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RECONSTRUCTION OF PIECEWISE CONSTANT POTENTIAL WITH PHASELESS REFLECTION COEFFICIENT

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This paper is dedicated to Prof. Hsiao-Yun Liu on the occasion of his retirement.

Abstract. Let us consider the scattering theory of one-dimensional Schrödinger equation defined by piecewise constant potentials with compact support. We construct the asymptotics of reflection coefficient by iterating the transition matrices. The jump points of the step potential appear in the asymptotics of reflection coefficient. Accordingly, we investigate the modulus of reflection coefficient. The modulus is connected to the zero set of reflection coefficient via Nevanlinna-Levin type of representation theorem. The growth theory of zero set of the reflection coefficient plays a role.

1. Introduction

We consider the one-dimensional Schrödinger equation

$$-\psi''(x,k)+q(x)\psi(x,k)=k^2\psi(x,k),\qquad x\in\mathbb{R},\tag{1.1}$$

where k^2 is energy, x is the space coordinate, and $q(x)$ is the potential belonging to $\mathcal{P}(\mathbb{R})$, the class of piecewise constant potentials with compact support. We assume there is no eigenvalue of (1.1) in this paper. In particular, $q(x)$ in (1.1) is piecewise-defined constant function with the real value q_n on the interval (x_{n-1},x_n) for $n=0,1,\cdots,N+1$, where $x_{-1}<0=x_0<\cdots<x_N<x_{N+1}=\infty$ and $0=q_0<q_1<\cdots<q_N<\infty$. We require that $q_0=q_{N+1}=0$ for the compactness of the support.

For such potentials, there are two linearly independent solutions of (1.1), $f_l(x,k)$ and $f_r(x,k)$, known as the Jost solutions from the left and from the right, respectively, satisfying the boundary conditions

$$f_l(x,k)=e^{ikx},\ x\to\infty;\tag{1.2}$$
$$f_r(x,k)=e^{-ikx},\ x\to-\infty.\tag{1.3}$$

The scattering matrix associated with (1.1) is defined as

$$S(k)=\left[\begin{array}{cc}T(k)&R(k)\\L(k)&T(k)\end{array}\right],\tag{1.4}$$

where $T(k)$ is the transmission coefficient, $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. These coefficients appear in the asymptotics of the Jost solutions in the form

$$\begin{aligned}f_l(x,k)&=\tfrac{1}{T(k)}e^{ikx}+\tfrac{L(k)}{T(k)}e^{-ikx},\ x\to-\infty;\\ f_r(x,k)&=\tfrac{1}{T(k)}e^{-ikx}+\tfrac{R(k)}{T(k)}e^{ikx},\ x\to\infty.\end{aligned}\tag{1.5}$$

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We may consider the scattering phenomenon in time-dependent model through the following one-dimensional wave equation

$$\begin{cases} \left(D_x^2 - D_y^2 + V(x)\right)A_\pm(x,y) = 0; \\ A_\pm(x,y) = \delta(x-y),\ \pm x \gg 0, \end{cases} \tag{1.6}$$

where $D_y A_-(x,y) = X(y-x) + Y(y+x)$. In particular [28, p. 727],

$$X(x) - \delta'(x) + \frac{\int_{\mathbb{R}} V(t)dt}{2}\delta(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}); \tag{1.7}$$

$$Y(y) - \frac{V(y/2)}{4} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{1.8}$$

and

$$\hat{X}(k) \sim ik - \frac{\int_{\mathbb{R}} V(t)dt}{2} + o(1),\ \Im k \geq 0; \tag{1.9}$$

$$\hat{Y}(k) \sim o(1),\ \Im k \geq 0. \tag{1.10}$$

Hence, $A_\pm(x,y)$ satisfies the wave equation with x taking the place of time (this choice is dictated by the forcing condition imposed). The uniqueness part then follows from the energy estimates of the wave equation [9, 35]. Accordingly, we can connect the scattering matrix in (1.4) in form of

$$S(k) := \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix} = \begin{pmatrix} \frac{ik}{\hat{X}(k)} & \frac{\hat{Y}(k)}{\hat{X}(k)} \\ \frac{\hat{Y}(-k)}{\hat{X}(k)} & \frac{ik}{\hat{X}(k)} \end{pmatrix}, \tag{1.11}$$

Inverse scattering problems for (1.1) with a variety of conditions about the potentials are abundant in the literature [6, 9, 8, 11, 24, 28, 32]. For integrable potential $q(x)$ on $\mathbb{R}$ with the finite first moment [11], Faddeev recovered $q(x)$ from the bound states, the bound state norming constants and the reflection coefficient. In [8], Deift and Trubowitz proved that if two potentials have no bound states but with the same $R(k)$, then the two potentials are identical. This paper addresses the inverse scattering problem of recovering information about the potential of (1.1) directly from the reflectivity $|R(k)| = |L(k)|$ without considering the bound states and the bound state norming constants. In this paper, we analyze the compactly-supported potentials with a finite number of jump discontinuities. Our aim is to determine the location and magnitude of the jumps from the scattering data.

