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METHODS OF FUNCTIONAL ANALYSIS AND TOPOLOGY

THE DUAL OF (p,σ)-WEAKLY SUMMABLE SEQUENCES SPACE AND ITS APPLICATION

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Abstract. In this paper, we prove that the space of strongly (r,p,σ)-summable sequences is the topological dual of the space of (p,σ)-weakly summable sequences. As an application, we provide a new proof that an operator $T : E \to F$ is (p,σ)-absolutely continuous if and only if its adjoint $T^* : F^* \to E^*$ is strongly (p^*,σ)-continuous.

1. Introduction and preliminaries

The many focus of this paper is the examination of various sequence spaces with elements in a Banach space. These sequence spaces are closely associated with the concept of summability for operators between Banach spaces. For instance, one notable type of operator is the absolutely p-summing operator Π_p, introduced by Pietsch [18], which refers to continuous operators transforming weakly p-summable sequences into absolutely p-summable sequences (as discussed in [12]). Numerous authors adopt this methodology to characterize specific classes of linear operators defined by their summability properties [3, 5, 10]. In this regard, Cohen, in [6], defined the Banach space $\ell_p \langle E\rangle$ of strongly p-summable sequences and utilized it to delineate the ideal of strongly p-summing operators $\mathcal{D}_p$, the related dual result is also established, an operator is absolutely p-summing operator if and only if its adjoint is strongly p^*-summing operator. Similarly, Apiola in [2] investigated the duality relation among the sequence spaces of weakly p-summable, absolutely p-summable, and strongly p-summable sequences. Furthermore, Apiola's work extended to applying this duality relation to characterize the dual ideals of certain classes. His notable demonstration revealed that the adjoint of absolutely p-summing operators is strongly p^*-summing operators. This discovery underscores the interconnectedness between different types of summability and their corresponding dual properties within the domain of linear operators.

The main goal of this paper is to study the duality between Banach spaces of vector-valued sequences, and to analyze and characterize certain ideals of linear operators associated with these sequence spaces.

After this introductory section, we delve into the following topics: In Section 2, we prove that the dual space of (p,σ)-weakly summable E-valued sequences is the space of strongly (r,p,σ)-summable E^*-valued sequences. In Section 3, we show that an operator $T : E \to F$ is (p,σ)-absolutely continuous if and only if its adjoint $T^* : F^* \to E^*$ is strongly (p^*,σ)-continuous.

Finally, in Section 4, we provide a reformulation of the spaces of (p,σ)-weakly summable sequences and strongly (p,q,σ)-summable sequences. In addition, we present some new results that follow from this reformulation.

Let's outline some definitions related to the topic: According to Pietsch [12], a linear operator T from a Banach space E into another Banach space F is called absolutely p-summing, where $1 \leq p < \infty$, if there exists a constant $C \geq 0$ such that for every

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sequence $(x_i)_{i=1}^{\infty} \in \ell_{p,w}(E)$, the inequality

$$\|(T(x_i))_{i=1}^{\infty}\|_p \leq C \|(x_i)_{i=1}^{\infty}\|_{p,w}$$

holds. The space of absolutely p-summing operators is denoted by $\Pi_p(E;F)$ and the absolutely p-summing norm of $T \in \Pi_p(E;F)$ is defined as the infimum of the positive numbers C satisfying the defining inequality and it will be denoted by $\pi_p(T)$. This is equivalent to say that, if it sends weakly p-summable sequences to absolutely p-summable sequences. On the other hand. A linear operator $T : E \to F$ is called strongly p-summing [2, 6], if there exists a constant $C \geq 0$ such that for all sequences $(x_i)_{i=1}^{\infty} \in \ell_p(E)$, the inequality

$$\|(T(x_i))_{i=1}^{\infty}\|_{\langle p \rangle} \leq C \|(x_i)_{i=1}^{\infty}\|_p$$

is satisfied. The space of all strongly p-summing operators is denoted by $\mathcal{D}_p(E,F)$, and the strongly p-summing norm of $T \in \mathcal{D}_p(E,F)$ is defined as the infimum of the numbers C satisfying the defining inequality, and it will be denoted by $d_p(T)$. This is equivalent to say that, if it sends absolutely p-summable sequences to strongly p-summable sequences.

Let's review some key notation and fundamental concepts concerning sequence spaces. Let $1 \leq p \leq \infty$. We write p^* for the extended real number that satisfies $1/p + 1/p^* = 1$. Let E be a Banach space (for further details see [11, 12]).

$\triangleright$ Let $\ell_p(E)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^{\infty} \subset E$ with the norm:

$$\|(x_i)_{i=1}^{\infty}\|_p := \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{\frac{1}{p}}.$$

$\triangleright$ We denote by $\ell_{p,w}(E)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^{\infty} \subset E$ with the norm:

$$\|(x_i)_{i=1}^{\infty}\|_{p,w} := \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{\infty} |\langle x^*, x_i \rangle|^p \right)^{\frac{1}{p}}.$$

$\triangleright$ The normed space of (p,σ)-weakly summable sequences $(\ell_{p,\sigma}(E), \|\cdot\|_{p,\sigma})$ was introduced by López Molina and Sánchez Pérez in [14] to provide a characterization of the ideal of (p,σ)-absolutely continuous operators. Here, we recall some properties of this space. Let $0 \leq \sigma < 1$. Let $(x_i)_{i=1}^{\infty} \subset E$, we define:

$$H_{p,\sigma}(E)$$
$$:= \left\{ (x_i)_{i=1}^{\infty} \subset E : \delta_{p,\sigma}((x_i)_{i=1}^{\infty}) = \sup_{x^* \in B_{E^*}} \| \left(|\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^{\sigma} \right)_{i=1}^{\infty} \|_{\frac{p}{1-\sigma}} < \infty \right\}.$$

