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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

ON THE RANGE OF A GENERALIZED DERIVATION

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Abstract. Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space H into itself. Given $A, B \in \mathcal{L}(H)$, the generalized derivation $\delta_{A,B} \in \mathcal{L}(\mathcal{L}(H))$ is defined by $\delta_{A,B}(X) = AX - XB$. For hyponormal operators A and B, L.A. Fialkow gave necessary conditions for which the range of $\delta_{A,B}$ is norm closed in $\mathcal{L}(H)$ [13]. In this paper, we present necessary or sufficient conditions under which the generalized derivation $\delta_{A,B}$ has closed range, where A and B are p-hyponormal operators. We give a characterization for the range of $\delta_{A,B}$ to have closed range in the case when A and B are cyclic subnormal operators.

Introduction

Let H denote a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H into itself, with identity I. For operators A and B in $\mathcal{L}(H)$, let L_A and $R_B \in \mathcal{L}(\mathcal{L}(H))$ denote respectively the operators

$$L_A(X) = AX \quad , \quad R_B(X) = XB$$

of left multiplication by A and right multiplication by B. The generalized derivation $\delta_{A,B} \in \mathcal{L}(\mathcal{L}(H))$, is defined by

$$\delta_{A,B}(X) = (L_A - R_B)(X) = AX - XB.$$

Recall that an operator $A \in \mathcal{L}(H)$ is said to be p-hyponormal, for some $0 < p \leq 1$, if $(A^*A)^p \geq (AA^*)^p$, where A^* denotes the adjoint of A. A p-hyponormal is called hyponormal if $p = 1$, and semi-hyponormal if $p = \frac{1}{2}$. An invertible operator A is called log-hyponormal if $log(A^*A) \geq log(AA^*)$.

Let $A = U|A|$ be the polar decomposition of A, we define the Aluthge transform of A by $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$. An operator A is said to be w-hyponormal if

$$|\widetilde{A}| \geq |A| \geq |(\widetilde{A})^*|.$$

It is well known that the class of w-hyponormal operators contains both the log- and p-hponormal operators.

An operator A on H is called subnormal if there exists a Hilbert space K with H subspace of K, and a normal operator $N \in \mathcal{L}(K)$ such that $N(H) \subseteq H$ and the restriction $N|H = A$. The operator A is cyclic if it has a cyclic vector.

The purpose of this paper is to present necessary or sufficient condition under which the generalized derivation $\delta_{A,B}$ has closed range, where A and B are p-hyponormal operators or A, B are cyclic subnormal operators. Many authors have studied the problem of characterizing when the range of $\delta_{A,B}$ is norm closed in $\mathcal{L}(H)$([15]).

In [6], J. Anderson and C. Foias proved that if A and B are scalar operators on a Banach space, then $R(\delta_{A,B})$ the range of $\delta_{A,B}$ is norm closed if and only if $\sigma(A) \cap \sigma(B)$ contains no limit point of $\sigma(A) \cup \sigma(B)$, where $\sigma(A)$ denote the spectrum of A.

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In particular, if $A \in \mathcal{L}(H)$ is similar to a normal operator, then $\delta_A = \delta_{A,A}$ has closed range if and only if $\sigma(A)$ is finite.

J. Stampfli [21] proved that if A is hyponormal, then $R(\delta_A)$ is closed if and only if $\sigma(A)$ is finite. Subsequently, C. Apostol and J. Stampfli studied inner derivation δ_A with closed range for the cases in which the operator A is compact, nilpotent or a weighted shift [5]. C. Apostol characterized the Hilbert space operators which induce inner derivations having closed range[4]. More precisely, he showed that the range of the derivation δ_A, is closed if and only if A is similar to a Jordan operator. A.Fialkow and D.Herrero [14] extended Apostol's results to the Calkin algebra $\mathcal{C}(H) = \mathcal{L}(H)/\mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the norm-closed ideal of all compact operators in $\mathcal{L}(H)$.

Our principal objective is to study the closedness of the range of the generalized derivation $\delta_{A,B}$, where A and B are p-hyponormal operators or cyclic subnormal operators. We shall present necessary or sufficient condition for $\delta_{A,B}$ to have closed range. Some related results are also given.

