times X\times X \to [0,\infty]$ *by* $\alpha(x,y,z) = 1$ for all $x,y,z \in X$. Therefore, T is a rectangular $G-\alpha-$admissible. Let $F(\theta) = ln\theta,\ \theta > 0$. Let $x_0 \in X$ and a sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$ for all $n \geq 0$. By 4.27, we get

$$x_{n+1}(t) = Tx_n(t) = y(t) + \frac{1}{6I}\int_0^I K(t,s,x_n(s))ds.$$

We prove that T is a generalized modified $F-$contraction *via* rectangular $G-\alpha-$admissible mapping. By 4.27 and 4.28, we have

$$\begin{aligned}\big|Tx(t)-Ty(t)\big| &= \frac{1}{6I}\Big|\int_0^I K(t,s,Tx(s))-K(t,s,Ty(s))ds\Big|\\ &\leq \frac{1}{6I}\int_0^I \big|K(t,s,Tx(s))-K(t,s,Ty(s))\big|ds.\end{aligned}$$

Taking the sup norm on both sides for all $t\in[0,I]$, we get

$$||Tx-Ty|| = \sup_{t\in[0,I]}\big|Tx(t)-Ty(t)\big|$$

$$\begin{aligned}&\leq \frac{1}{6I}\sup_{t\in[0,I]}\int_0^I \max\big\{|x(s)-y(s)|,|x(s)-Ty(s)|,|y(s)-Ty(s)|,|x(s)-Tx(s)|,\\ &\quad |Tx(s)-Ty(s)|,|x(s)-T^2x(s)|,|Tx(s)-T^2x(s)|,|y(s)-T^2y(s)|,\\ &\quad |Ty(s)-T^2y(s)|,|y(s)-T^2x(s)|,|T^2x(s)-T^2y(s)|\big\}ds\\ &= \frac{1}{6I}\sup_{t\in[0,I]}\big\{|x(t)-y(t)|,|x(t)-Ty(t)|,|y(t)-Ty(t)|,|x(t)-Tx(t)|,|Tx(t)-Ty(t)|,\\ &\quad |x(t)-T^2x(t)|,|Tx(t)-T^2x(t)|,|y(t)-T^2y(t)|,\\ &\quad |Ty(t)-T^2y(t)|,|y(t)-T^2x(t)|,|T^2x(t)-T^2y(t)|\big\}\int_0^I ds\\ &= \frac{1}{6}\max\big\{||x-y||,||x-Ty||,||y-Ty||,||x-Tx||,||Tx-Ty||,\\ &\quad ||x-T^2x||,||Tx-T^2x||,||y-T^2y||,||Ty-T^2y||,||y-T^2x||,||T^2x-T^2y||\big\}\\ &= \frac{1}{3}\phi(\mu(x,y)),\end{aligned}$$

where $\phi(t)=\dfrac{t}{2}$ for $t>0$.

It follows that, $||Tx-T^2y||\leq \dfrac{2}{3}\phi(\mu(x,y))$ *and* $||Ty-T^2y||\leq\dfrac{1}{3}\phi(\mu(x,y))$.

Therefore, $G(Tx,Ty,T^2y)\leq\dfrac{2}{3}\phi(\mu(x,y))$. Thus, we have

$$\alpha(x,y,Ty)G(Tx,Ty,T^2y)\leq\frac{2}{3}\phi(\mu(x,y)).$$

Now, applying the natural logarithm on both sides, we have

$$ln\frac{3}{2}+ln(\alpha(x,y,Ty)G(Tx,Ty,T^2y))\leq ln\phi(\mu(x,y)).$$

Hence, we get

$$\tau+F(\alpha(x,y,Ty)G(Tx,Ty,T^2y))\leq F(\phi(\mu(x,y))),$$

where $\tau=ln\dfrac{3}{2}$. As a result, T is a generalized modified $F-$contraction *via* rectangular $G-\alpha-$admissible mapping. Hence, all the conditions of Theorem 3.2 are satisfied. Thus, T has a fixed point.

Therefore, the integral equation 4.26 has a solution in X. □

5. Conclusion

In this study, we introduced a generalized modified $F-$contraction via rectangular $G-\alpha-$admissible mapping under Wardowski's $F-$contractive condition $(F1)$ together with an auxiliary function ϕ. Within this framework, we established new fixed point theorems in the setting of complete $G-$metric spaces. Our new results generalize and extend several related fixed point theorems in the existing literature. In addition, we provided some non-trivial examples to verify our results. Finally, we applied one of our main results to show the existence of solutions to an integral equation in complete $G-$metric spaces.

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Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in this manuscript.

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