The phaseless problem arises from inverse scattering theory in many settings, e.g., electron microscopy, crystallography, medical imaging, astronomy, and nano-optics [6, 10, 12, 13, 16, 17, 18, 19, 20, 23, 27, 29, 30, 31, 33, 34]. To image the objects in these micro/nano-structures, it is impractical to measure the phase of scattered waves. Therefore, we are interested to study the inverse problem of (1.1) without the phase information. That is to determine the potential V by the measurements partially or fully on functional modulus $\{|R(k)|, |L(k)|, |T(k)|\}$ in (1.4). In optical imaging, when the light (plane wave) originates from a laser, it is natural that the detection device, e.g., charge-coupled device (CCD) cameras, photosensitive films, and human eyes, can not measure the phase information of light wave. The optical devices rely on converting photons to electrons (current), say, photodiodes, and do not allow for direct recording of the phase. The electromagnetic field oscillates at rates of $\sim 10^{14}$ to 10^{16} Hz, to which sometimes no electronic measurement device can follow. A regular crystal frequency commonly used in watches is 2^{15} Hz. In this case, measuring the phase of optical waves could involve additional complexity and budget. When there are no bound states, the potential is determined by the reflection coefficients. However, it is known that in general one cannot uniquely determine the potential solely neither from the reflectivity $|L(k)|$ nor $|R(k)|$. Let us refer to [4, 7, 19, 20]. For $V = V^1 + V^2$, there is a series of uniqueness by including the knowledge of known non-trivial potential V_1 which is placed to the left of the unknown potential V_2 [2, 3, 14, 15, 19, 20, 25, 26].

Now we state the main result of this paper.

Theorem 1.1. *Let $V^j \in \mathcal{P}(\mathbb{R})$ be potential function with reflection coefficient $R^j(k)$, $j = 1, 2$ such that (1.1) has no eigenvalue. If reflectivity $|R^1(k)| = |R^2(k)|$ for $k \in \mathbb{R}$, then $V^1 = V^2$.*

This paper is organized as follows. In Section 2, we will use Aktosun's construction [1] to establish the reflection coefficient $R(k)$ step by step through each discontinuous point. In Section 3, we list a few results from complex analysis to investigate the zero distribution of a function and its modulus. In Section 4, we will prove the main result of this paper.

2. Direct Scattering Asymptotics

For the direct scattering problem, we construct the asymptotics of reflection coefficient $R(k)$. In particular, we consider the potential $q(x)$ in (1.1) as a piecewise constant function with the real value q_n on the interval (x_{n-1}, x_n) for $n = 0, 1, \cdots, N+1$, where $-\infty = x_{-1} < x_0 < \cdots < x_N < x_{N+1} = \infty$. We note that $q_0 = q_{N+1} = 0$ for the compactness of the support. Let $\Delta x_n = x_n - x_{n-1}$ for $n = 1, \cdots, N$, in particular, and let $\Delta x_0 = x_0$.

For each $n = 0, 1, \cdots, N+1$, on the interval (x_{n-1}, x_n), (1.1) becomes

$$-\psi''(x,k) = \rho_n^2(k)\psi(x,k), \qquad x \in (x_{n-1}, x_n), \tag{2.12}$$

where $\rho_n^2(k) = k^2 - q_n$. More precisely, we define $\rho_n(k)$ for $\Im k \geq 0$ by choosing the branch of $\rho_n(k) = \sqrt{k^2 - q_n}$ with $\Im \rho_n(k) \geq 0$. Thus, $\rho_n(k)$ is analytic for $\Im k > 0$ and continuous on $\Im k \geq 0$. For the case $q_n > 0$, the map $k \to \rho_n(k)$ is one-to-one from $\{k : \Im k > 0\}$ onto

$$\{\rho_n : \Im \rho_n > 0\} \setminus \{\rho_n = it : 0 < t \leq \sqrt{q_n}\}.$$

For the case $q_n < 0$, the map $k \to \rho_n(k)$ is one-to-one from

$$\{k : \Im k > 0\} \setminus \{k = it : 0 < t \leq \sqrt{-q_n}\}$$

onto

$$\{\rho_n : \Im \rho_n > 0\}.$$

We shall write $k_n = k(\rho_n)$ for the inverse function, and note that $k(\rho_n)$ is analytic on $\{\rho_n : \Im \rho_n > 0 \text{ and } |\rho_n| \gg 1\}$.