We say that the sequence $(x_i)_{i=1}^{\infty} \subset E$ is (p,σ)-weakly summable if it belongs to the vector space $\ell_{p,\sigma}(E)$ generated by the set $H_{p,\sigma}(E)$. The space $\ell_{p,\sigma}(E)$ is a normed space under the norm defined as follows:

$$\|(x_i)_{i=1}^{\infty}\|_{p,\sigma} := \inf \sum_{j=1}^{k} \delta_{p,\sigma}((x_i^j)_{i=1}^{\infty}),$$

where the infimum is taken over all representations of $(x_i)_{i=1}^{\infty}$ of the form $(x_i)_{i=1}^{\infty} = \sum_{j=1}^{k} (x_i^j)_{i=1}^{\infty}$ with $(x_i^j)_{i=1}^{\infty} \in H_{p,\sigma}(E)$, $k \in \mathbb{N}$. The relationship between these spaces is as follows:

$$\ell_{\frac{p}{1-\sigma}}(E) \subset \ell_{p,\sigma}(E) \subset \ell_{\frac{p}{1-\sigma},w}(E),$$

and for each $(x_i)_{i=1}^{\infty} \in \ell_{\frac{p}{1-\sigma}}(E)$ we have

$$\|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma},w} \leq \|(x_i)_{i=1}^{\infty}\|_{p,\sigma} \leq \|(x_i)_{i=1}^{\infty}\|_{\frac{p}{1-\sigma}}.$$

▷ A sequence $(x_i)_{i=1}^{\infty}$ in E is said to be strongly p-summable if $(\langle x_i^*, x_i\rangle)_{i=1}^{\infty} \in \ell_1$ for all $(x_i^*)_{i=1}^{\infty} \in \ell_{p^*,w}(E^*)$. We denote by $\ell_p \langle E\rangle$ the vector space of all strongly p-summable sequences in E. This is a Banach space, with the norm (see [6])

$$\|(x_i)_{i=1}^{\infty}\|_{\langle p\rangle} := \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p^*,w} \leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle|.$$

We know that [2] for $1 \leq p < \infty$,

$$(\ell_{p,w}(E))^* = \ell_{p^*} \langle E^*\rangle \text{ and } (\ell_p(E))^* = \ell_{p^*}(E^*).$$

▷ Let $1 < p, q < \infty$ and $0 \leq \sigma < 1$ such that $\frac{1}{p} + \frac{1-\sigma}{q} = 1$. A sequence $(x_i)_{i=1}^{\infty}$ in E is said to be strongly (p, q, σ)-summable if $(\langle x_i^*, x_i\rangle)_{i=1}^{\infty} \in \ell_1$ for all $(x_i^*)_{i=1}^{\infty} \in \ell_{q,\sigma}(E^*)$. We denote by $\ell_p^{q,\sigma} \langle E\rangle$ the vector space of all strongly (p, q, σ)-summable sequences in E. This is a Banach space, with the norm (see [10])

$$\|(x_i)_{i=1}^{\infty}\|_{\langle p,q,\sigma\rangle} := \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{q,\sigma} \leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle|.$$

2. Main Theorem

The main result of this section is the following:

Theorem 2.1. *Let $1 < r, p < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p} = 1$. The spaces $(\ell_{p,\sigma}(E))^*$ and $\ell_r^{p,\sigma} \langle E^*\rangle$ are isometrically isomorphic through the mapping*

$$J : \ell_r^{p,\sigma} \langle E^*\rangle \to (\ell_{p,\sigma}(E))^*$$

$$(x_i^*)_{i=1}^{\infty} \mapsto J((x_i^*)_{i=1}^{\infty})$$

given by $J((x_i^*)_{i=1}^{\infty})((x_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty} x_i^*(x_i)$, for every $(x_i)_{i=1}^{\infty} \in \ell_{p,\sigma}(E)$.

Before proceeding with the proof of the main theorem, we first establish the following lemma and recall a weak principle of local reflexivity [11].

Lemma 2.2. *Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. Then, for every operator $S \in \mathcal{L}(E; F)$ and any sequence $(x_i)_{i=1}^{\infty}$ in $H_{p,\sigma}(E)$, the sequence $(S(x_i))_{i=1}^{\infty}$ belongs to $H_{p,\sigma}(F)$ and satisfies the inequality*

$$\delta_{p,\sigma}\big((S(x_i))_{i=1}^{\infty}\big) \leq \|S\|\, \delta_{p,\sigma}\big((x_i)_{i=1}^{\infty}\big).$$

As a consequence, the canonical injection $J_E : E \to E^{**}$, defined by

$$J_E(x)(x^*) = \langle x^*, x\rangle,$$

satisfies the following: if $(x_i)_{i=1}^{\infty} \in H_{p,\sigma}(E)$, then $(J_E(x_i))_{i=1}^{\infty} \in H_{p,\sigma}(E^{**})$.

Proof. Let $S \in \mathcal{L}(E;F)$ and $(x_i)_{i=1}^{\infty} \in H_{p,\sigma}(E)$,

$$\begin{aligned}
\delta_{p,\sigma}((S(x_i))_{i=1}^{\infty}) &= \sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^{\infty}\left(|\langle S(x_i), y^*\rangle|^{1-\sigma}\|S(x_i)\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= \sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^{\infty}\left(|\langle x_i, S^*(y^*)\rangle|^{1-\sigma}\|S(x_i)\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&= \sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^{\infty}\left(\left|\left\langle x_i, \frac{S^*(y^*)}{\|S\|}\right\rangle\right|^{1-\sigma}\|S\|^{1-\sigma}\|S(x_i)\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|S\| \sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^{\infty}\left(\left|\left\langle x_i, \frac{S^*(y^*)}{\|S\|}\right\rangle\right|^{1-\sigma}\|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \\
&\leq \|S\| \sup_{\|x^*\|\leq 1}\left(\sum_{i=1}^{\infty}\left(|\langle x_i, x^*\rangle|^{1-\sigma}\|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} = \|S\|\delta_{p,\sigma}((x_i)_{i=1}^{\infty}).
\end{aligned}$$

□

Theorem 2.3. *Let M and F be normed spaces, $\dim(M) < \infty$, $R \in \mathcal{L}(M, F^{**})$, and N a subspace of F^* with $\dim(N) < \infty$. Then for every $\varepsilon > 0$, there exists an operator $S \in \mathcal{L}(M, F)$ such that:*
(a) $\|S\| \leq (1+\varepsilon)\|R\|$.
(b) $\varphi(S(y)) = R(y)(\varphi)$ for all $\varphi \in N$ and $y \in M$.