We shall use the following notations and terminology. For a bounded linear operator T, we shall denote the kernel, the orthogonal complement of the kernel, the range of T by $\ker(T)$, $\ker^{\perp}(T)$ and $R(T)$ respectively. The spectrum, the point spectrum, the approximate point spectrum, the approximate defect spectrum (or the surjectivity spectrum) and the resolvent set of T will be denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_s(T)$ and $\rho(T)$ respectively. The spectrum, the point spectrum, the approximate point spectrum and the approximate defect spectrum of T are defined as follows:

$$\begin{aligned}
\sigma(T) &= \{\lambda \in \mathbb{C} \,:\, T - \lambda I \quad \text{is not invertible } \}\\
\sigma_p(T) &= \{\lambda \in \mathbb{C} \,:\, T - \lambda I \quad \text{is not injective } \}\\
\sigma_{ap}(T) &= \{\lambda \in \mathbb{C} \,:\, T - \lambda I \quad \text{is not bounded below}\}\\
\sigma_s(T) &= \{\lambda \in \mathbb{C} \,:\, T - \lambda I \quad \text{is not surjective}\}
\end{aligned}$$

The spectral radius of T is defined by

$$r(T) = \sup \Big\{ |\lambda| : \lambda \in \sigma(T) \Big\}$$

Recall that for $x, y \in H$, the rank one operator $x \otimes y$ is defined by $(x \otimes y)z = \big\langle z, y \big\rangle x$ for all $z \in H$. Given $T \in \mathcal{L}(H)$, the lower bound of T is defined by

$$m(T) = \inf \Big\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \Big\}$$

It is well known that :

$$i(T) = \sup_n m(T^n)^{\frac{1}{n}} = \lim_n m(T^n)^{\frac{1}{n}}$$

(see ([18]) for further details). Any other notation will be explained as and when required.

1. Preliminaries

We start this section by recalling the following definitions.

Definition 1.1. ([20]) The joint point spectrum $\sigma_{jp}(A)$ is the subset of $\sigma_p(A)$ that consists of $\lambda \in \mathbb{C}$ such that there exists a nonzero vector x with $Ax = \lambda x$, $A^*x = \overline{\lambda} x$.

Definition 1.2. ([20]) A scalar λ is in the joint approximate point spectrum $\sigma_{ja}(A)$, if there exists a sequence $\{x_n\}$ of unit vectors in H such that $(A - \lambda)x_n \longrightarrow 0$ and $(A^* - \overline{\lambda})x_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 1.3. ([20]) A scalar $\lambda \in \mathbb{C}$ is in the reducing essential eigenvalues $R_e(A)$, if there is an orthonormal sequence $\{x_n\}$ in H such that

$$\|(A-\lambda)x_n\| \longrightarrow 0 \text{ and } \|(A^*-\overline{\lambda})x_n\| \longrightarrow 0.$$

Evidently, we have the inclusion $R_e(A) \cup \sigma_{jp}(A) \subset \sigma_{ja}(A)$. The opposite inclusion also holds from Theorem 6.1 [20].

Proposition 1.4. *([20]) Let $A \in \mathcal{L}(H)$, then $\sigma_{ja}(A) = R_e(A) \cup \sigma_{jp}(A)$.*

Proposition 1.5. *Let $A \in \mathcal{L}(H)$. If $\ker(A-\lambda)$ is a reducing space of A, then we have*

$$\mathcal{R}(A-\lambda) \text{ closed } \Rightarrow \lambda \text{ is isolated in } \sigma_{ap}(A).$$

Proof. Suppose that $\ker(A-\lambda)$ is a nontrivial reducing space of A. With respect to the decomposition $H = \ker(A-\lambda) \oplus \ker(A-\lambda)^\perp$ we have

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & A' \end{pmatrix}.$$

It is easily seen that $\sigma_{ap}(A) = \{\lambda\} \cup \sigma_{ap}(A')$. The closedness of the range of

$$A-\lambda = \begin{pmatrix} 0 & 0 \\ 0 & A'-\lambda \end{pmatrix},$$

implies that $\mathcal{R}(A'-\lambda)$ is closed. It follows that $(A'-\lambda)$ is injective with closed range, hence $\lambda \notin \sigma_{ap}(A')$. Since $\sigma_{ap}(A')$ is compact and $\lambda \notin \sigma_{ap}(A')$, then $V = \mathbb{C} \setminus \sigma_{ap}(A')$ is an open neighborhood of λ and $\sigma_{ap}(A) \cap V = \{\lambda\}$. Thus, λ is isolated in $\sigma_{ap}(A)$. □

Corollary 1.6. *Let $A \in \mathcal{L}(H)$ be a w-hyponormal operator. If $0 \neq \lambda \in \sigma_{ap}(A)$ and $\mathcal{R}(A-\lambda)$ is closed, then λ is isolated in $\sigma_{ap}(A)$. Furthermore, if A is p-hyponormal, the preceding assertion holds also for $\lambda = 0$.*

Proof. If A is p-hyponormal, then for every $\lambda \in \mathbb{C}$, the subspace $\ker(A-\lambda)$ is a reducing space of A. The same if A is w-hyponormal except possibly $\lambda = 0$ (see [3]). □

The following proposition is due to L.A. Fialkow [13, Corollary 4.5]. This proposition plays important roles in our arguments.