Furthermore, we observe that the solution of (2.1) must be of form

$$\psi(x,k) = \begin{cases} \alpha_0 e^{ikx} + \beta_0 e^{-ikx}, \ x \in (x_{-1}, x_0); \\ \alpha_n e^{i\rho_n(x - x_{n-1})} + \beta_n e^{-i\rho_n(x - x_{n-1})}, \ x \in (x_{n-1}, x_n), \end{cases} \tag{2.13}$$

for $n = 1, 2, \ldots, N+1$. There is a linear map from the coefficients (α_n, β_n) to $(\alpha_{n+1}, \beta_{n+1})$, which can be computed using the continuity of ψ and ψ' at x_n. Here we refer to [1]. Precisely,

$$\begin{cases} \psi(x_n, k) = \alpha_{n+1} + \beta_{n+1} = \alpha_n e^{i\rho_n \Delta x_n} + \beta_n e^{-i\rho_n \Delta x_n}, \\ \psi'(x_n, k) = i\rho_{n+1}(\alpha_{n+1} - \beta_{n+1}) = i\rho_n(\alpha_n e^{i\rho_n \Delta x_n} - \beta_n e^{-i\rho_n \Delta x_n}), \end{cases}$$

we can find

$$\begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = M_n(k) \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \tag{2.14}$$

where

$$M_n(k) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \rho_n/\rho_{n+1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{i\rho_n \Delta x_n} & 0 \\ 0 & e^{-i\rho_n \Delta x_n} \end{bmatrix} \tag{2.15}$$

for $n = 0, 1, \cdots, N$. Note that $\rho_0 = \rho_{N+1} = k$ and $\Delta x_0 = x_0$. Moreover, we have

$$\begin{bmatrix} \alpha_{N+1} \\ \beta_{N+1} \end{bmatrix} = \Lambda_N(k) \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}$$

where $\Lambda_N(k) = M_N(k) \cdots M_0(k)$. Since $\det(M_n(k)) = \rho_n/\rho_{n+1}$, thus $\det(\Lambda_N(k)) = 1$.

To precisely express the transition matrix $\Lambda_N(k)$, we shall set

$$\Lambda_N(k) = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix}.$$

For the Jost solution $f_l(x,k)$, by the boundary conditions (1.3) and (1.2), we have

$$\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{T(k)} \\ \frac{L(k)}{T(k)} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \alpha_{N+1} \\ \beta_{N+1} \end{bmatrix} = \begin{bmatrix} e^{ikx_N} \\ 0 \end{bmatrix},$$

and hence

$$\begin{bmatrix} \frac{1}{T(k)} \\ \frac{L(k)}{T(k)} \end{bmatrix} = \Lambda_N(k)^{-1} \begin{bmatrix} \alpha_{N+1} \\ \beta_{N+1} \end{bmatrix} = \begin{bmatrix} D(k) & -B(k) \\ -C(k) & A(k) \end{bmatrix} \begin{bmatrix} e^{ikx_N} \\ 0 \end{bmatrix},$$

thus, we know that

$$D(k) = \frac{1}{T(k)} e^{-ikx_N} \qquad \text{and} \qquad C(k) = \frac{-L(k)}{T(k)} e^{-ikx_N}. \tag{2.16}$$

On the other hand, for the Jost solution $f_r(x,k)$, by (1.2) and (1.3), we have

$$\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \alpha_{N+1} \\ \beta_{N+1} \end{bmatrix} = \begin{bmatrix} \frac{R(k)}{T(k)} e^{ikx_N} \\ \frac{1}{T(k)} e^{-ikx_N} \end{bmatrix},$$

and hence

$$\begin{bmatrix} \frac{R(k)}{T(k)} e^{ikx_N} \\ \frac{1}{T(k)} e^{-ikx_N} \end{bmatrix} = \Lambda_N(k) \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

thus we also have

$$B(k) = \frac{R(k)}{T(k)} e^{ikx_N} \qquad \text{and} \qquad D(k) = \frac{1}{T(k)} e^{-ikx_N}. \tag{2.17}$$

For k real, it is well known that

$$T(-k)R(k) + T(k)L(-k) = 0$$

and

$$T(k)T(-k) + L(k)L(-k) = 1,$$

we conclude that $A(k) = \frac{1}{T(-k)} e^{ikx_N}$. Thus

$$\Lambda_N(k) = \begin{bmatrix} A(k) & B(k) \\ C(k) & D(k) \end{bmatrix} = \begin{bmatrix} e^{ikx_N} & 0 \\ 0 & e^{-ikx_N} \end{bmatrix} \begin{bmatrix} \frac{1}{T(-k)} & \frac{R(k)}{T(k)} \\ \frac{-L(k)}{T(k)} & \frac{1}{T(k)} \end{bmatrix}$$

and hence

$$\begin{bmatrix} \frac{1}{T(-k)} & \frac{R(k)}{T(k)} \\ \frac{-L(k)}{T(k)} & \frac{1}{T(k)} \end{bmatrix} = \begin{bmatrix} e^{-ikx_N} & 0 \\ 0 & e^{ikx_N} \end{bmatrix} M_N(k) M_{N-1}(k) \cdots M_0(k).$$