Proof Theorem 2.1. First, let's show that the operator J is well-defined. By Lemma 2.2,

$$\begin{aligned}
\left|\sum_{i=1}^{n} x_i^*(x_i)\right| \leq \sum_{i=1}^{n} |x_i^*(x_i)| &\leq \delta_{p,\sigma}((x_i)_{i=1}^{\infty}) \sum_{i=1}^{n}\left|x_i^*\left(\frac{x_i}{\delta_{p,\sigma}((x_i)_{i=1}^{\infty})}\right)\right| \\
&= \delta_{p,\sigma}((x_i)_{i=1}^{\infty}) \sum_{i=1}^{n}\left|J_E\left(\frac{x_i}{\delta_{p,\sigma}((x_i)_{i=1}^{\infty})}\right)(x_i^*)\right| \\
&\leq \delta_{p,\sigma}((x_i)_{i=1}^{\infty}) \sup_{(\psi_i)_{i=1}^{\infty}\in B_{H_{p,\sigma}(E^{**})}} \sum_{i=1}^{n}|\psi_i(x_i^*)| \\
&\leq \delta_{p,\sigma}((x_i)_{i=1}^{\infty}) \|(x_i^*)_{i=1}^{n}\|_{\langle r,p,\sigma\rangle}
\end{aligned}$$

for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^{\infty} \in H_{p,\sigma}(E)$. Letting $n \to \infty$, we obtain

$$\left|\sum_{i=1}^{\infty} x_i^*(x_i)\right| \leq \delta_{p,\sigma}((x_i)_{i=1}^{\infty}) \|(x_i^*)_{i=1}^{\infty}\|_{\langle r,p,\sigma\rangle}.$$

Now, let $(x_i)_{i=1}^{\infty} \in \ell_{p,\sigma}(E)$ and $\varepsilon > 0$, there exists $(x_i^j)_{i=1}^{\infty} \in H_{p,\sigma}(E)$ such that

$$(x_i)_{i=1}^{\infty} = \sum_{j=1}^{k}(x_i^k)_{i=1}^{\infty} \text{ and } \sum_{j=1}^{k}\delta_{p,\sigma}((x_i^j)_{i=1}^{\infty}) \leq \|(x_i)_{i=1}^{\infty}\|_{p,\sigma} + \varepsilon.$$

So, we have

$$\begin{aligned}\left|\sum_{i=1}^{\infty} x_i^*\left(x_i\right)\right| &\leq \sum_{j=1}^{k}\sum_{i=1}^{\infty}\left|x_i^*\left(x_i^j\right)\right| \\ &\leq \sum_{j=1}^{k} \delta_{p,\sigma}((x_i^j)_{i=1}^{\infty}) \left\|(x_i^*)_{i=1}^{\infty}\right\|_{\langle r,p,\sigma\rangle} \\ &\leq \left(\|(x_i)_{i=1}^{\infty}\|_{p,\sigma}+\varepsilon\right) \left\|(x_i^*)_{i=1}^{\infty}\right\|_{\langle r,p,\sigma\rangle}.\end{aligned}$$

Since this holds for all $\varepsilon > 0$, we obtain

$$\left|\sum_{i=1}^{\infty} x_i^*\left(x_i\right)\right| \leq \|(x_i)_{i=1}^{\infty}\|_{p,\sigma} \left\|(x_i^*)_{i=1}^{\infty}\right\|_{\langle r,p,\sigma\rangle}.$$

Hence, $J((x_i^*)_{i=1}^{\infty})$ is well-defined, continuous and $\|J((x_i^*)_{i=1}^{\infty})\| \leq \|(x_i^*)_{i=1}^{\infty}\|_{\langle r,p,\sigma\rangle}$, proving that J is continuous and $\|J\| \leq 1$.

Let $I_i : E \to \ell_{p,\sigma}(E)$ be the i-th canonical injection, that is,

$$I_i(x) = (\delta_{ij}x)_{j=1}^{\infty} = (0,...,0,x,0,\ldots),$$

(with x placed in the i-th position) where δ_{ij} is the Kronecker delta. Let $I : (\ell_{p,\sigma}(E))^* \to \ell_r^{p,\sigma}\langle E^*\rangle$ be the operator given by $I(\varphi) = (\varphi \circ I_i)_{i=1}^{\infty}$. Let us see that I is well defined. Let $(\xi_i)_{i=1}^{\infty} \in H_{p,\sigma}(E^{**}), M = span\{\xi_1,\ldots,\xi_n\}$, $N = span\{\psi_i,\ldots,\psi_n\}, \psi_i = \varphi\circ I_i, \varphi \in (\ell_{p,\sigma}(E))^*, i = 1,\ldots,n, n \in \mathbb{N}$, $R = id_{E^{**}}|_M$ and $\varepsilon > 0$. Then by Theorem 2.3, there exists $S \in \mathcal{L}(M,E)$ such that

$$\begin{cases}\|S\| \leq (1+\varepsilon)\|R\| = 1+\varepsilon \\ \psi_i(S(\xi_i)) = \xi_i(\psi_i)\end{cases}$$

for every $i = 1,\ldots,n$. Therefore,

$$\begin{aligned}\left|\sum_{i=1}^{n} \xi_i\left(\psi_i\right)\right| &= \left|\sum_{i=1}^{n} \psi_i\left(S\left(\xi_i\right)\right)\right| \\ &= \left|\sum_{i=1}^{n} (\varphi\circ I_i)\left(S\left(\xi_i\right)\right)\right| \\ &= \left|\sum_{i=1}^{n} \varphi((\delta_{ij}S\left(\xi_i\right))_{j=1}^{n})\right| \\ &= \left|\varphi\left((S\left(\xi_i\right))_{i=1}^{n}\right)\right| \\ &\leq \|\varphi\|\delta_{p,\sigma}((S\left(\xi_i\right))_{i=1}^{n}) \\ &\leq \|\varphi\|\|S\|\delta_{p,\sigma}((\xi_i)_{i=1}^{n}) \\ &\leq \|\varphi\|(1+\varepsilon)\delta_{p,\sigma}((\xi_i)_{i=1}^{n}).\end{aligned}$$