Proposition 1.7. *Let A, $B \in \mathcal{L}(H)$. If the range $\mathcal{R}(\delta_{A,B})$ is closed, then we have*

i) *$\mathcal{R}(A-\lambda)$ is closed for all $\lambda \in R_e(B)$.*
ii) *$\mathcal{R}(B-\lambda)$ is closed for all $\lambda \in R_e(A)$.*

Theorem 1.8. *Let A be a bounded operator and J a curve of the complex plane. We suppose that J and $\sigma_{ap}(A)$ are disjoint, then we have*

$$J \cap \sigma(A) \neq \emptyset \Rightarrow J \subset \sigma(A).$$

Proof. J is defined by a continuous map γ from $[a,b]$ to $\mathbb{C}$ ($\gamma([a,b]) = J$).
First, we shall prove that the following set of intervals ordered by inclusion has a maximal element by Zorn's Lemma

$$\mathcal{I} = \{[a',b'] \subset [a,b] \ : \ \gamma([a',b']) \subset \sigma(A)\}.$$

Next we prove that $\mathcal{I}$ is not void. Since $J \cap \sigma(A) \neq \emptyset$, there exists $t_0 \in [a,b]$ such that $\gamma(t_0) \in \sigma(A)$. The assumption $J \cap \sigma_{ap}(A) = \emptyset$ implies that $J \cap \partial\sigma(A) = \emptyset$ and hence $\gamma(t_0) \in \overset{\circ}{\sigma}(A)$. By continuity of γ, there exists an interval L neighborhood of t_0 such that $\gamma(L) \subset \sigma(A)$.

Let $\mathcal{C} = \{[a_i,b_i]\}_i \subset \mathcal{I}$ be a nonempty chain of intervals. Put $I = [\inf_i a_i, \sup_i b_i] \subset [a,b]$, then $I \in \mathcal{I}$ and it is an upper bound of $\mathcal{C}$. Indeed, evidently $[a_i,b_i] \subset I$ for all i. Let $\inf_i a_i < t < \sup_i b_i$, then there exists i and j such that $a_i \leqslant t$ and $t \leqslant b_j$. Since $\mathcal{C}$ is a chain,

$[a_i, b_i]$ and $[a_j, b_j]$ are comparable with inclusion ($[a_j, b_j] \subset [a_i, b_i]$ or $[a_i, b_i] \subset [a_j, b_j]$). We get $t \in [a_i, b_i]$ or $t \in [a_j, b_j]$, hence $\gamma(t) \in \sigma(A)$.

We have proved that $\gamma(]\inf_i a_i, \sup_i b_i[) \subset \sigma(A)$. By closedness of $\sigma(A)$ and continuity of γ, we get $I \in \mathcal{I}$.

We conclude that $\mathcal{I}$ has a maximal element $[a', b'] \in \mathcal{I}$. If $b' < b$, by the continuity of γ and $\gamma(b') \in \overset{\circ}{\sigma(A)}$, we can find $\epsilon > 0$ such that $\gamma([b' - \epsilon, b' + \epsilon]) \subset \sigma(A)$. It follows that $[a', b' + \epsilon] \in \mathcal{I}$ which contradicts the maximality of $[a', b']$. Hence necessarily $b' = b$ and similarly $a' = a$. Consequently $J \subset \sigma(A)$, and the proof is complete. □

Corollary 1.9. *Let $A \in \mathcal{L}(H)$ and U an unbounded open connected set of $\mathbb{C}$. If $\sigma_{ap}(A) \cap \overline{U} = \emptyset$, then $\sigma(A) \cap \overline{U} = \emptyset$.*

Proof. Suppose that $\sigma(A) \cap U \neq \emptyset$. Let z_n be an unbounded sequence of U and $z \in \sigma(A) \cap U$. The connectedness of U implies that z and z_n can be joined by a curve $J \subset U$. Since $J \cap \sigma_{ap}(A) = \emptyset$ and $J \cap \sigma(A) \neq \emptyset$, Theorem 1.8 implies that $J \subset \sigma(A)$. In particular $\sigma(A)$ contains an unbounded sequence which contradicts the boundedness of A. We have proved that $\sigma(A) \cap U = \emptyset$. In fact $\overset{\circ}{\sigma(A)} \cap \overline{U} = \emptyset$ and $\partial\sigma(A) \cap \overline{U} = \emptyset$, thus $\sigma(A) \cap \overline{U} = \emptyset$. □

Definition 1.10. ([10],[16]) A vector $e_\circ \in H$ is cyclic for $A \in \mathcal{L}(H)$, if H is the smallest invariant subspace for A that contains e_0. The operator A is said to be cyclic if it has a cyclic vector.

Definition 1.11. ([10],[16]) An operator $A \in \mathcal{L}(H)$ is called subnormal, if there exists a Hilbert space K and a normal operator $N \in L(K)$, such that H is a subspace of K and $A = N|H$. The operator N is called a normal extension of A.