The analysis for piecewise constant potentials begins from the study of the asymptotics of $M_n(k)$ for sufficiently large k in $\Im k \geq 0$. Following from

$$\frac{\rho_n}{\rho_{n+1}} = \sqrt{\frac{k^2 - q_n}{k^2 - q_{n+1}}} = 1 + \frac{q_{n+1} - q_n}{2k^2} + O\left(\frac{1}{k^4}\right), \qquad |k| \to \infty,$$

we know that

$$\begin{bmatrix} 1 & 0 \\ 0 & \rho_n/\rho_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{q_{n+1} - q_n}{2k^2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + O\left(\frac{1}{k^4}\right). \tag{2.18}$$

Using the one-to-one correspondence between ρ_n and k, we use Newton's binomial theorem to deduce that, for $\Im k \geq 0$ and $|k| \to \infty$,

$$\rho_n = k - \frac{q_n}{2k} - \frac{(q_n)^2}{8k^3} + O\left(\frac{1}{k^5}\right). \tag{2.19}$$

Moreover, we have

$$\begin{bmatrix} e^{i\rho_n \Delta x_n} & 0 \\ 0 & e^{-i\rho_n \Delta x_n} \end{bmatrix} = O\left(e^{|\Im k| \Delta x_n}\right). \tag{2.20}$$

Thus, by (2.6) and (2.8), (2.3) has the asymptotic behaviour

$$M_n(k) = J_n + \frac{q_{n+1} - q_n}{4k^2} F J_n + O\left(\frac{e^{|\Im k| \Delta x_n}}{k^4}\right),$$

where

$$J_n = \begin{bmatrix} e^{i\rho_n \Delta x_n} & 0 \\ 0 & e^{-i\rho_n \Delta x_n} \end{bmatrix} \qquad \text{and} \qquad F = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

for $n = 0, 1, \cdots, N$. A simple computation gives that

$$\begin{aligned} \Lambda_N(k) &= \prod_{n=0}^{N} \left\{ J_n + \tfrac{q_{n+1} - q_n}{4k^2} F J_n + O\left(\tfrac{e^{|\Im k| \Delta x_n}}{k^4}\right) \right\} \\ &= \prod_{n=0}^{N} J_n + \sum_{n=0}^{N} \tfrac{q_{n+1} - q_n}{4k^2} \left(\prod_{m=n+1}^{N} J_m \right) F \left(\prod_{m=0}^{n} J_m \right) + O\left(\tfrac{e^{|\Im k| x_N}}{k^4} \right). \end{aligned} \tag{2.21}$$

Using two simple facts

$$\prod_{n=l}^{m} J_n = \begin{bmatrix} e^{i \sum_{n=l}^{m} \rho_n \Delta x_n} & 0 \\ 0 & e^{-i \sum_{n=l}^{m} \rho_n \Delta x_n} \end{bmatrix},$$

and

$$\begin{bmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{ib} & 0 \\ 0 & e^{-ib} \end{bmatrix} = \begin{bmatrix} e^{i(a+b)} & -e^{i(a-b)} \\ -e^{-i(a-b)} & e^{-i(a+b)} \end{bmatrix},$$

we have

$$\begin{aligned} &\left(\prod_{m=n+1}^{N} J_m \right) F \left(\prod_{m=0}^{n} J_m \right) \\ &= \begin{bmatrix} e^{i \sum_{m=0}^{N} \rho_m \Delta x_m} & -e^{i\left(\sum_{m=n+1}^{N} - \sum_{m=0}^{n} \right) \rho_m \Delta x_m} \\ -e^{-i\left(\sum_{m=n+1}^{N} - \sum_{m=0}^{n} \right) \rho_m \Delta x_m} & e^{-i \sum_{m=0}^{N} \rho_m \Delta x_m} \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned}\Lambda_N(k) \quad &= \quad \begin{bmatrix} e^{i\sum\limits_{m=0}^{N}\rho_m\Delta x_m} & 0 \\ 0 & e^{-i\sum\limits_{m=0}^{N}\rho_m\Delta x_m} \end{bmatrix} \\ &+\sum_{n=0}^{N}\frac{q_{n+1}-q_n}{4k^2}\begin{bmatrix} 0 & -e^{i\left(\sum\limits_{m=n+1}^{N}-\sum\limits_{m=0}^{n}\right)\rho_m\Delta x_m} \\ -e^{-i\left(\sum\limits_{m=n+1}^{N}-\sum\limits_{m=0}^{n}\right)\rho_m\Delta x_m} & 0 \end{bmatrix} \\ &+O\left(\frac{e^{|\Im k|x_N}}{k^4}\right).\end{aligned}$$