Since this holds for every $\varepsilon > 0$ and $n \to \infty$, we obtain

$$\left|\sum_{i=1}^{\infty} \xi_i\left(\varphi\circ I_i\right)\right| = \left|\sum_{i=1}^{\infty} \xi_i\left(\psi_i\right)\right| \leq \|\varphi\|\delta_{p,\sigma}((\xi_i)_{i=1}^{\infty}) < \infty.$$

Now let $(\xi_i)_{i=1}^{\infty} \in \ell_{p,\sigma}(E^{**})$ and $\varepsilon > 0$, there exists $(\xi_i^j)_{i=1}^{\infty} \in H_{p,\sigma}(E^{**})$ such that

$$(\xi_i)_{i=1}^{\infty} = \sum_{j=1}^{k}(\xi_i^j)_{i=1}^{\infty} \text{ and } \sum_{j=1}^{k}\delta_{p,\sigma}((\xi_i^j)_{i=1}^{\infty}) \leq \|(\xi_i)_{i=1}^{\infty}\|_{p,\sigma}+\varepsilon.$$

So, we have

$$\left|\sum_{i=1}^{\infty} \xi_i \left(\varphi \circ I_i\right)\right| = \left|\sum_{j=1}^{k}\sum_{i=1}^{\infty} \xi_i^j \left(\psi_i\right)\right| \leq \|\varphi\| \sum_{j=1}^{k} \delta_{p,\sigma}((\xi_i^j)_{i=1}^{\infty}) \leq \|\varphi\|(\|(\xi_i)_{i=1}^{\infty}\|_{p,\sigma} + \varepsilon).$$

Since this holds for all $\varepsilon > 0$, we obtain

$$\left|\sum_{i=1}^{\infty} \xi_i \left(\varphi \circ I_i\right)\right| \leq \|\varphi\| \|(\xi_i)_{i=1}^{\infty}\|_{p,\sigma},$$

for every $(\xi_i)_{i=1}^{\infty} \in \ell_{p,\sigma}\left(E^{**}\right)$, where I is well defined. I is clearly linear, and

$$\|I(\varphi)\|_{\langle r,p,\sigma\rangle} \leq \|\varphi\|.$$

Therefore, I is continuous with $\|I\| \leq 1$. Note that $I \circ J = \mathrm{id}_{\ell_r^{p,\sigma}\langle E^*\rangle}$ and $J \circ I = \mathrm{id}_{(\ell_{p,\sigma}(E))^*}$. Hence,

$$\|\varphi\| = \|J \circ I(\varphi)\| = \|J(I(\varphi))\| \leq \|I(\varphi)\| \leq \|\varphi\|,$$

Hence, J is an isometric isomorphism, which completes the proof. □

Corollary 2.4. *The duality identification $(\ell_{p,\sigma}(E))^* = \ell_r^{p,\sigma}\left\langle E^*\right\rangle$ yields a new formula for the norm $\|\cdot\|_{p,\sigma}$,*

$$\|(x_i)_{i=1}^{\infty}\|_{p,\sigma} = \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{\langle r,p,\sigma\rangle} \leq 1} \left|\sum_{i=1}^{\infty} \langle x_i^*, x_i\rangle\right|.$$

3. Applications

Many authors have studied ideals generated by the transformation of vector-valued sequences, and among these studies, [1, 7, 8, 9, 10].

We say that $T \in \mathcal{L}(E,F)$ is a (p,σ)-absolutely continuous operator, in symbols $T \in \Pi_{p,\sigma}(E,F)$, if there exist a Banach space G and an operator $S \in \Pi_p(E,G)$ such that

$$\forall x \in E, \quad \|Tx\| \leq \|x\|^{\sigma} \|Sx\|^{1-\sigma}. \tag{3.1}$$

In such case, we put $\pi_{p,\sigma}(T) = \inf\left\{\pi_p(S)^{1-\sigma}\right\}$, taking the infimum over all Banach spaces G and $S \in \Pi_p(E,G)$ such that (3.1) holds.

The class of (p,σ)-absolutely continuous linear operators is due to Matter [15]. In [14], Molina and Sánchez-Pérez presented a characterization for these operators by using the space of (p,σ)-weakly summable sequences.

Theorem 3.1. *For a linear operator $T \in \mathcal{L}(E,F)$, the following statements are equivalent:*

(i) $T \in \Pi_{p,\sigma}(E,F)$

(ii) *There is a constant $C > 0$ such that for every $(x_i)_{i=1}^{\infty} \in \ell_{p,\sigma}(E)$,*

$$\|(T(x_i))_{i=1}^{\infty}\|_{\frac{p}{1-\sigma}} \leq C \cdot \|(x_i)_{i=1}^{\infty}\|_{p,\sigma}. \tag{3.2}$$

(iii) *The induced map $\hat{T} : \ell_{p,\sigma}(E) \to \ell_{\frac{p}{1-\sigma}}(F)$, is a well-defined continuous linear operator,*

In addition, $\pi_{p,\sigma}(T) = \|\hat{T}\| = \inf\{C > 0 : C \text{satisfies } (3.2)\}$.