Corollary 1.12. *Let $A \in \mathcal{L}(H)$ be a cyclic subnormal operator and J a Jordan curve with $J \cap \sigma_{ap}(A) = \emptyset$. If J surrounds elements of $\sigma_{ap}(A)$ (i.e the intersection of the bounded component of $\mathbb{C} \setminus J$ with $\sigma_{ap}(A)$ is not empty), then $J \cap \sigma(A) = \emptyset$.*

Proof. Suppose that $J \cap \sigma(A) \neq \emptyset$. It follows from Theorem 1.8 that J is contained in $\sigma(A)$. But $\sigma_{ap}(A) \cap J = \emptyset$ which yields that J is subset of the open set $\sigma(A) \setminus \sigma_{ap}(A)$. J is contained in a connected component U of the open set $\sigma(A) \setminus \sigma_{ap}(A)$. It results from [10, Proposition 7.12] that U is simply connected. Thus the bounded component of $\mathbb{C} \setminus J$ is contained in U. Then we get $U \cap \sigma_{ap}(A) \neq \emptyset$, which contradicts the fact that $U \subset \sigma(A) \setminus \sigma_{ap}(A)$. □

2. Main results

Theorem 2.1. *Let $A, B \in \mathcal{L}(H)$ be p-hyponormal operators. We consider the following assertions*

(i) *The range $\mathcal{R}(\delta_{A,B})$ is closed.*
(ii) *$\mathcal{R}(A - \lambda)$ and $\mathcal{R}(B - \lambda)$ are closed for all $\lambda \in \sigma_{ap}(A) \cap \sigma_{ap}(B)$ (whenever the intersection is not empty).*
(iii) *Every number in $\sigma_{ap}(A) \cap \sigma_{ap}(B)$ is isolated in $\sigma_{ap}(A) \cup \sigma_{ap}(B)$ (whenever the intersection is not empty).*

Then we have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (ii) We prove first that $\mathcal{R}(A - \lambda)$ is closed for all $\lambda \in \sigma_{jp}(B)$. Since $\delta_{A-\lambda, B-\lambda} = \delta_{A,B}$, we can suppose $0 \in \sigma_{jp}(B)$. Let x be a unit vector such that $Bx = B^*x = 0$ and $x_n, y \in H$ such that $Ax_n \to y$. We put $X_n = x_n \otimes x$ and $Y = y \otimes x$, then for all $z \in H$ we have

$$\begin{array}{rcl}\delta_{A,B}(X_n)(z) - Y(z) & = & \langle z; x\rangle Ax_n - \langle z; B^*x\rangle x_n - \langle z; x\rangle y \\ & = & \langle z; x\rangle(Ax_n - y).\end{array}$$

Hence, we get

$$\|\delta_{A,B}(X_n) - Y\|_{\mathcal{L}(H)} \leq \|Ax_n - y\|_H.$$

From this inequality, we obtain $\delta_{A,B}(X_n) \to Y$. Since $\mathcal{R}(\delta_{A,B})$ is closed, then there exists $X \in \mathcal{L}(H)$ such that $\delta_{A,B}(X) = Y$. We have

$$AX(x) - XB(x) = AX(x) - 0 = Y(x) = y.$$

By Proposition 1.7 and Proposition 1.4 it follows that $\mathcal{R}(A-\lambda)$ is closed for all $\lambda \in R_e(B) \cup \sigma_{jp}(B) = \sigma_{ja}(B)$. Since B is p-hyponormal, it results from [8, Theorem B] that $\sigma_{ap}(B) = \sigma_{ja}(B)$. Then $\mathcal{R}(A-\lambda)$ is closed for all $\lambda \in \sigma_{ap}(B)$.
We will show that the range of δ_{B^*,A^*} is closed. Indeed let $X_n, Y \in \mathcal{L}(H)$ such that $\delta_{B^*,A^*}(X_n) \to Y$, then

$$\|\delta_{A,B}(X_n^*) - (-Y^*)\| = \|[\delta_{A,B}(X_n^*) - (-Y^*)]^*\| = \|\delta_{B^*,A^*}(X_n) - Y\| \to 0.$$

Thus there exists X such that $\delta_{A,B}(X) = -Y^*$. It results $\delta_{B^*,A^*}(X^*) = Y$, i.e $\mathcal{R}(\delta_{B^*,A^*})$ is closed. Using a similar argument as above we can show that $\mathcal{R}(B^* - \overline{\lambda})$ is closed, which is equivalent to the closedness of $\mathcal{R}(B-\lambda)$, for every $\overline{\lambda} \in \sigma_{jp}(A^*)$ (equivalent to $\lambda \in \sigma_{jp}(A)$).
Using Proposition 1.7 and Proposition 1.4 , we get $\mathcal{R}(B-\lambda)$ is closed for all $\lambda \in R_e(A) \cup \sigma_{jp}(A) = \sigma_{ja}(A)$. Since A is p-hyponormal, it follows from [8, Theorem B] that $\sigma_{ap}(A) = \sigma_{ja}(A)$. Finally $\mathcal{R}(A-\lambda)$ and $\mathcal{R}(B-\lambda)$ are closed for all $\lambda \in \sigma_{ap}(A) \cap \sigma_{ap}(B)$.