Following from

$$\begin{bmatrix} \frac{1}{T(-k)} & \frac{R(k)}{T(k)} \\ \frac{-L(k)}{T(k)} & \frac{1}{T(k)} \end{bmatrix} = \begin{bmatrix} e^{-ikx_N} & 0 \\ 0 & e^{ikx_N} \end{bmatrix}\Lambda_N(k),$$

and comparing the entries of the above matrices, we deduce that

$$\frac{\hat{X}(k)}{ik} \quad = \quad \frac{1}{T(k)} = \exp\left\{i\left(\sum_{n=1}^{N}(k-\rho_n)\Delta x_n\right)\right\} + O\left(\frac{e^{|\Im k|x_N}}{k^4}\right); \tag{2.22}$$

$$\begin{aligned}\frac{\hat{Y}(k)}{ik} \quad &= \quad \frac{R(k)}{T(k)} = -\sum_{n=0}^{N}\frac{q_{n+1}-q_n}{4k^2}\exp\left\{i\left(\sum_{m=n+1}^{N}\rho_m\Delta x_m-\sum_{m=0}^{n}\rho_m\Delta x_m-kx_N\right)\right\} \\ &+O\left(\frac{e^{|\Im k|x_N}}{k^4}\right),\ k\in\mathbb{C}.\end{aligned} \tag{2.23}$$

Finally, using

$$R(k) = \frac{R(k)}{T(k)}/\frac{1}{T(k)},$$

we obtain

$$-4k^2R(k) = e^{-2ikx_N}\sum_{n=0}^{N}(q_{n+1}-q_n)\exp\left\{2i\sum_{m=n+1}^{N}\rho_m\Delta x_m\right\} + O\left(\frac{e^{|\Im k|x_N}}{k^2}\right), \tag{2.24}$$

that is, outside the poles of $R(k)$,

$$-4k^2R(k) = \sum_{n=0}^{N}(q_{n+1}-q_n)\exp\left\{-2ikx_n + 2i\sum_{m=n+1}^{N}(\rho_m-k)\Delta x_m\right\} + O\left(\frac{e^{|\Im k|x_N}}{k^2}\right).$$

Example 2.1. Let us examine the case for $N = 3$. That is,

$$\begin{aligned}-4k^2R(k) \quad &= \quad q_1e^{-2ikx_0+2i\{(\rho_1-k)\Delta x_1+(\rho_2-k)\Delta x_2+(\rho_3-k)\Delta x_3)\}} \\ &+(q_2-q_1)e^{-2ikx_1+2i\{(\rho_2-k)\Delta x_2+(\rho_3-k)\Delta x_3)\}} \\ &+(q_3-q_2)e^{-2ikx_2+2i(\rho_3-k)\Delta x_3} \\ &-q_3e^{-2ikx_3}+O\left(\frac{e^{|\Im k|x_3}}{k^2}\right),\end{aligned}$$

where

$$\rho_m\Delta x_m = k\Delta x_m - \frac{q_m\Delta x_m}{2k} - \frac{(q_m)^2\,\Delta x_m}{8k^3} + O\left(\frac{1}{k^5}\right). \tag{2.25}$$

3. Zeros of Exponential Polynomial

For reader's connivence, we review some results from complex analysis [5, 21].

Definition 3.1. Let $f(z)$ be an entire function. Let

$$M_f(r) := \max_{|z|=r} |f(z)|.$$

An entire function of $f(z)$ is said to be a function of finite order if there exists a positive constant k such that the inequality

$$M_f(r) < e^{r^k}$$

is valid for all sufficiently large values of r. The greatest lower bound of such numbers k is called the order of the entire function $f(z)$. By the type σ of an entire function $f(z)$ of order ρ, we mean the greatest lower bound of positive number A for which asymptotically we have

$$M_f(r) < e^{Ar^\rho}.$$

That is,

$$\sigma_f := \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}.$$

If $0 < \sigma_f < \infty$, then we say $f(z)$ is of normal type or mean type. For $\sigma_f = 0$, we say $f(z)$ is of minimal type.

We refer the details to [21].

Definition 3.2. Let $f(z)$ be an integral function of finite order ρ in the angle $[\theta_1, \theta_2]$. We call the following quantity as the indicator function of function $f(z)$.

$$h_f(\theta) := \lim_{r\to\infty} \frac{\ln |f(re^{i\theta})|}{r^\rho}, \, \theta_1 \leq \theta \leq \theta_2.$$

Lemma 3.3. *Let f, g be two entire functions. Then the following two inequalities hold.*

$$h_{fg}(\theta) \leq h_f(\theta) + h_g(\theta), \text{ if one limit exists}; \tag{3.26}$$

$$h_{f+g}(\theta) \leq \max_\theta \{h_f(\theta), h_g(\theta)\}, \tag{3.27}$$

where the equality in (3.26) holds if one of the functions is of completely regular growth, and secondly the equality (3.27) holds if the indicator of the two summands are not equal at some θ_0.