Definition 3.2. Let $1 < p, r < \infty$ and $0 \leq \sigma < 1$, such that $\frac{1}{r} + \frac{1-\sigma}{p^*} = 1$. An operator $T \in \mathcal{L}(E,F)$ between Banach spaces is strongly (p,σ)-continuous if there is a constant $C > 0$ such that for every $(x_i)_{i=1}^{\infty} \in \ell_r(E)$ and $(y_i^*)_{i=1}^{\infty} \in \ell_{p^*,\sigma}(F^*)$, the following inequality holds:

$$\|(\langle T(x_i), y_i^*\rangle)_{i=1}^{\infty}\|_1 \leq C \, \|(x_i)_{i=1}^{\infty}\|_r \, \|(x_i)_{i=1}^{\infty}\|_{p^*,\sigma}. \tag{3.3}$$

The class of all strongly (p,σ)-continuous linear operators from E into F is denoted by $\mathcal{D}_p^\sigma(E,F)$, and by $d_{p,\sigma}(T)$ the strongly (p,σ)-continuous norm, which is defined by

$$d_{p,\sigma}(T)=\inf\{C>0 : C\text{satisfies } (3.3)\}.$$

The class of strongly (p,σ)-continuous linear operators introduced and studed by Achour et al. in [1]. In [10], E. Dahia et al. presented a characterization for these operators by using the space of strongly (r,p,σ)-summable sequences.

Theorem 3.3. *Let $1<p,r<\infty$ and $0\leq\sigma<1$, such that $\frac{1}{r}+\frac{1-\sigma}{p^*}=1$. For $T\in\mathcal{L}(E,F)$, the following statements are equivalent:*

(i) $T\in\mathcal{D}_p^\sigma(E,F)$

(iii) *The induced map $\hat{T}:\ell_r(E)\to\ell_r^{p^*,\sigma}\langle F\rangle$, is a well-defined continuous linear operator,*

In addition, $d_{p,\sigma}(T)=\|\hat{T}\|$.

Theorem 3.4. *Let $1<p,r<\infty$ and $0\leq\sigma<1$, such that $\frac{1}{r}+\frac{1-\sigma}{p}=1$. the following statements are equivalent:*

(i) $T\in\Pi_{p,\sigma}(E,F)$

(ii) $T^*\in\mathcal{D}_{p^*}^\sigma(F^*,E^*)$.

In addition, $\pi_{p,\sigma}(T)=d_{p^*,\sigma}(T^*)$.

Proof. By Theorems 2.1, 3.1 and 3.3 we have,

$$\begin{aligned}
T\in\Pi_{p,\sigma}(E,F) &\Leftrightarrow \hat{T}\in\mathcal{L}(\ell_{p,\sigma}(E),\ell_{\frac{p}{1-\sigma}}(F))\\
&\Leftrightarrow \hat{T^*}\in\mathcal{L}((\ell_{\frac{p}{1-\sigma}}(F))^*,(\ell_{p,\sigma}(E))^*)\\
&\Leftrightarrow \hat{T^*}\in\mathcal{L}(\ell_{(\frac{p}{1-\sigma})^*}(F^*),\ell_r^{p,\sigma}\langle E^*\rangle)\\
&\Leftrightarrow \hat{T^*}\in\mathcal{L}(\ell_r(F^*),\ell_r^{p,\sigma}\langle E^*\rangle)\\
&\Leftrightarrow T^*\in\mathcal{D}_{p^*}^\sigma(F^*,E^*).
\end{aligned}$$

Morover $\pi_{p,\sigma}(T)=\|\hat{T}\|=\|\hat{T^*}\|=d_{p^*,\sigma}(T^*)$. □

4. Reformulation of Sequence Spaces and Their Consequences

Inspired by the ideas presented in [16, Section 4], we propose a restructuring of the spaces of (p,σ)-weakly summable sequences and strongly (p,q,σ)-summable sequences. Additionally, we present new results that naturally arise from this reformulation.

4.1. Weakly $p|\sigma$-summable sequences.

In this subsection, we reformulate the concept of (p,σ)-weakly summable sequences and present some results related to this reformulation. Let $1\leq p<\infty$ and $0\leq\sigma\leq 1$. Let E be a Banach space and take a sequence $(x_i)_{i=1}^\infty$ in E. We set

$$\delta_{p|\sigma}((x_i)_{i=1}^\infty):=\sup_{x^*\in B_{E^*}}\left(\sum_{i=1}^\infty\left(|\langle x_i,x^*\rangle|^{1-\sigma}\|x_i\|^\sigma\right)^p\right)^{\frac{1}{p}}.$$

We define the set $H_{p|\sigma}(E)$ by

$$H_{p|\sigma}(E):=\left\{(x_i)_{i=1}^\infty\subset E,\ \text{such that } \delta_{p|\sigma}((x_i)_{i=1}^\infty)<\infty\right\}.$$

It can be observed that for $p=\infty$, we have

$$\begin{aligned}
\delta_{\infty|\sigma}((x_i)_{i=1}^\infty) &= \sup_{x^*\in B_{E^*}}\sup_{i\in\mathbb{N}}|\langle x_i,x^*\rangle|^{1-\sigma}\|x_i\|^\sigma\\
&=\sup_{i\in\mathbb{N}}\|x_i\|^\sigma\sup_{x^*\in B_{E^*}}|\langle x_i,x^*\rangle|^{1-\sigma}\\
&=\sup_{i\in\mathbb{N}}\|x_i\|^\sigma\|x_i\|^{1-\sigma}=\sup_{i\in\mathbb{N}}\|x_i\|
\end{aligned}$$

that is, $\ell_\infty(E) = H_{\infty|\sigma}(E) = \ell_{\infty,w}(E)$ and $\delta_{\infty|\sigma}(\cdot) = \|\cdot\|_{\infty,w} = \|\cdot\|_\infty$.

In the following proposition, we present the most significant possible inclusions.

Proposition 4.1. *Let $0 \le \sigma_1, \sigma_2 \le 1$ and $1 \le p_2, p_1 \le \infty$ such that $\sigma_1 \le \sigma_2$ and $p_2 \le p_1$. Then*

$$\ell_{p_2}(E) = H_{p_2|1}(E) \subset H_{p_2|\sigma_2}(E) \subset H_{p_1|\sigma_1}(E) \subset H_{p_1|0}(E) = \ell_{p_1,w}(E).$$

Moreover,

$$\|\cdot\|_{p_1,w} = \delta_{p_1|0}(\cdot) \le \delta_{p_1|\sigma_1}(\cdot) \le \delta_{p_2|\sigma_2}(\cdot) \le \delta_{p_2|1}(\cdot) = \|\cdot\|_{p_2}.$$

To prove this proposition, we use the following lemma:

Lemma 4.2. *Let $1 \le p \le \infty$, $0 \le \sigma \le 1$ and $(x_i)_{i=1}^\infty \in H_{p|\sigma}(E)$. Then the function*

$$\begin{aligned} f : &[0,1] \to \mathbb{R} \\ &\sigma \mapsto f(\sigma) = \left(\sum_{i=1}^\infty \left(|\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^\sigma \right)^p \right)^{\frac{1}{p}} \end{aligned}$$

is increasing.