(ii) $\Rightarrow$ (iii) Let $\lambda \in \sigma_{ap}(A) \cap \sigma_{ap}(B)$ and assume that $\mathcal{R}(A-\lambda)$ and $\mathcal{R}(B-\lambda)$ are closed, then Corollary 1.6 implies that λ is isolated in $\sigma_{ap}(A)$ and $\sigma_{ap}(B)$. $\square$

The following Corollary is a generalization of Stampfli's Result [21] to the larger class of p-hyponormal operators.

Corollary 2.2. *Let $A \in \mathcal{L}(H)$ be a p-hyponormal operator, then the range of δ_A is closed if and only if $\sigma(A)$ is finite.*

Proof. Suppose that the range of δ_A is closed. We deduce from Theorem 2.1 that every element in $\sigma_{ap}(A)$ is isolated, then $\sigma_{ap}(A)$ is a finite set. Since $\partial\sigma(A) \subset \sigma_{ap}(A)$, we get that $\sigma(A)$ is finite.

Now if $\sigma(A)$ is finite, we deduce from [9, Theorem 5] that A is normal with finite spectrum. We conclude immediately the closedness of the range by the following general statement: If $A \in \mathcal{L}(H_1)$ and $B \in \mathcal{L}(H_2)$ are normal operators with finite spectrum, then the range of $\delta_{A,B}$ (acting in the space $\mathcal{L}(H_2, H_1)$) is closed. Indeed, with the decomposition $H_1 = \ker(A-\lambda_1) \oplus \ldots \oplus \ker(A-\lambda_n)$ and $H_2 = \ker(B-\mu_1) \oplus \ldots \oplus \ker(B-\mu_p)$, we can write A and B as diagonal operators with spectrum respectively $\lambda_1, \cdots \lambda_n$ and $\mu_1, \cdots \mu_p$. Let $X = (X_{ij})_{1\leq i\leq n, 1\leq j\leq p} \in \mathcal{L}(H_2, H_1)$. By a simple matrix calculation, $\delta_{A,B}(X)$ has the matrix representation $((\lambda_i - \mu_j)X_{ij})_{1\leq i\leq n, 1\leq j\leq p}$ which leads obviously to the closedness of the range of $\delta_{A,B}$. $\square$

Definition 2.3. ([7]) Let E be a Banach space. An operator $A \in \mathcal{L}(E)$ possesses a generalized inverse, if there exists an operator $T \in \mathcal{L}(E)$ such that

$$ATA = A \text{ and } TAT = T.$$

In this case we will say that T is the generalized inverse of A.

Remark 2.4. 1) To prove the existence of a generalized inverse for A, it is sufficient to prove the existence of $S \in \mathcal{L}(E)$ such that $ASA = A$. We note that if A has a generalized inverse, then $\mathcal{R}(A)$ is closed.

2) In Theorem 2.1 the equivalence (i) $\Leftrightarrow$ (iii) cannot hold in general [13, Example 4.9]. But what about the equivalence (i) $\Leftrightarrow$ (ii)? This latter question is raised by L.A. Fialkow in the case of hyponormal operators [13, Question 4.11]. In what follows, we shall give a positive answer to this problem for a restricted class of operators.

Proposition 2.5. *Let $A \in \mathcal{L}(H_1)$ and $B \in \mathcal{L}(H_2)$ be cyclic subnormal operators, and $\delta_{A,B}(X) = AX - XB$ defined in $\mathcal{L}(H_2, H_1)$. If $\sigma_{ap}(A) \cap \sigma_{ap}(B) = \emptyset$, then the range of $\delta_{A,B}$ is closed.*

Proof. We shall prove that $\delta_{A,B}$ has a generalized inverse (the proof is inspired by Rosenblum's integral representation, see [19] and [12, Theorem 3.1]).
If $\sigma_{ap}(A) \cap \sigma_{ap}(B) = \emptyset$, there is a Cauchy domain D such that $\sigma_{ap}(A) \subset D$ and $\overline{D} \subset \mathbb{C} \setminus \sigma_{ap}(B)$ (left resolvent of B).
Allan's Theorem ([1], [2]) ensures the existence of analytic left inverses $L_A(\lambda)$ and $L_B(\lambda)$ of A and B respectively such that

$$L_A(\lambda)(A - \lambda) = I_{H_1},\ \lambda \in \mathbb{C} \setminus \sigma_{ap}(A).$$

$$L_B(\lambda)(B - \lambda) = I_{H_2},\ \lambda \in \mathbb{C} \setminus \sigma_{ap}(B).$$