Proof. We can find the details in [21]. ☐

Definition 3.4. The following quantity is called the width of the indicator diagram of entire function f:

$$d = h_f(\frac{\pi}{2}) + h_f(-\frac{\pi}{2}).$$

Theorem 3.5 (Cartwright)**.** *Let f be an entire function of exponential type with zero set $\{a_k\}$. We assume f satisfies one of the following conditions:*

$$\text{the integral } \int_{-\infty}^{\infty} \frac{\ln^+ |f(x)|}{1+x^2} dx \text{ exists.}$$

$$|f(x)| \text{ is bounded on the real axis.}$$

Then

(1) *all of the zeros of the function $f(z)$, except possibly those of a set of zero density, lie inside arbitrarily small angles $|\arg z| < \epsilon$ and $|\arg z - \pi| < \epsilon$, where the density*

$$\Delta_f(-\epsilon, \epsilon) = \Delta_f(\pi - \epsilon, \pi + \epsilon) = \lim_{r\to\infty} \frac{N(f, -\epsilon, \epsilon, r)}{r} = \lim_{r\to\infty} \frac{N(f, \pi - \epsilon, \pi + \epsilon, r)}{r}, \quad (3.28)$$

is equal to $\frac{d}{2\pi}$, where d is the width of the indicator diagram in (3.4). Furthermore, the limit $\delta = \lim_{r\to\infty} \delta(r)$ exists, where

$$\delta(r) := \sum_{\{|a_k|<r\}} \frac{1}{a_k};$$

(2) *moreover,*

$$\Delta_f(\epsilon, \pi - \epsilon) = \Delta_f(\pi + \epsilon, -\epsilon) = 0;$$

(3) *the function $f(z)$ can be represented in the form*

$$f(z) = cz^m e^{i\kappa z} \lim_{r\to\infty} \prod_{\{|a_k|<r\}} (1 - \frac{z}{a_k}),$$

where c, m, κ are constants and κ is real;

(4) *the indicator function of f is of the form*

$$h_f(\theta) = \sigma|\sin\theta|. \quad (3.29)$$

We refer the last statement to Levin [22, p. 126].

Lemma 3.6. *Let Blaschke product of zeros of $\hat{Y}^j(k)$ be defined as*

$$B^j(k) := \prod_{n=1}^{\infty} \frac{1 - \frac{k}{\bar{b}^j_n}}{1 - \frac{k}{b^j_n}}$$

where $\{b^j_n\}$ are the zeros of $\hat{Y}^j(k)$ in $\mathbb{C}^+$. Then, the indicator function

$$h_{B^j}(\theta) = 0.$$

Proof. We refer the theoretical details to Boas [5, p. 115], and Levin [21, p. 139; p. 237]. □

Lemma 3.7. *The Fourier transform $\hat{X}(z)$ as in (1.6) is of Cartwright class, and the function can be represented in the form*

$$\hat{X}(z) = cz^m e^{i\delta z} \lim_{R\to\infty} \prod_{|\sigma_n|<R} (1 - \frac{z}{\sigma_n}),\ z = x + iy,$$

where $\delta \in \mathbb{R}$, and the following integral converges:

$$\int_{-\infty}^{\infty} \frac{\ln^+ |\hat{X}(x)|}{1 + x^2} dx < \infty. \quad (3.30)$$

Similar results hold for $\hat{Y}(k)$.

Proof. We refer the definition of Cartwright class to [21, 22]. □

Theorem 3.8 (Nevanlinna-Levin)**.** *If the function $F(z)$ is holomorphic and of exponential type in the half-plane $\Im z \geq 0$, and if (3.30) holds, then*

(1)

$$F(z) \prod_{k=1}^{\infty} \frac{1 - \frac{z}{\bar{a}_k}}{1 - \frac{z}{a_k}} = e^{i\gamma} e^{u(z) + iv(z)}, \quad (3.31)$$

where

$$u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\ln |F(t)|}{(t - x)^2 + y^2} dt + \sigma_F^+ y,$$

$\sigma_F^+ = h_F(\frac{\pi}{2})$, $v(z)$ is the harmonic conjugate of $u(z)$, and $\{a_k\}$ are the zeros of the function $F(z)$ in the half-plane $\Im z > 0$;

(2)
$$\ln|F(z)| = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\ln|F(t)|}{(t-x)^2+y^2}dt + \sigma_F^+ y + \ln|\chi(z)|,\ z = x+iy,$$

where
$$\chi(z) = \prod_{k=1}^{\infty}\frac{1-\frac{z}{a_k}}{1-\frac{z}{\bar{a}_k}}.$$

Proof. We refer the proof to [21, p. 240].