Proof. To simplify the writing, we take the following functions:

$$g_i(\sigma) = |\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^\sigma, h(\sigma) = \sum_{i=1}^\infty g_i(\sigma)^p \text{ and } f(\sigma) = h(\sigma)^{1/p}.$$

Then

$$\begin{aligned} f'(\sigma) &= (h(\sigma)^{1/p})' \\ &= \frac{1}{p} h(\sigma)^{1/p-1} h'(\sigma) = \frac{1}{p} h(\sigma)^{1/p-1} \left(\sum_{i=1}^\infty g_i(\sigma)^p \right)' \\ &= \frac{1}{p} h(\sigma)^{1/p-1} \sum_{i=1}^\infty (g_i(\sigma)^p)' \\ &= \frac{1}{p} h(\sigma)^{1/p-1} \sum_{i=1}^\infty p g_i(\sigma)^{p-1} g_i'(\sigma) = h(\sigma)^{1/p-1} \sum_{i=1}^\infty g_i(\sigma)^{p-1} (|\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^\sigma)' \\ &= h(\sigma)^{1/p-1} \sum_{i=1}^\infty g_i(\sigma)^{p-1} |\langle x_i, x^* \rangle|^{1-\sigma} \|x_i\|^\sigma \left(\ln(\|x_i\|) - \ln(|\langle x_i, x^* \rangle|) \right) \\ &= \frac{\sum_{i=1}^\infty g_i(\sigma)^p \left(\ln(\frac{\|x_i\|}{|\langle x_i, x^* \rangle|}) \right)}{h(\sigma)^{1/p^*}}. \end{aligned}$$

For every $x^* \in B_{E^*}$ we have $\ln(\frac{\|x_i\|}{|\langle x_i, x^* \rangle|}) \geqslant 0$ this implies that $f'(\sigma) \geqslant 0$. Finally, we conclude that function f is increasing. $\square$

Proof. Proposition 4.1. Since the function f is increasing, we can conclude that $f(0) \leq f(\sigma_1) \leq f(\sigma_2) \leq f(1)$ implies that

$$\begin{aligned}\left(\sum_{i=1}^{\infty} |\langle x_i, x^*\rangle|^{p_1}\right)^{\frac{1}{p_1}} &\leq \left(\sum_{i=1}^{\infty}\left(|\langle x_i, x^*\rangle|^{1-\sigma_1}\|x_i\|^{\sigma_1}\right)^{p_1}\right)^{\frac{1}{p_1}} \\ &\leq \left(\sum_{i=1}^{\infty}\left(|\langle x_i, x^*\rangle|^{1-\sigma_2}\|x_i\|^{\sigma_2}\right)^{p_1}\right)^{\frac{1}{p_1}} \\ &\leq \left(\sum_{i=1}^{\infty}\left(|\langle x_i, x^*\rangle|^{1-\sigma_2}\|x_i\|^{\sigma_2}\right)^{p_2}\right)^{\frac{1}{p_2}} \\ &\leq \left(\sum_{i=1}^{\infty}\|x_i\|^{p_2}\right)^{\frac{1}{p_2}}.\end{aligned}$$

By taking the supremum over all $x^* \in E^*$ with $\|x^*\| \leq 1$, we obtain the desired result. $\square$

A sequence $(x_i)_{i=1}^{\infty} \subset E$ is said to be weakly $p|\sigma$-summable if it is contained in the vector space spanned by $H_{p|\sigma}(E)$. When $p = 1$, it is said that $(x_i)_{i=1}^{\infty}$ in E is weakly σ-summable. We denote by $\ell_{p|\sigma}(E)$ the space of all weakly $p|\sigma$-summable sequences of elements $(x_i)_{i=1}^{\infty} \subset E$. We define the norm $\|\cdot\|_{p|\sigma}$ as follows

$$\|(x_i)_{i=1}^{\infty}\|_{p|\sigma}$$
$$:= \inf\left\{\sum_{j=1}^{k}\delta_{p|\sigma}((x_i^j)_{i=1}^{\infty}) : (x_i)_{i=1}^{\infty} = \sum_{j=1}^{k}(x_i^j)_{i=1}^{\infty},\ (x_i^j)_{i=1}^{\infty} \in H_{p|\sigma}(E),\ k \in \mathbb{N}\right\}.$$

Based on Proposition 4.1, the following corollary can be derived:

Corollary 4.3. *Let $0 \leq \sigma_1, \sigma_2 \leq 1$ and $1 \leq p_2, p_1 \leq \infty$ such that $\sigma_1 \leq \sigma_2$ and $p_2 \leq p_1$. Then*

$$\ell_{p_2}(E) = \ell_{p_2|1}(E) \subset \ell_{p_2|\sigma_2}(E) \subset \ell_{p_1|\sigma_1}(E) \subset \ell_{p_1|0}(E) = \ell_{p_1,w}(E),$$

moreover, for every $(x_i)_{i=1}^{\infty} \in \ell_{p_2}(E)$ we have

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{p_1,w} = \|(x_i)_{i=1}^{\infty}\|_{p_1|0} &\leq \|(x_i)_{i=1}^{\infty}\|_{p_1|\sigma_1} \\ &\leq \|(x_i)_{i=1}^{\infty}\|_{p_2|\sigma_2} \\ &\leq \|(x_i)_{i=1}^{\infty}\|_{p_2|1} = \|(x_i)_{i=1}^{\infty}\|_{p_2}.\end{aligned}$$