D is the disjoint union of finite number of components D_i, all of them are Cauchy domains and $\overline{D_i} \cap \overline{D_j} = \emptyset$ for $i \neq j$. $L_B(\lambda)$ is analytic in a neighborhood of the bounded closed region $\overline{D_i}$. Cauchy's Theorem implies that $\int_{\partial D_i} L_B(\lambda) d\lambda = 0$, hence $\int_{\partial D} L_B(\lambda) d\lambda = 0$. Let us define the bounded operator

$$R(Y) = \frac{1}{2\pi i} \int_{\partial D} L_A(\lambda) Y L_B(\lambda) d\lambda, \quad Y \in \mathcal{L}(H_2, H_1),$$

∂D is positively oriented. Given $X \in \mathcal{L}(H_2, H_1)$, then

$$\begin{aligned}
R\delta_{A,B}(X) &= \frac{1}{2\pi i} \int_{\partial D} L_A(\lambda)[(A - \lambda)X - X(B - \lambda)] L_B(\lambda) d\lambda \\
&= \frac{1}{2\pi i} \int_{\partial D} X L_B(\lambda) d\lambda - \frac{1}{2\pi i} \int_{\partial D} L_A(\lambda) X (B - \lambda) L_B(\lambda) d\lambda \\
&= 0 - \frac{1}{2\pi i} \int_{\partial D} L_A(\lambda) X (B - \lambda) L_B(\lambda) d\lambda \\
\delta_{AB} R\delta_{A,B}(X) &= \frac{-1}{2\pi i} \int_{\partial D} (A - \lambda + \lambda) L_A(\lambda) X (B - \lambda) L_B(\lambda) d\lambda \\
& \quad + \frac{1}{2\pi i} \int_{\partial D} L_A(\lambda) X (B - \lambda) L_B(\lambda)(B - \lambda + \lambda) d\lambda \\
&= \frac{-1}{2\pi i} \int_{\partial D} (A - \lambda) L_A(\lambda) X (B - \lambda) L_B(\lambda) d\lambda \\
& \quad + \frac{1}{2\pi i} \int_{\partial D} L_A(\lambda) X (B - \lambda) d\lambda.
\end{aligned}$$

The boundary ∂D consists of disjoint Jordan rectifiable curves $J_1 \cup J_2 \cdots \cup J_n$.

- If J_i does not wind around points of $\sigma_{ap}(A)$ and $\sigma_{ap}(B)$, then a simple application of Cauchy's Theorem gives

$$\int_{J_i} (A - \lambda) L_A(\lambda) X (B - \lambda) L_B(\lambda) d\lambda = 0.$$

- If J_i winds around some elements of $\sigma_{ap}(A)$, but doesn't do so for elements of $\sigma_{ap}(B)$, we deduce by Corollary 1.12 that $J_i \cap \sigma(A) = \emptyset$. Then, it results from Cauchy's Theorem that

$$\begin{aligned}\int_{J_i}(A-\lambda)L_A(\lambda)X(B-\lambda)L_B(\lambda)d\lambda &= \int_{J_i}(A-\lambda)(A-\lambda)^{-1}X(B-\lambda)L_B(\lambda)d\lambda \\ &= \int_{J_i}X(B-\lambda)L_B(\lambda)d\lambda \\ &= 0.\end{aligned}$$

  The same argument for the case in which J_i surrounds elements of $\sigma_{ap}(B)$, but not so for $\sigma_{ap}(A)$.
- If J_i winds around elements of $\sigma_{ap}(A)$ and $\sigma_{ap}(B)$, then again Corollary 1.12 implies that $J_i \cap \sigma(A) = J_i \cap \sigma(B) = \emptyset$. Immediately we have

$$\begin{aligned}\int_{J_i}(A-\lambda)L_A(\lambda)X(B-\lambda)L_B(\lambda)d\lambda &= \int_{J_i}(A-\lambda)(A-\lambda)^{-1}X(B-\lambda)(B-\lambda)^{-1}d\lambda \\ &= \int_{J_i}Xd\lambda = 0.\end{aligned}$$

Consequently, we get

$$\int_{\partial D}(A-\lambda)L_A(\lambda)X(B-\lambda)L_B(\lambda)d\lambda = 0.$$