□

4. Proof of Theorem 1.1

From theorem assumption, we use the unitary property of (1.11). Thus,
$$|T(k)|^2 + |R(k)|^2 = 1.$$
If $|R^1(k)| = |R^2(k)|$, then
$$|T^1(k)| = |T^2(k)|,\ k \in \mathbb{R},$$
Hence, by analytic extension,
$$R^1(k)R^1(-k) = R^2(k)R^2(-k);\ T^1(k)T^1(-k) = T^2(k)T^2(-k),\ k \in \mathbb{C}. \tag{4.32}$$
From $T^1(k)T^1(-k) = T^2(k)T^2(-k)$, we deduce that
$$\hat{X}^1(k)\hat{X}^1(-k) = \hat{X}^2(k)\hat{X}^2(-k),\ k \in \mathbb{C}. \tag{4.33}$$
We have assumed that V^j has no eigenvalue, so the zeros of $\hat{X}^1(k)$ are all in $\mathbb{C}^-$. In this case, we deduce from (4.33) that $\hat{X}^1(k)$ and $\hat{X}^2(k)$ have the same zero set in $\mathbb{C}^-$. Moreover, we use Theorem 3.8 in $\mathbb{C}^+$ to deduce that
$$\hat{X}^1(k)\prod_{n=1}^{\infty}\frac{1-\frac{k}{\bar{a}_n^1}}{1-\frac{k}{a_n^1}} = e^{i\gamma}\hat{X}^2(k)\prod_{n=1}^{\infty}\frac{1-\frac{k}{\bar{a}_n^2}}{1-\frac{k}{a_n^2}}, \tag{4.34}$$
where $\{a_n^j\}$ are the zeros of $\hat{X}^j(k)$ in $\mathbb{C}^+$ which is empty by assumption, that is,
$$\hat{X}^1(k) = e^{i\gamma}\hat{X}^2(k). \tag{4.35}$$
Using (1.9), we obtain that
$$\hat{X}^1(k) = \hat{X}^2(k). \tag{4.36}$$
Moreover,
$$R^j(k) = \frac{\hat{Y}^j(k)}{\hat{X}^j(k)},\ j = 1, 2. \tag{4.37}$$
Hence, we have
$$\hat{Y}^1(k)\hat{Y}^1(-k) = \hat{Y}^2(k)\hat{Y}^2(-k),\ k \in \mathbb{C}, \tag{4.38}$$
to which we use Theorem 3.8 again in $\mathbb{C}^+$ to deduce that
$$\hat{Y}^1(k)\prod_{n=1}^{\infty}\frac{1-\frac{k}{\bar{b}_n^1}}{1-\frac{k}{b_n^1}} = e^{i\gamma'}\hat{Y}^2(k)\prod_{n=1}^{\infty}\frac{1-\frac{k}{\bar{b}_n^2}}{1-\frac{k}{b_n^2}},\ \text{for some } \gamma' \in \mathbb{R}, \tag{4.39}$$
where $\{b_n^j\}$ are the zeros of $\hat{Y}^j(k)$ in $\mathbb{C}^+$. Using Cartwright's theorem Theorem 3.5, there is root set $\{k_0\}$ solves
$$\hat{Y}^1(k_0)\prod_{n=1}^{\infty}\frac{1-\frac{k_0}{\bar{b}_n^1}}{1-\frac{k_0}{b_n^1}} = \hat{Y}^2(k_0)\prod_{n=1}^{\infty}\frac{1-\frac{k_0}{\bar{b}_n^2}}{1-\frac{k_0}{b_n^2}}. \tag{4.40}$$

That is $\gamma' = 0$. Using (4.36), (4.37) and (4.39), we deduce that

$$R^1(k)\prod_{n=1}^{\infty}\frac{1-\frac{k}{\overline{b}_n^1}}{1-\frac{k}{b_n^1}} = R^2(k)\prod_{n=1}^{\infty}\frac{1-\frac{k}{\overline{b}_n^2}}{1-\frac{k}{b_n^2}}. \tag{4.41}$$

Using (2.25), for $j=1,2$,

$$-4k^2R^j(k) = \sum_{n=0}^{N}(q_{n+1}^j - q_n^j)\exp\left\{-2ikx_n^j + 2i\sum_{m=n+1}^{N}(\rho_m^j - k)\Delta x_m^j\right\} + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right),\ \Im k \geq 0. \tag{4.42}$$

We use (2.25) to deduce, for $j=1,2$,

$$-4k^2R^j(k) = \sum_{n=0}^{N}(q_{n+1}^j - q_n^j)\exp\left\{-2ikx_n^j + O(\frac{1}{k})\right\} + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right), \tag{4.43}$$

that is,

$$-4k^2R^j(k) = \sum_{n=0}^{N}(q_{n+1}^j - q_n^j)\exp\left\{-2ikx_n^j\right\}\left(1+O(\frac{1}{k})\right) + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right). \tag{4.44}$$