As a special case

$$\ell_p(E) \subset \ell_{p|\sigma}(E) \subset \ell_{p,w}(E), \text{ moreover } \|\cdot\|_{p,w} \leq \|\cdot\|_{p|\sigma} \leq \|\cdot\|_p. \tag{4.4}$$

We establish a relationship between the sequence spaces of weakly $p|\sigma$-summable and (p,σ)-weakly summable sequences as follows:

Remark 4.4. For $0 \leq \sigma < 1$ and $1 \leq p < \infty$. We have:

$$\ell_{p,\sigma}(E) = \ell_{\frac{p}{1-\sigma}|\sigma}(E) \text{ and } \|\cdot\|_{p,\sigma} = \|\cdot\|_{\frac{p}{1-\sigma}|\sigma}. \tag{4.5}$$

4.2. **Strongly $p|\sigma$-summable sequences.** In this subsection, we reformulate the concept of strongly (p,q,σ)-summable sequences defined by Dahia et al. in [10] and discuss some results associated with this reformulation. Let $1 \leq p \leq \infty$ and $0 \leq \sigma \leq 1$. A sequence $(x_i)_{i=1}^{\infty}$ in a Banach space E is strongly $p|\sigma$-summable if $(\langle x_i^*, x_i\rangle)_{i=1}^{\infty} \in \ell_1$ for all $(x_i^*)_{i=1}^{\infty} \in \ell_{p^*|\sigma}(E^*)$. We denote the vector space of all strongly $p|\sigma$-summable sequences in E by $\ell_{p|\sigma}\langle E\rangle$.

For $(x_i)_{i=1}^{\infty} \in \ell_{p|\sigma}\langle E\rangle$. We define the norm $\|.\|_{\langle p|\sigma\rangle}$ as follows:

$$\|(x_i)_{i=1}^{\infty}\|_{\langle p|\sigma\rangle} := \sup\left\{\sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle| : \quad (x_i^*)_{i=1}^{\infty} \in B_{\ell_{p^*|\sigma}(E^*)}\right\}.$$

From the definitions it is clear that, for every $0 \leq \sigma \leq 1$, $\ell_1(E) = \ell_{1|\sigma}\langle E\rangle = \ell_1\langle E\rangle$ and $\|\cdot\|_{\langle 1\rangle} = \|\cdot\|_{\langle 1|\sigma\rangle} = \|\cdot\|_1$.

In the following proposition, we present the most notable possible inclusions.

Proposition 4.5. *Let $0 \leq \sigma_1, \sigma_2 \leq 1$ and $1 \leq p_2, p_1 \leq \infty$ such that $\sigma_1 \leq \sigma_2$ and $p_1 \leq p_2 (\Leftrightarrow p_2^* \leq p_1^*)$. Then*

$$\ell_{p_1|0}\langle E\rangle = \ell_{p_1}\langle E\rangle \subset \ell_{p_1|\sigma_1}\langle E\rangle \subset \ell_{p_2|\sigma_2}\langle E\rangle \subset \ell_{p_2}(E) = \ell_{p_2|1}\langle E\rangle. \tag{4.6}$$

Moreover, for each $(x_i)_{i=1}^{\infty} \in \ell_{p_1}\langle E\rangle$ we have

$$\begin{aligned}\|(x_i)_{i=1}^{\infty}\|_{p_2} = \|(x_i)_{i=1}^{\infty}\|_{p_2|1} &\leq \|(x_i)_{i=1}^{\infty}\|_{\langle p_2|\sigma_2\rangle}\\ &\leq \|(x_i)_{i=1}^{\infty}\|_{\langle p_1|\sigma_1\rangle}\\ &\leq \|(x_i)_{i=1}^{\infty}\|_{\langle p_1\rangle} = \|(x_i)_{i=1}^{\infty}\|_{\langle p_1|0\rangle}.\end{aligned}$$

As a special case

$$\ell_p\langle E\rangle \subset \ell_{p|\sigma}\langle E\rangle \subset \ell_p(E), \text{ moreover } \|\cdot\|_p \leq \|\cdot\|_{\langle p|\sigma\rangle} \leq \|\cdot\|_{\langle p\rangle}. \tag{4.7}$$

Proof. According to Corollary 4.3 we obtain

$$\ell_{p_2^*}(E^*) = \ell_{p_2^*|1}(E^*) \subset \ell_{p_2^*|\sigma_2}(E^*) \subset \ell_{p_1^*|\sigma_1}(E^*) \subset \ell_{p_1^*|0}(E^*) = \ell_{p_1^*,w}(E^*).$$

So, for $(x_i)_{i=1}^{\infty} \in \ell_{p_1}\langle E\rangle$ we have

$$\begin{aligned}&\|(x_i)_{i=1}^{\infty}\|_{p_2} = \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p_2^*}\leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle| = \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p_2^*|1}\leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle| = \|(x_i)_{i=1}^{\infty}\|_{\langle p_2|1\rangle}\\ &\leq \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p_2^*|\sigma_2}\leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle| = \|(x_i)_{i=1}^{\infty}\|_{\langle p_2|\sigma_2\rangle}\\ &\leq \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p_1^*|\sigma_1}\leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle| = \|(x_i)_{i=1}^{\infty}\|_{\langle p_1|\sigma_1\rangle}\\ &\leq \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p_1^*|0}\leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle| = \sup_{\|(x_i^*)_{i=1}^{\infty}\|_{p_1^*,w}\leq 1} \sum_{i=1}^{\infty} |\langle x_i^*, x_i\rangle|\\ &= \|(x_i)_{i=1}^{\infty}\|_{\langle p_1|0\rangle} = \|(x_i)_{i=1}^{\infty}\|_{\langle p_1\rangle}.\end{aligned}$$

For $p_1 = p_2 = p$ and $\sigma_1 = \sigma_2 = \sigma$, we conclude the special case. □

We establish a connection between the sequence spaces of strongly $r|\sigma$-summable and strongly (r, p, σ)-summable sequences as follows:

Remark 4.6. Let $0 \leq \sigma < 1$ and $1 < r, p < \infty$ with $\frac{1}{r} + \frac{1-\sigma}{p} = 1$. We have:

$$\ell_r^{p,\sigma}\langle E\rangle = \ell_{r|\sigma}\langle E\rangle \quad \text{and} \quad \|\cdot\|_{\langle r,p,\sigma\rangle} = \|\cdot\|_{\langle r|\sigma\rangle}.$$

Indeed, since $\frac{1}{r}+\frac{1}{\frac{p}{1-\sigma}}=1 \Leftrightarrow r^*=\frac{p}{1-\sigma}$ we obtain

$$\begin{aligned}
\|(x_i)_{i=1}^\infty\|_{\langle r,p,\sigma\rangle} &= \sup_{\|(x_i^*)_{i=1}^\infty\|_{p,\sigma}\le 1} \sum_{i=1}^\infty |\langle x_i, x_i^*\rangle| \\
&\overset{(4.5)}{=} \sup_{\|(x_i^*)_{i=1}^\infty\|_{\frac{p}{1-\sigma}|\sigma}\le 1} \sum_{i=1}^\infty |\langle x_i, x_i^*\rangle| \\
&= \sup_{\|(x_i^*)_{i=1}^\infty\|_{r^*|\sigma}\le 1} \sum_{i=1}^\infty |\langle x_i, x_i^*\rangle| \\
&= \|(x_i)_{i=1}^\infty\|_{\langle r|\sigma\rangle}.
\end{aligned}$$

From (4.4) and (4.7) we find

$$\ell_p\langle E\rangle \subset \ell_{p|\sigma}\langle E\rangle \subset \ell_p(E) \subset \ell_{p|\sigma}(E) \subset \ell_{p,w}(E).$$

Moreover

$$\|\cdot\|_{p,w} \le \|\cdot\|_{p|\sigma} \le \|\cdot\|_p \le \|\cdot\|_{\langle p|\sigma\rangle} \le \|\cdot\|_{\langle p\rangle}.$$

The proof of the following theorem is similar to the proof of Theorem 2.1, and is therefore omitted.

Theorem 4.7. *Let $1 \le p < \infty$ and $0 \le \sigma \le 1$. The spaces $(\ell_{p|\sigma}(E))^*$ and $\ell_{p^*|\sigma}\langle E^*\rangle$ are isometrically isomorphic through the mapping*

$$J : \ell_{p^*|\sigma}\langle E^*\rangle \to (\ell_{p|\sigma}(E))^*$$
$$(x_i^*)_{i=1}^\infty \mapsto J((x_i^*)_{i=1}^\infty)$$

given by $J((x_i^*)_{i=1}^\infty)((x_i)_{i=1}^\infty) = \sum_{i=1}^\infty x_i^*(x_i)$, for every $(x_i)_{i=1}^\infty \in \ell_{p|\sigma}(E)$.

As a special case of Theorem 4.7, Theorems 2.1 and 2.2, presented by Apiola in [2], can be derived.

Corollary 4.8. *If $1 \le p < \infty$, then by Corollary 4.3, Proposition 4.5 and Lemma 2.1 we obtain:*

$$(\ell_{p,w}(E))^* = (\ell_{p|0}(E))^* = \ell_{p^*|0}\langle E^*\rangle = \ell_{p^*}\langle E^*\rangle,$$

and

$$(\ell_p(E))^* = (\ell_{p|1}(E))^* = \ell_{p^*|1}\langle E^*\rangle = \ell_{p^*}(E^*).$$

Corollary 4.9. *The duality identification $(\ell_{p|\sigma}(E))^* = \ell_{p^*|\sigma}\langle E^*\rangle$ yields a new formula for the norm $\|\cdot\|_{p|\sigma}$,*

$$\|(x_i)_{i=1}^\infty\|_{p|\sigma} = \sup_{\|(x_i^*)_{i=1}^\infty\|_{\langle p^*|\sigma\rangle}\le 1} \left|\sum_{i=1}^\infty \langle x_i^*, x_i\rangle\right|.$$

Based on the developments presented in Section 4, several new definitions can be introduced, particularly those related to sequence spaces, such as:

- The space of Roshdi strongly $(p,q|\sigma)$-summable sequences:

$$\ell_{p,q|\sigma}\langle E\rangle := \left\{(x_i)_{i=1}^\infty \subset E:\ \|(x_i)_{i=1}^\infty\|_{\langle p,q|\sigma\rangle} := \sup_{\|(x_i^*)_{i=1}^\infty\|_{q^*|\sigma}\le 1} \|(\langle x_i^*, x_i\rangle)_{i=1}^\infty\|_p < \infty\right\}.$$

- The space of mixed $(p|\sigma,q)$-summable sequences:

$$\ell^{\text{mix}}_{p|\sigma,q}(E)$$
$$:= \left\{(x_i)_{i=1}^\infty \subset E:\ \|(x_i)_{i=1}^\infty\|_{m(p|\sigma,q)} := \inf_{x_i=\lambda_i x_i^0} \left\{\|(\lambda_i)_{i=1}^\infty\|_s\ \|(x_i^0)_{i=1}^\infty\|_{p|\sigma}\right\} < \infty\right\}.$$

- The space of anisotropic $(p,q,r|\sigma)$-summable sequences:

$$\ell_{p,q}^{r|\sigma}(E)$$
$$:= \left\{ (x_i)_{i=1}^{\infty} \subset E : \ \|(x_i)_{i=1}^{\infty}\|_{p,q,r|\sigma} := \sup_{\|(x_j^*)_{j=1}^{\infty}\|_{r|\sigma} \leq 1} \left\| \left((\langle x_j^*, x_i \rangle)_{j=1}^{\infty} \right)_{i=1}^{\infty} \right\|_{\ell_q(\ell_p)} < \infty \right\}.$$

These spaces generalize the notions of strongly (p,q)-summable sequences introduced by Roshdi [13], (p,q)-mixed summable sequences introduced by Pietsch [19] and by Pérez [17], and anisotropic $(p,q,r|\sigma)$-summable sequences introduced by Campos et al. [4, 5], each of which comes with its own structural properties and consequences. Likewise, various new linear and nonlinear operator ideals associated with these sequence spaces can also be introduced and studied.

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