We can remark above the crucial role played by Corollary 1.12 (and hence the hypothesis of cyclic subnormality) to eliminate the last integral.
Since D is a Cauchy domain, the open set $\mathbb{C}\setminus\overline{D}$ consists of a unique unbounded component U and a finite number of bounded components U_k, each one of them is a Cauchy domain. Applying Cauchy's Theorem in every U_k gives

$$\int_{\partial U_k} L_A(\lambda)X(B-\lambda)d\lambda = 0.$$

Since $\partial D = \partial(\mathbb{C}\setminus\overline{D}) = \cup_k \partial U_k \cup \partial U = \cup_k \partial U_k \cup \partial(\mathbb{C}\setminus\overline{U})$, one obtains

$$\delta_{AB}R\delta_{A,B}(X) = \frac{1}{2\pi i}\int_{\partial(\mathbb{C}\setminus\overline{U})} L_A(\lambda)X(B-\lambda)d\lambda.$$

U is an unbounded connected set and $\sigma_{ap}(A)\cap\overline{U} = \emptyset$. We get by Corollary 1.9 that $\sigma(A)\cap\overline{U} = \emptyset$, that is, $\sigma(A)$ is contained in the Cauchy domain $\mathbb{C}\setminus\overline{U}$ (we can use the Riez-Dunford functional calculus in this domain).

$$\begin{aligned}\delta_{A,B}R\delta_{A,B}(X) &= \frac{1}{2\pi i}\int_{\partial(\mathbb{C}\setminus\overline{U})}(A-\lambda)^{-1}X(B-\lambda)d\lambda \\ &= (\frac{1}{2\pi i}\int_{\partial(\mathbb{C}\setminus\overline{U})}(A-\lambda)^{-1}d\lambda)XB - (\frac{1}{2\pi i}\int_{\partial(\mathbb{C}\setminus\overline{U})}\lambda(A-\lambda)^{-1}d\lambda)X \\ &= -XB + AX = \delta_{A,B}(X).\end{aligned}$$

. □

Theorem 2.6. *Let $A, B \in \mathcal{L}(H)$ be cyclic subnormal operators. Then the following statements are equivalent:*

(i) *The range $\mathcal{R}(\delta_{A,B})$ is closed.*
(ii) $\sigma_{ap}(A)\cap\sigma_{ap}(B) = \emptyset$, *or $\mathcal{R}(A-\lambda)$ and $\mathcal{R}(B-\lambda)$ are closed for all $\lambda \in \sigma_{ap}(A)\cap\sigma_{ap}(B)$.*

Proof. (ii) $\Rightarrow$ (i) If $\sigma_{ap}(A) \cap \sigma_{ap}(B) = \emptyset$, using Proposition 2.5, $\mathcal{R}(\delta_{A,B})$ is closed.
If $\sigma_{ap}(A) \cap \sigma_{ap}(B) \neq \emptyset$, using Theorem 2.1, $\sigma_{ap}(A) \cap \sigma_{ap}(B)$ consists of finite number of scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$.
Put $H_1 = \ker(A - \lambda_1) \oplus \ldots \oplus \ker(A - \lambda_n)$ and $H_2 = \ker(B - \lambda_1) \oplus \ldots \oplus \ker(B - \lambda_n)$. Since A and B are subnormal operators, the sums are orthogonal (hence H_1 and H_2 are closed subspaces). Furthermore, H_1 and H_2 reduces A and B respectively

$$A = \begin{array}{c} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix} \end{array} \quad \text{and} \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{matrix} H_2 \\ H_2^{\perp} \end{matrix},$$

where A_1 and B_1 are normal operators with finite spectrum. A_i and B_i remain cyclic subnormal operators. The hypothesis of closedness of $\mathcal{R}(B - \lambda_i)$ and $\mathcal{R}(A - \lambda_i)$ implies that $\sigma_{ap}(A_2) \cap \sigma_{ap}(B_2) = \sigma_{ap}(A_1) \cap \sigma_{ap}(B_2) = \sigma_{ap}(A_2) \cap \sigma_{ap}(B_1) = \emptyset$. With respect to this decomposition, let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H_2 \oplus H_2^{\perp}, H_1 \oplus H_1^{\perp})$. We have

$$\delta_{A,B}(X) = \begin{pmatrix} \delta_{A_1,B_1}(X_1) & \delta_{A_1,B_2}(X_2) \\ \delta_{A_2,B_1}(X_3) & \delta_{A_2,B_2}(X_4) \end{pmatrix}.$$

Proposition 2.5 implies that $\mathcal{R}(\delta_{A_1,B_2})$, $\mathcal{R}(\delta_{A_2,B_1})$ and $\mathcal{R}(\delta_{A_2,B_2})$ are closed. In addition A_1 and B_1 are normal operators with finite spectrum, then $\mathcal{R}(\delta_{A_1,B_1})$ is closed (see the proof of Corollary 2.2). We conclude that $\mathcal{R}(\delta_{A,B})$ is closed.
For the remaining implication (i)$\Rightarrow$ (ii) it suffices to apply Theorem 2.1. $\square$