Using (4.41) and (4.44), we deduce that

$$\begin{aligned}&\prod_{n=1}^{\infty}\frac{1-\frac{k}{\overline{b}_n^1}}{1-\frac{k}{b_n^1}}\left\{\sum_{n=0}^{N}(q_{n+1}^1 - q_n^1)\exp\left\{-2ikx_n^1\right\}\left(1+O(\frac{1}{k})\right) + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right)\right\}\\ &= \prod_{n=1}^{\infty}\frac{1-\frac{k}{\overline{b}_n^2}}{1-\frac{k}{b_n^2}}\left\{\sum_{n=0}^{N}(q_{n+1}^2 - q_n^2)\exp\left\{-2ikx_n^2\right\}\left(1+O(\frac{1}{k})\right) + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right)\right\}.\end{aligned}$$

That is

$$\begin{aligned}&B^1(k)\left\{\sum_{n=0}^{N}(q_{n+1}^1 - q_n^1)\exp\left\{-2ikx_n^1\right\}\left(1+O(\frac{1}{k})\right) + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right)\right\}\\ &= B^2(k)\left\{\sum_{n=0}^{N}(q_{n+1}^2 - q_n^2)\exp\left\{-2ikx_n^2\right\}\left(1+O(\frac{1}{k})\right) + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right)\right\}.\end{aligned} \tag{4.45}$$

Let us define

$$H^j(k) = \sum_{n=0}^{N}(q_{n+1}^j - q_n^j)\exp\left\{-2ikx_n^j\right\}\left(1+O(\frac{1}{k})\right) + O\left(\frac{e^{|\Im k|x_N^j}}{k^2}\right).$$

Using Lemma 3.3 and Lemma 3.6, we deduce that the indicator function

$$h_{H^j}(\theta) = 2x_N^j\sin\theta; \tag{4.46}$$

$$h_{B^j}(\theta) = 0,\ \theta\in(0,\pi). \tag{4.47}$$

We deduce that $x_N^1 = x_N^2$ from (4.45) and (4.46). Therefore, we multiply (4.45) by $e^{2ikx_N^j}$, and deduce

$$B^1(k)\left\{\sum_{n=0}^{N}(q_{n+1}^1 - q_n^1)\exp\left\{-2ikx_n^1 + 2ikx_N^1\right\}\left(1 + O(\frac{1}{k})\right) + O\left(\frac{e^{-|\Im k|x_N^1}}{k^2}\right)\right\}$$
$$= B^2(k)\left\{\sum_{n=0}^{N}(q_{n+1}^2 - q_n^2)\exp\left\{-2ikx_n^2 + 2ikx_N^2\right\}\left(1 + O(\frac{1}{k})\right) + O\left(\frac{e^{-|\Im k|x_N^2}}{k^2}\right)\right\}.$$

Let compute the indicator function in $\mathbb{C}^+$ again from both sides of (4.48), and deduce $x_N^1 - x_{N-1}^1 = x_N^2 - x_{N-1}^2$. Consecutively, so we obtain that

$$\{x_n^1\} = \{x_n^2\}.$$

Now we obtain

$$B^1(k)\left\{\sum_{n=0}^{N}(q_{n+1}^1 - q_n^1)\exp\left\{-2ikx_n^1 + 2ikx_N^1\right\}\left(1 + O(\frac{1}{k})\right) + O\left(\frac{e^{-|\Im k|x_N^1}}{k^2}\right)\right\}$$
$$= B^2(k)\left\{\sum_{n=0}^{N}(q_{n+1}^2 - q_n^2)\exp\left\{-2ikx_n^1 + 2ikx_N^1\right\}\left(1 + O(\frac{1}{k})\right) + O\left(\frac{e^{-|\Im k|x_N^1}}{k^2}\right)\right\},$$

where $B^1(k)$ and $B^2(k)$ have no zero on real axis. Therefore, we deduce that

$$\sum_{n=0}^{N}\left(B^1(k)(q_{n+1}^1 - q_n^1) - B^2(k)(q_{n+1}^2 - q_n^2)\right)\exp\left\{-2ikx_n^1 + 2ikx_N^1\right\}(1 + O(\frac{1}{k}))$$
$$+O(\frac{e^{-|\Im k|x_N^1}}{k^2}) = 0. \quad (4.48)$$

We use Lemma 3.3 to compare the indictor function on both sides of (4.48), and deduce consecutively that

$$x_0 = x_1 = \cdots = x_N, \quad (4.49)$$

which is not true unless $B^1(k)(q_{n+1}^1 - q_n^1) - B^2(k)(q_{n+1}^2 - q_n^2) = 0$. Since $|B^1(k)| = |B^2(k)| = 1$ for real k,

$$q_{n+1}^1 - q_n^1 = q_{n+1}^2 - q_n^2,\ n = 0, 1, \cdots, N. \quad (4.50)$$

Then,

$$q_n^1 = q_n^2,\ n = 0, 1, \cdots, N.$$

This proves the theorem.

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