Let B denote a weighted right shift on l^2_+ defined by $Be_n = s_n e_{n+1}$, where (e_n) is an orthonormal basis of l^2_+, $n = 1, 2, \cdots$. Then we have the following formulae(see[18]):

$$i(B) = \lim_n \inf_k |s_{k+1} \cdots s_{k+n}|^{\frac{1}{n}}$$

$$r(B) = \lim_n \sup_k |s_{k+1} \cdots s_{k+n}|^{\frac{1}{n}}.$$

We end the present paper with the following characterization:

Theorem 2.7. *Let $A, B \in \mathcal{L}(H)$ be hyponormal weighted right shifts, then we have the equivalence :*

$$\mathcal{R}(\delta_{A,B}) \textit{ is closed } \Leftrightarrow \sigma_{ap}(A) \cap \sigma_{ap}(B) \subset \{0\}.$$

Proof. Suppose that $\mathcal{R}(\delta_{A,B})$ is closed. If $0 \neq z \in \sigma_{ap}(A) \cap \sigma_{ap}(B)$, then the circle with radius $|z| \neq 0$ is contained in $\sigma_{ap}(A)$ [16, Problem 94]. By Theorem 2.1, we get a contradiction with the fact that z is isolated in $\sigma_{ap}(A)$. Hence necessarily $\sigma_{ap}(A) \cap \sigma_{ap}(B) \subset \{0\}$.

Conversely, suppose $\sigma_{ap}(A) \cap \sigma_{ap}(B) \subset \{0\}$. We know that the hyponormality of the weighted right shift $Ae_k = \alpha_k e_{k+1}$ is equivalent to $|\alpha_k| \leqslant |\alpha_{k+1}|$ for all k [18, p. 48]. Using this later equivalence, we can verify easily that the closed linear span of e_k for which we have $\alpha_k = 0$ (we denote it by H_1) is a reducing space for A. With respect to the decomposition $H = H_1 \oplus H_1^{\perp}$, we have

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix},$$

where A' is a hyponormal weighted shift with nonzero weights (whenever $H_1^{\perp} \neq 0$). Similarly, we get a decomposition for B with respect to $H = H_2 \oplus H_2^{\perp}$ by setting

$$B = \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix} \begin{matrix} H_2 \\ H_2^{\perp} \end{matrix},$$

where B' is a hyponormal weighted shift with nonzero weights (whenever $H_2^\perp \neq 0$). Let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H_2 \oplus H_2^\perp, H_1 \oplus H_1^\perp)$, we get

$$\delta_{A,B}(X) = \begin{pmatrix} 0 & R_{B'}(X_2) \\ L_{A'}(X_3) & \delta_{A',B'}(X_4) \end{pmatrix},$$

where $L_{A'}(X_3) = A'X_3$ and $R_{B'}(X_2) = X_2B'$ are respectively the left multiplication by A' and the right multiplication by B'.

It follows from [10, Theorem 6.7] that the range of A' is closed. Then there exists a generalized inverse S of A', that is, $SA'S = S$ and $A'SA' = A'$. We verify easily that L_S is a generalized inverse of $L_{A'}$, in particular the range of $L_{A'}$ is closed. Similar argument entails the closedness of the range of $R_{B'}$.

It is easily seen that if one of the spaces $H_1^\perp$ or $H_2^\perp$ is the zero space, then $\delta_{A',B'} = 0$. Suppose none of theme is the zero space. Then A' and B' are weighted right shifts with nonzero weights with approximate point spectrum $\{z \in \mathbb{C} \mid 0 < i(A') \leq |z| \leq r(A')\}$ and $\{z \in \mathbb{C} \mid 0 < i(B') \leq |z| \leq r(B')\}$ respectively ([18]). The hypothesis $\sigma_{ap}(A) \cap \sigma_{ap}(B) \subset \{0\}$ entails that these annulus are disjoint. Three cases can occur:

Case 1 : $\sigma(A') \cap \sigma(B') = \emptyset$, then it follows from Rosenblum's Theorem [19] that $\delta_{A',B'}$ is invertible.

Case 2 : $\sigma_{ap}(A') \cap \sigma(B') = \emptyset$, in particular we have $\sigma_{ap}(A') \cap \sigma_\delta(B') = \emptyset$, hence Davis-Rosenthal Theorem [11] implies that $\delta_{A',B'}$ is bounded below.

Case 3 : $\sigma(A') \cap \sigma_{ap}(B') = \emptyset$, in particular we get $\sigma_\delta(A') \cap \sigma_{ap}(B') = \emptyset$, then Davis-Rosenthal Theorem [11] implies that $\delta_{A',B'}$ is surjective.

Consequently, the range of $\delta_{A,B}$ is closed. □

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